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NONLINEAR SYSTEMS

Consensus in Nonlinear Stationary Networks with Identical Agents

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Abstract—For the multiagent networks with arbitrary-order identical agents and nonlinear uncertain couplings, satisfying the sector inequalities consideration was given to the problem of reaching consensus (asymptotic synchronization) The network topology was assumed to be time-invariant. A frequency-domain consensus criterion extending the Popov criterion for the absolute stability of Lurie systems with one scalar-valued nonlinearity was proposed.

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1. INTRODUCTION

The last decades witnessed an appreciable growth of interest in various problems of determination the desired cooperative behavior of multiagent systems (the terms such as "complex network" etc. are used as well) through local interaction of simple subsystems called the system agents (or network "nodes"). Local property of interaction implies that the agent does not have or use information about the system as a whole and interacts only with a relatively narrow circle of its "neighbors." The asymptotic synchronization or *consensus* between the agent outputs represents one of the basic models of the cooperative behavior.

Originally the problems of consensus were studied in connection with making coordinated decisions by groups of experts. For the history of this question, the readers are referred to the reviews and monographs [1–4] and their bibliographies. Synchronization is the basic principle of operation of numerous natural and artificial systems including the self-organizing biological populations such as swarms, flocks, etc. [5], groups of mobile robots, smart power grids, and so on. Examples of consensus and related problems of controlling the multiagent systems can be found in [2, 3, 6–12].

The majority of the results on consensus belong to the networks with linear agent couplings. For the time-invariant topology and linear couplings, consensus is defined as a rule by the spectral properties of the graph [2, 4]; in the case of first-order agents, by availability of an oriented spanning tree [13]. For the case of time-varying topology, mostly studied is the case of the first and second order agents. The main tool here is represented by the results on convergence of the products of stochastic matrices and the Lyapunov methods [1–3, 14]. Some consensus problems of the agents of higher orders can be reduced to the case of the first order [15].

At the same time, it is only natural that in many applications the agent networks with nonlinear couplings arise. They are exemplified by the oscillator networks and those of mobile agents whose communication is defined by the distance [8, 17]. In practice, the linear algorithms may become nonlinear owing to the measurement errors and other distortions of the data. The existing criteria for reaching consensus in the networks with nonlinear couplings mostly refer to special agents. To study the contractive algorithms for discrete-time first-order consensus where each agent moves inwards a convex hull of its neighbors, one of the pioneering works [14] suggested the Lyapunov function which is valid also for continuous time [17]. The method of contractive maps [18] presents an alternative to the Lyapunov methods. Some consensus algorithms for the second-order agents with nonlinear couplings like the hyperbolic tangent were considered in the monographs [2, 3]. Some nonlinear consensus algorithms were examined in [20] for the case where the agents are passive objects of control [19].

Therefore, the majority of the existing results on consensus in the multiagent systems presume either special agent dynamics or linear couplings between them. Results of another kind were considered by the present author earlier in [21–23] where the agents are described by an arbitrary model in the state space and their couplings are nonlinear and need not to be known. It is assumed that they satisfy the sectorial [24] or more general quadratic inequalities, as well as the symmetry conditions. Under the above assumptions, a consensus criterion was established which is a direct generalization of the circle criterion for stability of the Lurie systems ensuring synchronization under any nonlinear couplings of the aforementioned kind. The consensus condition of [21, 23] is applicable in the case of an arbitrary time-varying topology and nonstationary couplings, but in the case of networks with fixed topology and stationary nonlinearities it proves to be conservative. It is common knowledge that conservatism of the ordinary circle criterion for the Lurie system with stationary nonlinearity can be essentially reduced by introducing an integral constraint (see, for example, [24, Theorem 1.1.6]) leading to the well-known Popov criterion [24, 25]. The present paper does a similar specification of the consensus criterion of [21, 23] for the case of stationary couplings and topology which is a network analog of the Popov criterion for stability of systems with one nonlinearity. It is shown that a number of consensus criteria can be deduced from this criterion for special agents such as the Ren criterion for synchronization of double integrators [26].

2. PROBLEM FORMULATION AND MAIN ASSUMPTIONS

The set of numbers $\{m, m+1, \ldots, n\}$ is denoted everywhere below by $m : n,\ m \leqslant n$. The components of the vector $y \in \mathbb{R}^m$, as well as the elements of the diagonal matrices $D = \mathrm{diag}(d^1, \ldots, d^N)$, are numerated by superscripts like $y = (y^1, \ldots, y^m)^{\mathrm{T}}$. For the given matrix L, the notation L^* refers to an Hermitian conjugate matrix. For the Hermitian matrix $L = L^*$, the notation $L \geqslant 0$ (correspondingly, $L > 0$) implies a nonnegative (correspondingly, positive) definiteness of L.

2.1. Auxiliary Notions and Notation

For the given $N \times N$ matrix $\Gamma = (\gamma_{jk})$ with nonnegative elements, let $G(\Gamma)$ denote a weighted graph[1] with the vertex set $1 : N$ where the edge $j \mapsto k$ from the vertex j to the vertex k exists if and only if $\gamma_{jk} \neq 0$ (γ_{jk} is treated as the weight of the edge $j \mapsto k$). If $\Gamma = \Gamma^{\mathrm{T}}$, then $G(\Gamma)$ is an undirected graph where an edge $k \mapsto j$ with the same weight corresponds to each edge $j \mapsto k$. In what follows, we confine ourselves to this case. A sequence of vertices $v_1, \ldots, v_k$ where any two neighbor terms are connected by an edge is called the *route* from v_1 to v_k. The graph is *connected* if there exists in it a route between any two distinct vertices. We denote by $D_j(\Gamma) = \sum_{k=1}^{N} \gamma_{jk}$ the sum of weights at the vertex j of the graph $G(\Gamma)$. The Kirchhoff matrix of the graph $G(\Gamma)$ which

[1] That is a graph with weights assigned to each its edge.

is also called the Laplace matrix or *Laplacian* obeys the formula

$$L(\Gamma) := \operatorname{diag}(D_1(\Gamma), \ldots, D_N(\Gamma)) - \Gamma = \begin{bmatrix} \sum\limits_{j=1}^{N} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1N} \\ -\gamma_{21} & \sum\limits_{j=1}^{N} \gamma_{2j} & \cdots & -\gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{N1} & -\gamma_{N2} & \cdots & \sum\limits_{j=1}^{N} \gamma_{Nj} \end{bmatrix}. \tag{1}$$

For $\Gamma = \Gamma^{\mathrm{T}}$, we have $L(\Gamma) = Ł(\Gamma)^{\mathrm{T}}$. At that, $L(\Gamma) \geqslant 0$ [4, 13], the least eigenvalue of the Laplacian $\lambda_1(\Gamma)$, is equal to 0, and positiveness of the second eigenvalue $\lambda_2(\Gamma)$ called the *algebraic connectivity* [27] of $G(\Gamma)$ amounts to connectivity of the graph $G(\Gamma)$ [2, 13, 27]. This value is one of the main characteristics defining the dynamics of multiagent system [4]. Although explicit calculation of the algebraic connectivity of a large graph may be a matter of some difficulty, some its estimates are known [27, 28]. In particular,

$$\lambda_2(\Gamma) = N \min_{z \in \mathfrak{Z}} \frac{\sum\limits_{j,k=1}^{N} \gamma_{jk}(z^k - z^j)^2}{\sum\limits_{j,k=1}^{N} (z^k - z^j)^2}, \quad \mathfrak{Z} := \left\{ z \in \mathbb{R}^N : \sum_{j,k=1}^{N} (z^k - z^j)^2 \neq 0 \right\}. \tag{2}$$

