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NAVIGATION AND CONTROL OF MOVING SYSTEMS

Problem of Uniform Deployment on a Line Segment for Second-Order Agents

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Abstract—Consideration was given to a special problem of controlling a formation of mobile agents, that of uniform deployment of several identical agents on a segment of the straight line. For the case of agents obeying the first-order dynamic model, this problem seems to be first formulated in 1997 by I.A. Wagner and A.M. Bruckstein as “row straightening.” In the present paper, the straightening algorithm was generalized to a more interesting case where the agent dynamics obeys second-order differential equations or, stated differently, it is the agent’s acceleration (or the force applied to it) that is the control.

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1. INTRODUCTION

Many complicated systems considered by the natural, social, and technical sciences are representable as unions of simpler subsystems or *agents* interacting each with several of its “neighbors” through exchange of information or physical actions. The graph describing possible directions of such interactions can be of complex and variable structure.

The oscillator networks, small power grids, networks of robots and sensors, models of economic and social interactions, and biological populations exemplify such systems that are called the *multiagent systems* and also known as *complex networks*, *cellular networks*, *interconnected systems*, and so on. Examples of such systems can be found in the recent monographs and reviews [1–6].

Self-sufficiency of agents, their *local interactions* without using the global information about the system as a whole, and *decentralization*, that is, lack of a central controller or module generating decisions that are common for all agents are the fundamental principle for design of the multiagent systems. Proliferation of the multiagent systems in engineering and industry, as well as rapid development of the corresponding mathematical theory, are due to their flexibility and cheapness as compared with the classical centralized approach.

One of the most important areas of application of the multiagent systems is represented by the *formation control* aiming at generating a group of agents of the desired fixed or moving geometrical images of a regular-shaped form. The problems of *dynamic* control of formations [5–9] are mostly concerned with control of mobile agents such as the wheeled robots, drones, submarines, and spacecraft. The desire to imitate behavior of biological formations [10] such as flock of birds of or insect swarm often is the incentive to design systems of this kind. The distributed algorithms to generate *static* formations are used in some problems of the sensor network theory such as the *problem of deployment* of agents in some domain or on a manifold [11–15], *coverage problem* [16], and *area partition problem* [17].

One of the simplest algorithms to deploy the agents on a static line segment was proposed in [18] under the name of “row straightening.” It relies on the concept of *averaging* where each agent moves towards the middle of the segment connecting two of its neighbors by measuring only the relative distances to them. Allied algorithms were suggested for the multiagent deployment of the discrete “ant like” agents on a ring [14, 15]. Another iterative procedure providing uniform deployment of agents on an ellipsis is the so-called “van Loan scheme” [19, 20]. Apart from the problem of formation control, the averaging model was used in [18] to describe the propagation of signal (voltage) along a cascade of RC chains. This algorithm was extended in [21, 22] to the case of perturbed agents. The nonlinear control law proposed in [21, 22] enables uniform deployment in *finite* time independently of the initial conditions. The structure of the protocol for uniform deployment of agents over a segment resembles the *consensus* algorithms in the multiagent system [1, 2, 4, 5, 23–28], but in distinction of them leads to a closed-loop system with a single globally stable equilibrium.

The assumption that the agent has the simplest dynamics of a single integrator, that is, that its velocity can be controlled directly, is a grave shortcoming of the algorithms considered in [15, 18, 20–22]. The present paper suggests a generalization of the averaging algorithm to the case of more realistic agent models obeying the second-order equations. It is shown that the control law of [18] can be used directly if the agent model has a damping velocity feedback. If friction in the agent model is negligible, it is possible to introduce in the control law a velocity feedback proposed in [29]. However, this approach assumes that each agent is capable of measuring its absolute velocity, which is very restrictive. The main results of the present paper are represented by the uniform deployment algorithms based only on the *relative measurements*. Each agent measures both its position and velocity relative to its two neighbors, rather than in the absolute coordinate system. The concept of such approach is rather common for the problems of multiagent control of the second-order agents [1, 5, 30]. In addition, it was shown that at the expense of some worsening in the characteristics of the transient process it becomes possible to avoid measuring the relative velocity and replace it by the output of the low-pass differentiating filter to which the agent’s relative position is fed (a similar idea was used, for example, in [31]). Thus, a solution of the problem of uniform deployment of agents with the double integrator model which does not use any velocity measurements and only those of relative position was obtained.