2.2. Problem Formulation

We consider a group of identical agents numerated from 1 to $N \geqslant 2$ and described by the following model of the state space

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t), \quad y_j(t) = Cx_j(t), \quad t \geqslant 0, \quad j \in 1:N, \tag{3}$$

where $x_j \in \mathbb{R}^n$, $u_j \in \mathbb{R}^m$, $y_j \in \mathbb{R}^m$ are, respectively, state, control, and output of the jth agent. System (3) is assumed to be controllable and observable. The control inputs are generated as the result of physical, informational, and so on interaction between the agents. The interaction topology is defined by the constant matrix $\Gamma = (\gamma_{jk}) = \Gamma^{\mathrm{T}}$ with nonnegative elements. In more precise terms, consideration is given to the distributed control algorithm (protocols) of the following form:

$$u_j^q(t) = \sum_{k=1}^{N} \gamma_{jk} \varphi_{jk}^q (y_k^q(t) - y_j^q(t)), \quad q \in 1:m. \tag{4}$$

We recall that u_j^q denotes the components of the vector $u_j = (u_j^1, \ldots, u_j^m) \in \mathbb{R}^m$. The functions $\varphi_{jk}^q : \mathbb{R} \to \mathbb{R}$ are assumed to be continuous and called *coupling functions* or simply *couplings* defining "intensity" of agents' interaction. The present paper aims at establishing the sufficient conditions for protocol (4) to establish *consensus* between the agents in the sense of [21].

Definition. Protocol (4) establishes *consensus* if all solutions of the closed-loop system (3), (4) are infinitely right prolongable and satisfy the condition

$$|x_j(t) - x_k(t)| \to 0 \quad \forall j, k \in 1:N \quad \text{for } t \to +\infty. \tag{5}$$

Under the above assumptions of controllability and observability and the assumption of coupling continuity $\varphi_{jk}^q(y)$ for $y = 0$, consensus amounts to a formally weaker requirement of consensus in output ($y_k(t) - y_j(t) \to 0$ for $t \to \infty$ under any initial data) [23, Remark 1].

At first glance, the considered below network system (3), (4) has a specific structure: the number of agent's outputs coincides with the number of inputs and the protocol is decoupled in the sense that the qth input of each agent is defined only by the values of the qth output of the neighbors. However, it is possible to reduce nominally to this form a more general class of the network systems where the agents like (3) may have different numbers of inputs $m = \dim u$ and outputs $l = \dim y$ and the protocol is given by

$$u_j^q(t) = \sum_{k=1}^{N} \gamma_{jk} \sum_{s=1}^{l} \psi_{jk}^{qs}(y_k^s(t) - y_j^s(t)), \quad q \in 1:m. \tag{6}$$

To reduce such protocol to the form of (4), one can introduce extended vectors of controls, $\xi_j \in \mathbb{R}^{ml}$, and outputs, $\sigma_j \in \mathbb{R}^{ml}$, where $\sigma_j^{ls+r} := y_j^r$ and $\xi_j^{ls+r} := \sum_k \psi_{jk}^{s+1,r}(y_k^r - y_j^r)$ for all $s \in 0 : (m-1)$ and $r \in 1 : l$. Since $u_j^q = \sum_{r=1}^{l} \xi_j^{(q-1)l+r}$, system (3), (6) may be considered as a network of controllable and observable agents

$$\dot{x}_j(t) = Ax_j(t) + \hat{B}\xi_j(t), \quad \sigma_j(t) = \hat{C}x_j(t)$$

related by a protocol like (4) where m must be replaced by ml, u_j by ξ_j, y_j by σ_j, and $\varphi_{jk}^{sl+r} := \psi_{jk}^{s+1,r}$ for all $s \in 0 : (m-1)$, $r \in 1 : l$. A practical example of reducing system (3), (6) to (3), (4) will be considered below in Section 4.2.

2.3. Main Assumptions

The main assumptions come to connectivity of the interaction graph, symmetry of the coupling functions, and sectorial constraints [24] on them.

Assumption 1. The interaction graph $G(\Gamma)$ is a connected one.

Obviously, with no connectivity, a group of agents falls down into more than one independent subgroup between which consensus generally is impossible.

Assumption 2. The equality $\varphi_{jk}^q(t, y) = -\varphi_{kj}^q(t, -y)$ is valid for all $j, k \in 1 : N$, $j \neq k$, $q \in 1 : m$, $t \geqslant 0$, and $y \in \mathbb{R}$.

This symmetry condition is usually satisfied for the couplings of physical nature such as oscillator networks, power grids, and so on. If it is satisfied, then $\sum_j u_j(t) = 0$ and consensus implies that

$$x_j(t) - \frac{1}{N} e^{tA} \sum_{j=1}^{N} x_j(0) \xrightarrow{t\to\infty} 0.$$

The present paper considers the case where the coupling functions may be unknown and only the fact be known that their graphs lie strictly within the given sector (the *sectorial conditions* are satisfied [24]) without approaching "too rapidly" on infinity to its boundaries. Strictly speaking, for $q \in 1 : m$, $j, k \in 1 : N$ we get $\varphi_{jk}^q \in S[\alpha^q; \beta^q]$ where $0 \leqslant \alpha^q < \beta^q \leqslant \infty$ and the class $S[\alpha; \beta]$ consists of continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{gathered} \alpha < \frac{\varphi(y)}{y} < \beta \; \forall y \neq 0, \quad \varphi(0) = 0, \\ \varliminf_{|y|\to\infty} |\varphi(y) - \alpha y| > 0, \quad \varliminf_{|y|\to\infty} |\beta^{-1}\varphi(y) - y| > 0. \end{gathered} \tag{7}$$

The consensus condition given below includes only the boundaries of the sectors $\alpha^1, \ldots, \alpha^m$, $\beta^1, \ldots, \beta^m$, algebraic connectivity of the graph $\lambda_2(\Gamma)$, and the coefficients of agent A, B, C. This condition guarantees consensus *for any* coupling functions satisfying the aforementioned conditions and in this sense representing the criterion for *robust* consensus in the aforementioned class of nonlinearities. Obviously, achievement of the robust consensus is possible only if consensus is reached for the linear coupling functions $\varphi_{jk}^q(y) = \kappa^q y$ which satisfy Assumption 2 and where $\kappa^q \in (\alpha^q; \beta^q)$. Stated differently, consensus is supported by any protocol

$$u_j(t) = K \sum_{j,k=1}^{N} \gamma_{jk}(y_k(t) - y_j(t)), \quad K := \mathrm{diag}(\kappa^1, \dots, \kappa^m), \quad \kappa^q \in (\alpha^q, \beta^q).$$

In virtue of [4, Theorem 8], the last assertion implies that $A - \mu BKC$ is a Hurwitz matrix for any eigenvalue $\mu \neq 0$ of the Laplace matrices $L(\Gamma)$, that is, all its eigenvalues lie on the left imaginary axis. We assume that a formally weaker but easier verifiable condition is satisfied.

Assumption 3. There are sequences of matrices $K_n = \mathrm{diag}(\kappa_n^1, \dots, \kappa_n^m)$ and $\tilde{K}_n = \mathrm{diag}(\tilde{\kappa}_n^1, \dots, \tilde{\kappa}_n^m)$ such that $\kappa_n^q, \tilde{\kappa}_n^q \in (\alpha^q, \beta^q)$, $\tilde{\kappa}_n^q \to \beta^q$, and $\kappa_n^q \to \alpha^q$ for $n \to \infty$ and $A - \mu BK_nC$ and $A - \mu B\tilde{K}_nC$ are Hurwitz matrices under all n for any eigenvalue $\mu \neq 0$ of the matrix $L(\Gamma)$.

3. MAIN RESULTS

The consensus condition takes the simplest form in the scalar case of $m = 1$ discussed in Section 3.1. This condition is similar in form to the Popov criterion and, as will be shown below, is its direct extension to the case of multiagent systems. For the general case, the consensus criterion is given in Section 3.2.

We assume throughout this section that fixed are the boundaries of the sectors $\alpha^1, \dots, \alpha^m$, $\beta^1, \dots, \beta^m$ where the graphs of the nonlinearities $\varphi_{jk}^q \in S[\alpha^q; \beta^q]$ lie and also Assumptions 1–3 are satisfied.

We introduce for convenience additional notation. Let $W_x(\lambda) := (\lambda I - A)^{-1}B$ and $W_y(\lambda) := CW_x(\lambda)$ be transfer functions of system (3) from u to x and y, respectively. Let $D_{\max} := \max_j D_j(\Gamma)$ be the maximal weight among the graph vertices and $\lambda_2 := \lambda_2(\Gamma)$ be its algebraic connectivity. We also assume by definition that

$$\delta^q := \frac{\alpha^q}{1 + \alpha^q(\beta^q)^{-1}}, \quad \rho^q := \frac{1}{2(\alpha^q + \beta^q)},$$
$$\Delta := \mathrm{diag}(\delta^1, \dots, \delta^m), \quad R := \mathrm{diag}(\rho^1, \dots, \rho^m).$$

3.1. Consensus Criterion for Scalar Agents

In this section we consider the case of $m = 1$, that is, $\dim y = \dim u = 1$, and for brevity omit the superscripts assuming that $\delta := \delta^1$, $\rho := \rho^1$, and so on. For all $\theta \in \mathbb{R}$ and $\omega \in \mathbb{R}$ such that $\det(\imath\omega I - A) \neq 0$, we assume by definition that

$$\pi_\theta(\omega) := \mathrm{Re}[(1 + \imath\omega\theta)W_y(\imath\omega)] + \lambda_2\delta|W_y(\imath\omega)|^2 + \rho D_{\max}^{-1}.$$

Theorem 1. *Assume that there exists $\theta \in \mathbb{R}$ such that $\pi_\theta(\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$ such that $\det(\imath\omega I - A) \neq 0$; additionally, if $\beta = \infty$, then $\theta \geqslant 0$. Then, protocol* (4) *provides attainment of consensus* (5)*. If the frequency inequality $\pi_\theta(\omega) \geqslant 0$ is satisfied "in the limit" for $\theta \to +\infty$, that is,* $\mathrm{Re}[\imath\omega W_y(\imath\omega)] \geqslant 0$, *protocol* (4) *supports boundedness of the functions $x_j(t) - x_k(t)$ for $t \geqslant 0$.*

Theorem 1 follows immediately from a more general criterion presented in the following section (Theorem 2), see Remark 1.

In the case of $\theta = 0$, Theorem 1 in essence coincides with the consensus criterion of [21–23] which is applicable to the case of time-varying topology, but assumes that $\gamma_{jk} = \gamma_{jk}(t) \in \{0, 1\}$. In common with the latter result generalizing the circle stability criterion [24] to the case of network systems, Theorem 1 is a counterpart of the Popov criterion [24, 25] in the following sense. Consider a system of $N = 2$ agents related by protocol (4) (we have $D_{\max} = \lambda_2 = 2\gamma_{12}$). Assuming that $X(t) := x_2(t) - x_1(t)$, $Y(t) := y_2(t) - y_1(t)$, and $U(t) := u_1(t) - u_2(t)$, we get

$$\dot{X}(t) = AX(t) - BU(t), \quad Y(t) = CX(t), \quad U(t) = 2\gamma\Phi(Y(t)). \tag{8}$$

In Eq. (8), $\Phi(Y) := 2\gamma_{12}\varphi_{12}(t, Y) \in S[0; \eta]$, where $\eta := 2\gamma_{12}\beta$.

According to the Popov criterion,[2] if Assumption 3 is satisfied, that is, $A - 2\gamma\kappa BC$ is a Hurwitz matrix for $\kappa \in (0; \beta)$, and for some $\theta \in \mathbb{R}$ satisfied is the inequality

$$\mathrm{Re}[(1 + \imath\omega\theta)W_y(\imath\omega)] + \frac{1}{\eta} \geqslant 0 \quad \forall \omega \in \mathbb{R} : \det(\imath\omega I - A) \neq 0$$

coinciding, as is readily verifiable, with the condition of Theorem 1, then the equilibrium $X = 0$ of the auxiliary system (8) is stable, that is, consensus between two agents is reached. Theorem 1 generalizes the Popov criterion to the case of network system of $N \geqslant 2$ agents which additionally may be exponentially unstable.

3.2. Consensus Criterion in the Multivariable Case

This section considers the case where dimensionality of control (and output) $m \geqslant 1$ can be arbitrary. Notations W_x, W_y, λ_2, $D_{\max}$, Δ, R have the same sense as in Section 3.1. We assume for the given matrices $\Psi, \Theta \in \mathbb{R}^{m \times m}$ that

$$\Pi_{\Psi,\Theta}(\omega) := (\Psi + \imath\omega\Theta)W_y(\imath\omega) + W_y(\imath\omega)^*(\Psi - \imath\omega\Theta) + \lambda_2 W_y(\imath\omega)^*\Psi\Delta W_y(\imath\omega) + D_{\max}^{-1}\Psi R. \tag{9}$$

Theorem 2. *Assume that there exist matrices*

$$\Psi := \mathrm{diag}(\psi^1, \ldots, \psi^m) \geqslant 0 \quad \textit{and} \quad \Theta := \mathrm{diag}(\theta^1, \ldots, \theta^m) \geqslant 0$$

such that the inequality $\Pi_{\Psi,\Theta}(\omega) \geqslant 0$ is satisfied for all $\omega \in \mathbb{R}$ for which $\det(\imath\omega I - A) \neq 0$. Then, the following assertions are true:

1) *if $\theta^q > 0$ for some $q \in 1:m$, then the functions $y_j^q(t) - y_k^q(t)$ are globally bounded over the entire interval of existence of solutions for all j, k;*
2) *if $\theta^q > 0$ for all q, then any solution of system (3), (4) is prolongable to the interval $(0; \infty)$ and at that $\sup_{t \geqslant 0} |x_j(t) - x_k(t)| < \infty$ for all j, k;*
3) *if $\psi^q > 0$ for some $q \in 1:m$, then either $y_j^q(t) - y_k^q(t) \to 0$ for $t \to \infty$ for all j, k or one of the functions $\dot{y}_j^q(t) - \dot{y}_k^q(t)$ is unrestricted;*
4) *under the conditions of item 2, if $\psi^q > 0$ for some $q \in 1:m$, then for any solution we have $y_j^q(t) - y_k^q(t) \to 0$ for $t \to \infty$;*
5) *if $\Psi > 0$ and $\Theta > 0$ or $\Theta = 0$, then protocol (4) provides consensus.*

For $\beta < \infty$, assertions 1–5 retain validity if the inequalities $\Theta \geqslant 0$, $\Theta > 0$, and $\theta^q > 0$ are replaced everywhere by the respective inequalities $\Theta \leqslant 0$, $\Theta < 0$, and $\theta^q < 0$.

Theorem 2 is proved in the Appendix.

Remark 1. The first assertion of Theorem 1 follows directly from item 5 as applied to $\Psi := 1$, $\Theta := \theta$; the second assertion follows directly from item 2 for $\Psi := 0$, $\Theta := 1$. Therefore, Theorem 1 follows from Theorem 2.