2. PRELIMINARY INFORMATION AND FORMULATION OF THE PROBLEM

In the present paper we deal with a group of $N \geqslant 1$ *mobile* agents numerated from 1 through N and two *static* agents with indices 0 and $N+1$. Position of the jth agent at a time instant is denoted by $x_j(t) \in \mathbb{R}^d$, $j = 0, 1, \dots, N+1$. It is required to determine a control algorithm, called also the *protocol*, which provides a uniform deployment of the mobile agents on the segment connecting the fixed points x_0 and x_{N+1}.

The above problem can be reduced formally to the classical problem of reachability or terminal control if the agents must reach the desired points in a finite time or to the stabilization problem if only the asymptotic convergence must be provided. In both cases, one needs to compute the desired positions on the segment. If the agents are arranged along the segment in the ascending order of their indices, the objective point of the jth agent is given by $x_j^0 := x_0 + x_{N+1}(j-1)/N$. After this preliminary procedure, each agent moves to the corresponding objective point independently of the rest of the agents. Under the assumption that the agent has a first-order model

$$\dot{x}_j(t) = u_j(t) \in \mathbb{R}^d, \quad j = 1, \dots, N, \tag{1}$$

the simplest control algorithm can be defined by the P-controller

$$\dot{x}_j(t) = \eta(x_j^0 - x_j), \quad x_j^0 := x_0 + x_{N+1}(j-1)/N. \tag{2}$$

Despite its apparent simplicity, the algorithm (2) relies on a very important restrictive assumption which in essence excludes the possibility of using the controller (2) for large agent formations. Namely, it is assumed for each agent that the position relative to the objective point can be measured. In particular, each agent has either to calculate itself its terminal point or recognize it somehow in the space using, for example, a dedicated transceiver. In both cases, the formation agents are not interchangeable and use different control laws to reach their objectives. If one or more agents fail, formation regrouping requires the total recalculation of the objective values x_j^0.

2.1. Decentralized Protocol for Uniform Deployment of Agents with Integrator Model

In contrast with the direct "centralized" solution of (2), a more promising *decentralized* protocol for the first-order agents (1) was suggested and considered in [18, 20, 29] which enables more uniform deployment of agents with the use only of the "local" interactions:

$$u_j(t) = \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)), \quad j = 1, \ldots, N. \tag{3}$$

The protocol (3) has many advantages over (2). The agents use only the relative measurements without accessing the full information about the group. Moreover, in the formation each agent needs to know only its "predecessor" and "successor" without its own number in the formation. If the jth agent fails, only one "reconnection" is required in the system to assign the agents $j-1$ and $j+1$ as neighbors after which protocol (3) deploys automatically the remaining $N-1$ agents. A similar reconnection enables one to add a new agent to the formation.

Since protocol (2) is coordinatewise decoupled, one can assume without loss of generality that $d = 1$: $x_j(t) \in \mathbb{R}$. By introducing the vector of system state $x = [x_1, x_2, \ldots, x_N]^\top$, its dynamics can be represented in the matrix terms as

$$\dot{x} = Ax + b, \tag{4}$$

where the matrix A and vector b are given by

$$A := \begin{bmatrix} -1 & 0.5 & 0 & \ldots & 0 \\ 0.5 & -1 & 0.5 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \ldots & 0.5 & -1 \end{bmatrix} \in \mathbb{R}^{N \times N}, \tag{5}$$

$$b := [x_0/2, 0, \ldots, 0, x_{N+1}/2]^\top \in \mathbb{R}^N. \tag{6}$$

The three-diagonal matrix A has the eigenvalues [32]

$$\lambda_k = -2\sin^2 \frac{k\pi}{2(N+1)}, \quad k = 1, \ldots, N. \tag{7}$$

Since $\lambda_k < 0$ for any $k = 1, \ldots, N$, A is a Hurwitz matrix and system (4) has one exponentially stable equilibrium

$$x_* := -A^{-1}b = x_0[1, \ldots, 1]^\top + \frac{x_{N+1} - x_0}{N+1}[1, 2, \ldots, N]^\top \in \mathbb{R}^N. \tag{8}$$

To put it in another way, independently of the initial conditions protocol (3) provides uniform deployment of agents on the segment with ends x_0 and x_{N+1}. The estimate of the rate of convergence follows immediately from (7)

$$\|x(t) - x_*\| \leqslant e^{-\hat{\lambda} t}\|x(0) - x_*\|. \tag{9}$$

It is valid for any solution of system (4), where $x(0)$ is the vector of initial position of agents, and the factor of convergence rate $\hat{\lambda}$ is given by

$$\hat{\lambda} = \min_k |\lambda_k| = 2\sin^2 \frac{\pi}{2(N+1)}. \tag{10}$$

It deserves noting that in this manner one can consider the discrete version of system (1), (3) considered in [18].