[2] The traditional formulation of the Popov criterion presumes that A is a Hurwitz matrix. By repeating the proof from [24], one can readily verify that this condition can be replaced by the Hurwitz stability of the matrix $A - \varepsilon BC$ under a certain $\varepsilon \in (0; 2\beta\gamma)$, which takes place in virtue of Assumption 3.

Remark 2. The frequency inequality $\Pi_{\Psi,\Theta}(\omega) \geqslant 0$ may be conveniently rearranged in nonnegativeness for any ω of the Hermitian form:

$$\mathrm{Re}[\tilde{u}^*(\Psi + \imath\omega\Theta)\tilde{y}] + \lambda_2 \tilde{y}^*\Psi\Delta\tilde{y} + D_{\max}^{-1}\tilde{u}^*\Psi R\tilde{u} \geqslant 0, \tilde{\forall} u \in \mathbb{C}^m, \quad \text{where} \quad \tilde{y} := W_y(\imath\omega)\tilde{u}. \tag{10}$$

We note that the network topology is involved in the conditions of the theorem only as the factor $\lambda_2 > 0$ in the function $\Pi_{\Upsilon,\Theta}$. Since by assumption $\Psi, \Delta \geqslant 0$, replacement of λ_2 by a smaller value retains sufficiency in Theorem 2. Whenever calculation of the algebraic connectivity presents difficulties, its lower estimates can be used [28].

4. EXAMPLES

In this section we illustrate application of Theorems 1 and 2 to the groups of agents with second-order dynamics. The interest the multiagent system of this kind [2, 3, 8, 29] is first of all due to the numerous applications to the control of mobile robots. In distinction to the single integrators, the convergence conditions for the consensus protocols for the agents of second and higher orders are known mostly in the case of linear couplings. One of the few results on convergence of the nonlinear protocols belongs to Ren [2, 26] and refers to the couplings like $\varphi_{jk}(y) = \tanh y$. Below we prove that this result follows from Theorem 2. It is also shown that Theorems 1 and 2 are essentially less conservative for the networks with constant topology than the synchronization criteria of [21–23].

The double integrator is the most popular type of the second-order dynamics. The consensus algorithms for the second-order agents are divided traditionally into three categories: protocols with measurement of the agent absolute speed, the proper position in the global coordinate being not accessible to the agent; protocols using only the relative speed; and protocols doing without speed measurements. The corresponding cases are considered in what follows.

4.1. Agents with Model of Double Integrator Measuring the Proper Speed

Let us consider a group of agents obeying an equation like

$$\ddot{z}_j(t) = v_j(t) \in \mathbb{R} \tag{11}$$

and a protocol like

$$v_j(t) = -\mu\dot{z}_j + \sum_{k=1}^{N} \gamma_{jk}\varphi_{jk}(z_k(t) - z_j(t)). \tag{12}$$

This protocol can be used if the agent's absolute speed $\dot{z}_j$ is accessible. At the same time, measurement of the absolute coordinate z_j can be inaccessible, and measured is only the position relative to the neighbor agents. This situation may happen, for example, for groups of mobile marine apparatuses having no GPS sensors, but equipped with sensors of relative position such as radars, sonars, and so on, as well as a log to measure speed.

Theorem 3. *Assume that Assumptions 1 and 2, and $\varphi_{jk} \in S[0;\infty]$ are satisfied. Then, protocol* (12) *provides consensus for any $\mu > 0$; moreover, $\dot{z}_j(t) \to 0$ and $z_j(t) \to z_*$ for $t \to \infty$, and the final position z_* is defined by the initial data.*

Proof. We introduce the output $y_j(t) := z_j(t)$ and new control $u_j(t) := v_j(t) + \mu\dot{z}_j$ to consider the closed-loop system (11), (12) as a network of agents like

$$\ddot{z}_j(t) + \mu\dot{z}_j(t) = u_j(t), \quad y_j(t) = z_j(t) \tag{13}$$

using protocol (4). Since we have $W_y(\imath\omega) = [\imath\omega(\imath\omega + \mu)]^{-1}$ for agents (13), for any $\theta \geqslant \mu^{-1}$ $\pi_\theta(\omega) = [\theta\mu - 1]/(\omega^2 + \mu^2) \geqslant 0$. Obviously, Assumption 3 is valid: the feedback $u_j = -\kappa y_j$ stabilizes system (13) for any $\kappa > 0$. Therefore, the conditions of Theorem 1 are satisfied for system (13), (4), that is, consensus is reached. Consequently, $u_j(t) \to 0$ and in virtue of (13) $\dot{z}_j(t) \to 0$ for $t \to \infty$. Since $\sum_j u_j = 0$ by Assumption 2, we get from (13) that $Z(t) := \sum_j \dot{z}_j(t) + \mu \sum_j z_j(t) \equiv Z(0)$, and in virtue of $z_j - z_k \to 0$ obtain $z_j(t) \to z_* := Z(0)/(\mu N)$, which proves the theorem.

We note that in contrast to the result of [21–23] which refers to the time-varying topology networks, Theorem 3 is applicable to an infinite sector ($\beta = \infty$).

4.2. Reaching Consensus with Measurements of Relative Speeds

If the agent measures only the relative speeds $\dot{z}_k - \dot{z}_j$, then it is only natural to generalize algorithm (12) by a protocol like

$$v_j(t) = \sum_{k=1}^{N} \gamma_{jk}\varphi_{jk}\big(z_k(t) - z_j(t) + \mu(\dot{z}_k(t) - \dot{z}_j(t))\big). \tag{14}$$

The following theorem gives sufficient conditions for consensus of protocol (14).

Theorem 4. *Let Assumptions 1 and 2 and $\varphi_{jk} \in S[0;\infty]$ be satisfied. Additionally, $\underline{\lim}_{|y|\to 0} \varphi_{jk}(y)/y > 0$ (for example, there exists $\varphi'_{jk}(0) > 0$). Then, for any $\mu > 0$ protocol* (14) *provides consensus and, moreover, $\dot{z}_j(t) \to \upsilon_*$, $z_j(t) - z_* - t\upsilon_* \to 0$ for $t \to \infty$, where z_* and υ_* depend on the initial data.*

Proof. Assume that $u_j(t) := v_j(t)$, $y_j(t) := z_j + \mu\dot{z}_j$ and obtain that system (11), (14) can be regarded as a network of agents like

$$\ddot{z}_j(t) = u_j(t), \quad y_j(t) = z_j + \mu\dot{z}_j \tag{15}$$

related by a protocol of form (4). Obviously, for such agents Assumption 3 is satisfied (the feedback $u_j = -\kappa y_j$ stabilizes the system for all $\kappa > 0$ since $\mu > 0$), moreover, $W_y(\lambda) = \lambda^{-2} + \mu\lambda^{-1}$. Let us assume that $\varphi_{jk} \in S[\varepsilon;\infty]$ for some $\varepsilon > 0$. For sufficiently great $\theta > 0$, in this case the condition $\pi_\theta \geqslant 0$ is, obviously, satisfied. Indeed, $\delta = \varepsilon$, $\rho = 0$, consequently,

$$\pi_\theta(\omega) = \theta\mu - \frac{1}{\omega^2} + \varepsilon\lambda_2\frac{1 + \mu^2\omega^2}{\omega^4} \geqslant 0 \quad \text{for} \quad 2\sqrt{\theta\mu\varepsilon\lambda_2} \geqslant |1 - \varepsilon\lambda_2\mu^2|.$$

Therefore, consensus is reached for a stronger assumption $\varphi_{jk} \in S[\varepsilon;\infty]$. Now, we consider an arbitrary collection of couplings φ_{jk} meeting the theorem conditions and fix some particular solution z_j, y_j, u_j of system (15), (4). Since $\mathrm{Re}[\imath\omega W_y(\imath\omega)] = \theta\mu > 0$, we have by Theorem 1 that $|y_j(t) - y_k(t)| \leqslant M$ for all $t \geqslant 0$ (the constant M depends on the chosen solution). Let us assume that $\varepsilon_0 := \min_{j,k} \inf_{|y|\leqslant M} \varphi_{jk}(y)/y$ in virtue of continuity of φ_{jk} and assumptions of the theorem $\varepsilon > 0$, and let $\tilde{\varphi}_{jk}(y) := y\max(\varphi_{jk}(y)/y, \varepsilon_0)$ (assume by definition that $\tilde{\varphi}_{jk}(0) = 0$). Obviously, $\tilde{\varphi}_{jk} \in S[\varepsilon;\infty]$ for $\varepsilon < \varepsilon_0$; moreover, $\tilde{\varphi}_{jk}$ satisfies Assumption 2, because $\tilde{\varphi}_{kj}(-y) = -y\max(-\varphi_{kj}(-y)/y, \varepsilon_0) = -y\max(\varphi_{jk}(y)/y, \varepsilon_0) = -\tilde{\varphi}_{jk}(y)$. We have $\tilde{\varphi}_{jk}(y_k(t) - y_j(t)) = \varphi_{jk}(y_k(t) - y_j(t))$ on the strength of choosing ε_0. Therefore, the chosen solution is also that of the closed-loop system (15), (4), where φ_{jk} are replaced by $\tilde{\varphi}_{jk}$. Consequently, we get in virtue of the proved above special case that $z_j - z_k \to 0$, $\dot{z}_j - \dot{z}_k \to 0$ for $t \to \infty$, which proves consensus, because solution can be selected arbitrarily. Since $\sum_j u_j = 0$ and, consequently, $\sum_j \ddot{z}_j = 0$, there exist limits $\upsilon_* = \lim_{t\to\infty} \dot{z}_j(t)$, $z_* = \lim_{t\to\infty}(z_j(t) - \upsilon_* t)$, which completes the proof.

For the agents with the second-order dynamics, Ren [26] considered a somewhat different class of the nonlinear protocols which are distinct in the structure of nonlinearities:

$$v_j(t) = \sum_{k=1}^{N} \gamma_{jk}\big[\varphi^1_{jk}(z_k(t) - z_j(t)) + \varphi^2_{jk}(\dot z_k(t) - \dot z_j(t))\big]. \tag{16}$$

Protocols of this kind arise, for example, if the relative positions and speed are measured by different sensors or these measurements are fed to different actuators with nonlinear characteristics of the saturation type. The criterion obtained by Ren guaranteed convergence of this protocol for the case where $\varphi^1_{jk}(z) = \varphi^2_{jk}(z) = \tanh z$. The following result obtained from Theorem 2, extends the Ren theorem to a wider class of nonlinearities.

Theorem 5. *Assume that Assumptions 1 and 2 and $\varphi^1_{jk}, \varphi^2_{jk} \in S[0;\infty]$ are satisfied; moreover, $0 < \underline{\lim}_{|y|\to 0}\, \varphi^q_{jk}(y)/y \leqslant \overline{\lim}_{|y|\to 0}\, \varphi^q_{jk}(y)/y < \infty$ for $q = 1, 2$, and the function $\varphi^2_{jk}(\cdot)$ is bounded. Then, the protocol* (14) *provides consensus and $z_j(t) \to \upsilon_*$, $z_j(t) - z_* - t\upsilon_* \to 0$ for $t \to \infty$ for all j, where z_*, υ_* depend on the initial data.*

Proof. Consider system (11), (16) as a group of agents like

$$\ddot z_j(t) = u^1_j(t) + u^2_j(t), \quad y_j(t) = (z_j(t), \dot z_j(t))^{\mathrm T}, \tag{17}$$

related by the protocol

$$u^q_j(t) := \sum_{k=1}^{N} \gamma_{jk}\varphi^q_{jk}(y^q_k(t) - y^q_j(t)), \quad q = 1, 2. \tag{18}$$

One can readily see that Assumption 3 is satisfied: the feedback of the form $u^q_j(t) = -\mu^q y^q_j$, $q = 1, 2$, stabilizes the system under any $\mu^1, \mu^2 > 0$. We note that for $\psi^1 = 0$, $\theta^1 = \psi^2 = 1$, $\theta^2 = 0$ satisfied is the frequency inequality (10) whose left side is given by

$$\operatorname{Re}\left[\imath\omega \frac{u^*_1(\tilde u_1 + \tilde u_2)}{(\imath\omega)^2} + \frac{u^*_2(\tilde u_1 + \tilde u_2)}{(\imath\omega)}\right] = 0$$

in virtue of the equality $W_y(\lambda) = (\lambda^{-2}; \lambda^{-1})^{\mathrm T}(1; 1)$. In virtue of assertion 1 of Theorem 2, we establish that the functions $y^1_j - y^1_k = z_j - z_k$ are globally bounded. Consequently, in virtue of (16) and boundedness of the functions φ^2_{jk} we obtain that $\ddot z_j$ are globally bounded and, in particular, solution is infinitely prolongable. Now, in virtue of assertion 3 of Theorem 2 we obtain that $\dot z_j - \dot z_k \to 0$ for any solution.

Taking advantage of the device used to prove Theorem 4, we can also construct for the given solution the nonlinearities $\tilde\varphi^1_{jk}, \tilde\varphi^2_{jk} \in S[\alpha;\beta]$, where $\alpha > 0, \beta < \infty$, such that this solution is generated not only by protocol (18), but also by the protocol

$$u^q_j(t) := \sum_{k=1}^{N} \gamma_{jk}\tilde\varphi^q_{jk}(y^i_k(t) - y^i_j(t)), \quad q = 1, 2, \tag{19}$$

where the function $\tilde\varphi^2_{jk}$ is no more bounded. For that, we assume that $\alpha < \alpha_0$, $\beta > \beta_0$, $\tilde\varphi^q_{jk}(y) := y \min\big(\max(\varphi^q_{jk}(y)/y, \alpha_0), \beta_0\big)$, where

$$\alpha_0 := \min_{q=1,2}\min_{j,k}\inf_{|y|\leqslant M}\varphi^q_{jk}(y)/y > 0, \quad \beta_0 := \max_{q=1,2}\max_{j,k}\sup_{|y|\leqslant M}\varphi^q_{jk}(y)/y < \infty, \quad \text{and}$$

$$M := \max_{j,k,q}\sup_{t\geqslant 0}(|y^q_k(t) - y^q_j(t)|), \quad q = 1, 2.$$

We apply Theorem 2 to protocol (19) and demonstrate that for some $\xi > 0$ under $\psi^1 = 1$, $\psi^2 = \theta^1 = \xi$ and $\theta^2 = 0$ inequality (10) is satisfied, where $\delta^1 = \delta^2 = \delta > 0$, $\rho^1 = \rho^2 = \rho > 0$ depend on α and β. Since $W_y(\imath\omega) = ((\imath\omega)^{-2}; (\imath\omega)^{-1})^{\mathrm{T}}$, with the use of $\mathrm{Re}[u_1^*(\tilde{u}_1+\tilde{u}_2)/(\imath\omega)+u_2^*(\tilde{u}_1+\tilde{u}_2)/(\imath\omega)] = 0$ Eq. (10) takes on form