The present paper deals with the problem of uniform deployment of agents with a more realistic *second order* dynamics

$$\ddot{x}_j + a\dot{x}_j = u_j, \quad j = 1, \ldots, N, \tag{11}$$

where $a \geqslant 0$ is the constant friction coefficient. For $a = 0$, the model (11) becomes a double integrator. The problems of consensus in the multiagent systems and formation control with second-order agents are of essential interest in connection with the applications to the multiagent groups of mobile robots (see, for example, [1, 5, 6]).

3. MAIN RESULTS

We describe the main results, the distributed control protocols capable of uniform deployment of the second-order agents (11) over the given segment. For a start, we are going to check whether the control protocol (3) may be used for the second-order agents (11) and introduce some notation. For two numbers $p, q \in \mathbb{R}$, let $h_1(p, q), h_2(p, q) \in \mathbb{C}$ be two (real or complex) roots of the equation $h^2 + hp + q = 0$ and $H(p, q) := \max(\operatorname{Re} h_1(p, q), \operatorname{Re} h_2(p, q))$. In other words,

$$H(p, q) = \begin{cases} -p/2, & p^2 - 4q < 0 \\ \dfrac{-p + \sqrt{p^2 - 4q}}{2}, & p^2 - 4q \geqslant 0. \end{cases} \tag{12}$$

The following theorem proves that if each agent has a velocity feedback ($a > 0$), then one can use for them protocol (3) and estimate the rate of convergence.

Theorem 1. *Let $a > 0$. Then, protocol* (3) *is capable of uniform deployment of agents* (11) *over the segment with the ends x_0 and x_{N+1}, that is, $x(t) \to x_*$ and $\dot{x}(t) \to 0$ for $t \to +\infty$. At that, convergence is exponential:*

$$\|x(t) - x_*\| + \|\dot{x}(t)\| \leqslant Ce^{-\mu t}, \tag{13}$$

where $C = C(x(0), \dot{x}(0))$ and $\mu := -H(a, \hat{\lambda}) > 0$.

For $a \to 0$, we get $\mu \to 0$. With protocol (3), therefore, the rate of convergence to equilibrium is retarded. Indeed, for $a = 0$ the protocol is incapable to provide convergence to the equilibrium because the corresponding linear system

$$\ddot{x} = Ax + b,$$

where the matrices A, b are the same as in Eq. (4), is not exponentially stable but only Lyapunov-stable. The solutions of the system are given by $x(t) = x_* + \operatorname{Re}[v_k e^{i\omega_k}]$, where v_k is the eigenvector A corresponding to the eigenvalue λ_k and $\omega_k := \sqrt{|\lambda_k|}$. At the same time, Theorem 1 admits the following modification of protocol (3) not only for $a = 0$ but also any unstable agent (11) ($a < 0$).

Corollary. *The control algorithm*

$$u_j(t) = -\varkappa \dot{x}_j(t) + \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)) \tag{14}$$

is capable of uniform deployment of agents over a segment with the ends x_0 *and* x_{N+1} *for* $a + \varkappa > 0$. *At that, the convergence rate is given by* (13)*, where* $\mu := -H(a + \varkappa, \hat{\lambda}) > 0$.

Proof. With protocol (14) used for agents (11), we obtain a closed-loop system similar to that which could have been obtained by applying the original algorithm (14) to the agents with modified "friction" coefficient $a \mapsto a + \varkappa$. Therefore, the corollary follows from Theorem 1.

In distinction to protocol (3), the control law (14) uses not only the relative measurements, but also the agent's *absolute* velocity. In some applications, measurement of the absolute velocity may be possible despite the fact that position measurement in a fixed system may be impossible. The seagoing ships where the velocity is measured by an electromagnetic or Doppler log provide examples. At that, in position estimation integration of velocity gives too high errors to be useful for control. Another example is provided by the inertial navigation systems that are capable of determining the object's absolute velocity with much higher precision than its coordinates. Nevertheless, the formation control algorithms based only on the relative measurements have a much wider circle of applications.

As applied to the problem discussed in the present paper, the following control protocol

$$\begin{aligned} u_j(t) = \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)) + \frac{p}{2}(\dot{x}_{j-1}(t) - \dot{x}_j(t)) \\ + \frac{p}{2}(\dot{x}_{j+1}(t) - \dot{x}_j(t)), \quad j = 1, \ldots, N, \end{aligned} \tag{15}$$

is proposed, where $p > 0$ is a constant coefficient. The following result demonstrates that protocol (15) enables one to arrange uniformly the agents over a segment with provision of an exponential convergence.