$$0 \leqslant \mathrm{Re}\left[(\tilde{u}^1)^*(\tilde{u}^1+\tilde{u}^2)\frac{\psi^1+\theta^1\imath\omega}{(\imath\omega)^2} + (\tilde{u}^2)^*(\tilde{u}^1+\tilde{u}^2)\frac{\psi^2+\theta^2\imath\omega}{(\imath\omega)}\right]$$
$$+\left(\frac{\psi^1}{\omega^4}+\frac{\psi^2}{\omega^2}\right)\delta\lambda_2|\tilde{u}^1+\tilde{u}^2|^2 + \psi^1\rho|\tilde{u}_1|^2 + \psi^2\rho|\tilde{u}^2|^2$$
$$= -\frac{1}{\omega^2}\mathrm{Re}[(\tilde{u}^1)^*(\tilde{u}^1+\tilde{u}^2)] + \left(\frac{1}{\omega^4}+\frac{\xi}{\omega^2}\right)\lambda_2\delta|\tilde{u}^1+\tilde{u}^2|^2 + \rho|\tilde{u}^1|^2 + \xi\rho|\tilde{u}^2|^2$$
$$= \frac{1}{\omega^2}\mathrm{Re}[\tilde{u}_2^*(\tilde{u}_1+\tilde{u}_2)] + \rho|\tilde{u}_1|^2 + \rho\xi|\tilde{u}_2|^2 + \frac{1}{\omega^4}\lambda_2\delta|\tilde{u}_1+\tilde{u}_2|^2 + \frac{\xi\lambda_2\delta-1}{\omega^2}|\tilde{u}_1+\tilde{u}_2|^2.$$

The last inequality is satisfied if $\xi\lambda_2\delta \geqslant \max(1, 1/(4\rho))$. By applying assertion 3 of Theorem 2 to the selected solution, we establish that $z_j(t) - z_k(t) = y_j^1(t) - y_k^1(t) \to 0$ for $t \to +\infty$ by virtue of the fact that all functions $\dot{z}_j(t) - \dot{z}_k(t)$ are bounded, which proves consensus. Existence of v_*, z_* is proved along the same lines as in Theorem 4.

4.3. Reaching Consensus without Speed Measurements

The last example refers to the case where it is impossible to measure either the absolute or relative speeds of the agents. A consensus protocol using only the relative agent positions was proposed in [30]:

$$\dot{w}_j = -\nu_1 w_j + \xi_j, \quad \xi_j = \sum_{k=1}^{N}\gamma_{jk}(z_k - z_j), \tag{20}$$
$$v_j(t) = \nu_2\dot{w}_j + \xi_j = -\nu_2\nu_1 w_j + (\nu_2+1)\xi_j, \quad j \in 1:N. \tag{21}$$

Here, ν_1, ν_2 are constant. Protocol (21) is similar to the aforementioned protocol (14) (with linear couplings $\varphi_{jk}(y) = y$), but instead of the nonobservable function $\dot{\xi}_j = \sum_{k=1}^{N}\gamma_{jk}(\dot{z}_k - \dot{z}_j)$ used is the derivative of the output of the low-frequency filter $w_j(t)$. Let us consider a nonlinear protocol which is similar to (20), (21):

$$\begin{gathered}\dot{w}_j(t) = -\nu_1 w_j(t) + \xi_j(t), \quad v_j(t) = \nu_2\dot{w}_j(t) + \xi_j(t),\\ \xi_j(t) = \sum_{k=1}^{N}\gamma_{jk}\varphi_{jk}(t, z_k(t) - z_j(t)).\end{gathered} \tag{22}$$

The next result gives a criterion for convergence of protocols like (22).

Theorem 6. *Let us assume that the conditions of Theorem 4 are satisfied. Then, protocol* (22) *ensures attainment of consensus for any* $\nu_1, \nu_2 > 0$.

Proof. Let us consider a closed-loop system as a network of agents like

$$\ddot{z}_j = \nu_2\dot{w}_j + u_j, \quad \dot{w}_j = -\nu_1 w_j + u_j, \quad y_j(t) = z_j(t), \tag{23}$$

using protocol (4). First, we assume that $\varphi_{jk} \in S[\alpha;\infty]$, where $\alpha > 0$. An agent of form (23) can be stabilized by a feedback like $u_j = -Ky_j$ for any $K > 0$. Therefore, Assumption 3 is satisfied. The transfer function of agent (23) has form

$$W_y(\lambda) = ((1+\nu_2)\lambda + \nu_1)/(\lambda^2(\lambda+\nu_1)).$$

Therefore,

$$\operatorname{Re} W_y(\imath\omega) = -\frac{1}{\omega^2} - \frac{\nu_2}{\nu_1^2+\omega^2}, \quad \operatorname{Re}\left[\imath\omega W_y(\imath\omega)\right] = \frac{\nu_2\nu_1}{\nu_1^2+\omega^2},$$
$$|W_y(\imath\omega)|^2 = \frac{(1+\nu_2)^2\omega^2+\nu_1^2}{\omega^4(\nu_1^2+\omega^2)}.$$

The frequency condition in Theorem 1 takes on form

$$0 \leqslant -\frac{(\nu_2+1)\omega^2+\nu_1^2}{\omega^2(\nu_1^2+\omega^2)} + \frac{\theta\nu_1\nu_2}{\nu_1^2+\omega^2} + \delta\lambda_2\frac{(1+\nu_2)^2\omega^2+\nu_1^2}{\omega^4(\nu_1^2+\omega^2)}$$

which can be rearranged in $(\theta\nu_1\nu_2 - \nu_2 - 1)\omega^4 + a\omega^2 + \delta\lambda_2\nu_1^2 \geqslant 0$, where a is independent of θ. Obviously, for a sufficiently great $\theta > 0$, the last inequality is satisfied for all $\omega \in \mathbb{R}$. Therefore, for $\varphi_{jk} \in S[\alpha;\infty]$, where $\alpha > 0$, the networks of Theorem 1 are satisfied, which proves consensus. The case of couplings from $S[0;\infty]$ boils down to that considered above similar to what was done in Theorem 4.

5. CONCLUSIONS

Consideration was given to the question of reaching state consensus in the fixed-topology networks of identical linear arbitrary-order agents. The functions of couplings between the agents are unknown, it is just assumed that their graphs lie within the given sector and the symmetry condition is satisfied. Established was the frequency consensus condition which is a direct generalization of the Popov criterion for the Lurie systems with one nonlinearity. Application of the criterion was illustrated by way of example of a second-order agent. In particular, it was demonstrated that the well-known result of Ren [26] about consensus in the network of second-order agents can be deduced from it.

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APPENDIX

Proof of Theorem 2 is based on the Lyapunov function which is classical for the theory of absolute stability and belongs to type of "quadratic forms plus nonlinearity integral" which exists in virtue of the sectorial constraints on couplings φ_{jk}^q, on the one hand, and Yakubovich–Kalman lemma (frequency theorem), on the other hand.

Assumptions of Theorem 2 are assumed to be satisfied everywhere below. In particular, determined were $\psi^1,\ldots,\psi^m \geqslant 0, \theta^1,\ldots,\theta^m$ (all θ^q are either nonnegative or nonpositive, the latter is possible for $\beta < \infty$) for which the frequency inequality $\Pi_{\Psi,\Theta}(\omega) \geqslant 0$ is valid. Let $\bar\zeta := (\zeta_1^\top,\ldots,\zeta_N^\top)^\top$ denote the unified column vector for the sequence of column vectors $\zeta_1,\ldots,\zeta_N$. Let $F_q(\sigma,\xi) := -\operatorname{Re}(\bar\sigma\xi) - \lambda_2\delta^q|\sigma|^2 - \rho^q D_{\max}^{-1}|\xi|^2$ (here $\sigma,\xi \in \mathbb{C}$ and $\delta^q,\rho^q,\lambda_2,D_{\max}$ are determined in Section 3). We assume for the vectors $u_1,\ldots,u_N,y_1,\ldots,y_N \in \mathbb{C}^m$ that

$$\mathcal{F}_q(\bar u,\bar y) := \sum_{j,k=1}^N F_q(u_k^q - u_j^q, y_k^q - y_j^q); \quad \mathcal{F}(\bar u,\bar y) := \sum_{q=1}^m \psi^q\mathcal{F}_q(\bar u,\bar y).$$

Now, we assume for an arbitrary $\sigma \in \mathbb{R}$ that by definition

$$\Phi_{jk}^q(\sigma) := \int\limits_0^\sigma (\varphi_{jk}^q(s) - \alpha^q s)\, ds, \quad \tilde{\Phi}_{jk}^q(\sigma) := \int\limits_0^\sigma (\beta^q s - \varphi_{jk}^q(s))\, ds$$

(the function $\tilde{\Phi}_{jk}^q$ is defined only for $\beta < \infty$). Finally, we assume

$$\mathfrak{S}(\bar{y}) := N \sum_{q=1}^m \sum_{j,k=1}^N \theta^q \gamma_{jk} \Phi_{jk}^q (y_k^q - y_j^q), \quad \tilde{\mathfrak{S}}(\bar{y}) := -N \sum_{q=1}^m \sum_{j,k=1}^N \theta^q \gamma_{jk} \tilde{\Phi}_{jk}^q (y_k^q - y_j^q)$$

for $\beta < \infty$.

Generation of the Lyapunov function used to prove Theorem 2 relies on the following technical Lemmas 1–3.

Lemma 1. *For any solution of system* (3), (4) *and any $q \in 1:m$, we have $\mathcal{F}_q(\bar{y}(t), \bar{u}(t)) \geqslant 0$. For any $\nu > 0$, there exists $\varkappa = \varkappa(\nu) > 0$ such that $\mathcal{F}_q(\bar{y}(t), \bar{u}(t)) > \varkappa$ for $\max_{1\leqslant j,k\leqslant N} |y_k^q(t) - y_j^q(t)| > \nu$. In particular, if $\psi^q > 0$, then $\mathcal{F}(\bar{y}(t), \bar{u}(t)) > \psi^q \varkappa(\nu) > 0$ for $\max_{1\leqslant j,k\leqslant N} |y_k^q(t) - y_j^q(t)| > \nu$.*

Proof of Lemma 1. Let $\xi_{jk}^q(t) := \gamma_{jk}\varphi_{jk}^q(y_k^q(t) - y_j^q(t))$, so $u_j^q(t) = \sum_{k=1}^N \xi_{jk}^q(t)$. In virtue of (7), we have $\sigma\varphi_{jk}^q(\sigma) - \delta^q|\sigma|^2 - \rho^q|\varphi_{jk}^q(\sigma)|^2 \geqslant 0$ for $j,k \in 1:N$, $q \in 1:m$, $\sigma \in \mathbb{R}$, and the left side is separated from 0 if the value of σ is separated from 0. By substituting $\sigma := y_k^q(t) - y_j^q(t)$, multiplying by γ_{jk}, and summing over j,k, we obtain

$$\underbrace{\sum_{j,k=1}^N \xi_{jk}^q (y_k^q - y_j^q)}_{=a} - \delta^q \underbrace{\sum_{j,k=1}^N \gamma_{jk} |y_k^q - y_j^q|^2}_{=b} - \rho^q \underbrace{\sum_{j,k=1}^N \gamma_{jk} |\varphi_{jk}^q (y_k^q - y_j^q)|^2}_{=c} \geqslant 0. \tag{A.1}$$

By Assumption 2, we have $\xi_{jk}^q(t) = -\xi_{kj}^q(t)$ and $\sum_j u_j^q(t) = 0$. Consequently,

$$\begin{aligned} a &= -\sum_{j,k} \xi_{jk}^q(t) y_j^q(t) - \sum_{j,k} \xi_{kj}^q(t) y_k^q(t) \\ &= -2\sum_j u_j^q y_j^q = -N^{-1} \sum_{j,k} (u_k^q - u_j^q)(y_k^q - y_j^q) \end{aligned}$$

in virtue of (2) $b \geqslant N^{-1}\lambda_2 \sum_{j,k} |y_k^q - y_j^q|^2$. By the Cauchy–Buniakowsky inequality we have

$$|u_j^q(t)|^2 = \sum_k \left|\gamma_{jk}\varphi_{jk}^q(y_k^q(t) - y_j^q(t))\right|^2 \leqslant \sum_k \gamma_{jk} \sum_k \gamma_{jk} (\varphi_{jk}^q(y_k^q(t) - y_j^q(t)))^2.$$

Consequently, $c \geqslant D_{\max}^{-1} \sum_j |u_j^q(t)|^2 = (ND_{\max})^{-1} \sum_{j,k} |u_j^q(t) - u_k^q(t)|^2$. By substituting into (A.1) the above estimates of a, b, c and multiplying by N, we obtain the assertion of lemma.

Lemma 2. *The relations $\Phi_{jk}^q(\sigma), \tilde{\Phi}_{jk}^q(\sigma) \geqslant 0$ and $\lim_{|\sigma|\to\infty} \Phi_{jk}^q(\sigma) = \infty, \lim_{|\sigma|\to\infty} \tilde{\Phi}_{jk}^q(\sigma) = \infty$ are valid for all $j,k \in 1:N$, $q \in 1:m$ and $\sigma \in \mathbb{R}$. If $\theta^q \geqslant 0$ (correspondingly, $\theta^q \leqslant 0$, at that $\beta < \infty$) for all q, then $\mathfrak{S}(\bar{y}) \geqslant 0$ (correspondingly, $\tilde{\mathfrak{S}}(\bar{y}) \geqslant 0$) for all $y_1, \ldots, y_N \in \mathbb{R}^m$. If at that we have $\theta^q > 0$ for some q (correspondingly, $\theta^q < 0$) and on some solution of system* (3), (4) *the function $\mathfrak{S}(\bar{y}(t))$ (correspondingly, $\tilde{\mathfrak{S}}(\bar{y}(t))$) is bounded, then the functions $y_j^q(t) - y_k^q(t)$ are bounded as well.*

Proof of Lemma 2. The two first assertions of the lemma follow from (7), the last one follows from them in virtue of graph connectivity: the function $y_j^q(t) - y_k^q(t)$ is bounded for $\gamma_{jk} > 0$, and, consequently, also for all j, k.

Lemma 3. *For some matrix $H_0 = H_0^\top$, the quadratic form $V_0(\bar{x}) := \sum_{j,k=1}^N (x_k - x_j)^* H_0 (x_k - x_j)$ on the solutions of system* (3) *satisfies the inequality*

$$\frac{d}{dt} V_0(\bar{x}(t)) + \mathcal{F}(\bar{y}(t), \bar{u}(t)) - \sum_{j,k=1}^N (u_k(t) - u_j(t))^\top \Theta (\dot{y}_k(t) - \dot{y}_j(t)) \leqslant 0. \tag{A.2}$$

Proof of Lemma 3. Direct calculations demonstrate that (A.2) is equivalent to the linear matrix inequality $2x^\top H_0(Ax + Bu) - u^\top \Psi y - \lambda_2 y^\top (\Psi\Delta) y - u^\top \Psi R u - u^\top \Theta \dot{y} \leqslant 0$ for any $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y := Cx,\ \dot{y} := C(Ax + Bu)$. According to the frequency theorem [24, 31], this inequality is solvable with respect to the matrix $H = H^\mathrm{T}$ if and only if for any $\tilde{u} \in \mathbb{C}^m$ we have

$$\begin{aligned} &- \operatorname{Re}\left[\tilde{u}^* W_y(\imath\omega)\tilde{u} + \imath\omega \tilde{u}^* \Theta W_y(\imath\omega)\tilde{u}\right] \\ &- \lambda_2 \tilde{u}^* W_y(\imath\omega)^* (\Psi\Delta) W_y(\imath\omega)\tilde{u} - D_{\max}^{-1} \tilde{u}^* \Psi R \tilde{u} \leqslant 0, \end{aligned}$$

which is equivalent to the frequency inequality $\Pi_{\Psi,\Theta} \geqslant 0$.

Proof of Theorem 2. We consider first the case of $\Theta \geqslant 0$ where $\beta \leqslant \infty$ and assume that

$$V_\alpha(\bar{x}) := V_0(\bar{x}) + \frac{N}{2} \sum_{q=1}^m \theta^q \alpha^q \sum_{j,k=1}^N \gamma_{jk} |y_k^q - y_j^q|^2$$

and

$$\begin{gathered} V(\bar{x}) = V_\alpha(\bar{x}) + \mathfrak{S}(\bar{y}) = V_0(\bar{x}) + N \sum_{q=1}^m \sum_{j,k=1}^N \theta^q \gamma_{jk} \int\limits_0^{(y_k^q - y_j^q)} \varphi_{jk}^q(s)\, ds, \\ y_j := Cx_j, x_1, \ldots, x_N \in \mathbb{R}^n, \end{gathered}$$

where H_0 and $V_0(\bar{x})$ are the matrix and quadratic form introduced in Lemma 3. Calculation of the derivative $V(\bar{x}(t))$ along the solutions of system (3), (4) provides

$$\begin{aligned} \dot{V}(\bar{x}(t)) &= \dot{V}_0(\bar{x}(t)) + N \sum_{q=1}^m \sum_{j,k=1}^N \theta^q \underbrace{\gamma_{jk} \varphi_{jk}^q (y_k^q(t) - y_j^q(t))}_{\xi_{jk}^q(t) = -\xi_{kj}^q(t)} (\dot{y}_k^q(t) - \dot{y}_j^q(t)) \\ &= \dot{V}_0(\bar{x}(t)) - 2N \sum_{j=1}^N u_j(t)^\mathrm{T} \Theta \dot{y}_j(t) \\ &= \dot{V}_0(\bar{x}(t)) - \sum_{j,k=1}^N (u_j(t) - u_k(t))^\mathrm{T} \Theta (\dot{y}_j(t) - \dot{y}_k(t)) \leqslant -\mathcal{F}(\bar{y}(t), \bar{u}(t)). \end{aligned} \tag{A.3}$$

Therefore, by Lemma 1 along the solutions of system (3), (4) the function $V(\bar{x}(t))$ does not grow for any nonlinearity φ_{jk}^q satisfying the assumptions of Theorem 2. Relation (A.3) is satisfied for any assembly of nonlinearities φ_{jk}^q defining the function V. We note that if protocol (4) provides consensus, then $V(\bar{x}(t)) \to 0$ for $t \to \infty$. In particular, the aforementioned properties take place for solutions of the linear system like (3), (4) with the functions $\varphi_{jk}^q(y) = \kappa_n^q y$, where κ_n^q is a sequence from Assumption 3. Let $\bar{x}^{[n]}$ be a solution corresponding to the coefficients κ_n^q and some initial datum $\bar{x}^{[n]}(0) = a$, and $V^{[n]}$ being the corresponding Lyapunov function. Then, $\mathfrak{S}(\bar{y}^{[n]}(t)) = W_n(\bar{x}^{[n]}(t))$, where $W_n(\bar{x})$ is a quadratic form with $W_n(\bar{x}) \to 0$ for $n \to \infty$ (and $\kappa_n^q \to \alpha^q$). Since consensus is reached for any n, $V^{[n]}(a) = V_\alpha(a) + W_n(a) \geqslant \lim_{t\to\infty} V^{[n]}(\bar{x}^{[n]}(t)) = 0$. For $n \to \infty$, we have $V_\alpha(a) \geqslant 0$. Therefore, by Lemma 2 for any assembly of couplings φ_{jk} we have $V \geqslant 0$.

Since the function $V(\bar{x}(t))$ is globally bounded on any solution of system (3), (4), the same is true for the function $\mathfrak{S}(\bar{y}(t))$, whence assertion 1 of Theorem 2 follows in virtue of Lemma 2. To prove assertion 2, it suffices to note that if $y_j(t) - y_k(t)$ are globally bounded functions for all j, k, then $u_j(t)$ are bounded as well. Consequently, solution is prolongable infinitely. Additionally, $x_j(t) - x_k(t)$ is also bounded by virtue of observability of agents (3).

Further, in virtue of (A.3) on system solutions we have

$$\psi^q \int\limits_0^\infty \mathcal{F}_q(\bar{y}(t), \bar{u}(t))dt \leqslant \int\limits_0^\infty \mathcal{F}(\bar{y}(t), \bar{u}(t))dt < \infty.$$

If $\psi^q > 0$ for some q, then we get summability of the function

$$\sum_{j,k=1}^{N} F_q(u_k^q(t) - u_j^q(t), y_k^q(t) - y_j^q(t)) \geqslant 0.$$

If $\dot{y}_k^q(t) - \dot{y}_j^q(t)$ is bounded for all j, k, then it is easy to verify that this function is also uniformly continuous. Consequently, by the Barbalat lemma [19] $F_q(u_k^q(t) - u_j^q(t), y_k^q(t) - y_j^q(t)) \to 0$ for $t \to \infty$, which in virtue of Lemma 1 entails $y_j^q(t) - y_k^q(t) \to 0$ for all j, k and gives rise to assertion 3. Assertion 4 follows from assertions 2 and 3. For $\Theta > 0$, assertion 5 follows from assertion 4 in virtue of agent's observability, because if $y_j(t) - y_k(t) \to 0$, then $u_j(t) - u_k(t) \to 0$ and, consequently, $x_j(t) - x_k(t) \to 0$ for $t \to \infty$. In the case of $\Theta = 0$, it follows from (A.2) that $H_0 > 0$. Indeed, by substituting in (A.2) $x(t) = x^{[n]}(t)$ under arbitrary n and integrating it we obtain $V_0(\bar{x}(0)) \geqslant V_0(\bar{x}^{[n]}(t)) \to 0$ for $t \to +\infty$, equality being reached only in the case of $\mathcal{F}(\bar{y}^{[n]}(t), \bar{u}^{[n]}(t)) \equiv 0$, which by Lemma 1 entails $y_1(t) \equiv \ldots \equiv y_N(t)$, that is, $V_0(\bar{x}(0)) = 0$ only if $x_1(0) = \ldots = x_N(0)$. Consequently, the functions $x_j(t) - x_k(t)$ are bounded and for $\Theta = 0$ assertion 5 also follows from assertion 3, which proves the theorem for $\Theta \geqslant 0$.

In the case of $\Theta \leqslant 0$, the proof follows the same lines with replacement of $V_\alpha(\bar{x})$ by $V_\beta(\bar{x}) := V_0(\bar{x}) - \frac{N}{2}\sum_{q=1}^m \theta^q \beta^q \sum_{j,k=1}^N \gamma_{jk}|y_k^q - y_j^q|^2$, $\mathfrak{S}(\bar{y})$ by $\tilde{\mathfrak{S}}(\bar{y})$, and $\kappa_n^q \to \alpha^q$ by $\tilde{\kappa}_n^q \to \beta^q$.

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