Theorem 2. *Let* $a = 0$ *and* $p > 0$. *Then, protocol* (15) *provides uniform deployment of agents* (11) *over a segment with ends* x_0 *and* x_{N+1}*, that is,* $x(t) \to x_*$ *and* $\dot{x}(t) \to 0$ *for* $t \to +\infty$. *Additionally, satisfied is* (13) *where*

$$\mu = -\max_k H(-p\lambda_k, -\lambda_k) > 0 \quad \text{and} \quad \lambda_k \text{ is established from (7).}$$

Protocol (15) enables one to deploy the agents using the relative measurements. At the same time, at the expense of reduced convergence rate one can avoid at all the need for velocity measurement. Denote $w_j(t) := (x_{j-1}(t) + x_{j+1}(t))/2 - x_j(t)$; then algorithm (15) can be rearranged in $u_j(t) = w_j(t) + p\dot{w}_j(t)$. The idea borrowed from [31] lies in replacing the derivative $\dot{w}_j(t)$ by the output of some differentiating low-frequency filter $\dot{w}_j(t) \approx \dot{y}_j(t)$ where

$$\dot{y}_j(t) = -\gamma y_j(t) + w_j(t), \quad \gamma > 0.$$

After this replacement, the control protocol (15) comes to

$$\begin{aligned} u_j(t) &= w_j(t) + p\dot{y}_j(t) = (1 + p)w_j(t) - p\gamma y_j(t), \\ \dot{y}_j(t) &= -\gamma y_j(t) + w_j(t), \\ w_j(t) &= \frac{1}{2}(x_{j-1}(t) - x_j(t)) + \frac{1}{2}(x_{j+1}(t) - x_j(t)), \end{aligned} \tag{16}$$

which also allows one to deploy uniformly the agents with exponential convergence, as is corroborated by the following theorem.

Theorem 3. *Let $a = 0$ and $p, \gamma > 0$. Then, the control protocol* (16) *provides uniform deployment of agents* (11) *over a segment with the ends x_0 and x_{N+1}, that is, $x(t) \to x_*$ and $\dot{x}(t) \to 0$ for $t \to +\infty$. Additionally, satisfied is Eq.* (13) *where*

$$\mu = -\max\left\{\operatorname{Re} z : z^3 + \gamma z^2 - (p+1)\lambda_k z - \gamma\lambda_k = 0\right\} > 0.$$

Remark 1. The special formation control algorithm at hand can be considered as a particular case of the problem of containment control where a group of mobile agent uses an algorithm retaining them within the convex hull of several "leaders" which are fixed in the simplest case [1]. This problem does not come to the conventional consensus protocol. The interaction graph has no spanning trees and is covered by the spanning forest of trees rooted in the fixed leaders. In distinction to the consensus protocol having an infinite number of equilibria, the protocols in the problem of containment control with fixed leaders as a rule provide agents' convergence to a unique equilibrium. Generally speaking, this equilibrium depends on the leaders' topology and positions. In the general case, calculation of the given final position of the agents is laborious even in the case of single integrators [1]. In the case under consideration, not only the stable agents' positions (the points arranged uniformly over the segment between two leaders) are calculated explicitly, but also the rates of convergence to these positions are estimated.

Remark 2. The structurally allied formation control algorithms for the second-order agent were considered in [33]. The formation stability criterion proposed there refers to the case of a more general interaction graph where the agent can use information not only about its two neighbors and has to solve a linear matrix inequality including the graph Laplacian and the coefficients of the agent transfer function. To determine the stability criteria, in [33] the method of generalized frequency variable was used which in essence is equivalent to the Polyak–Tsypkin method [34] used in the present paper. At the same time, as applied to the agents with second-order dynamics the results of [33] have some limitations. For instance, Corollaries 5.1 and 5.2 of [33], where consideration is given to the distributed algorithms that are structurally similar to (15), assume that the friction coefficient is strictly positive, $a > 0$. In the case of agents with dynamics of the double integrator, proved was convergence of a structurally more complicated algorithm comprising an integrating unit (Corollary 5.3 in [33]). In distinction to [33] for the frictionless model, the present paper establishes convergence of algorithm (15). Additionally, consideration is also given to an algorithm with "differentiator" (16) enabling to do without measuring the relative velocities of the neighboring agents. Apart from this, Theorems 1–3 give precise estimates of the algorithm's rate of convergence which were not established in [33].

4. EXAMPLES

We are going to demonstrate efficiency of the proposed control protocols for uniform deployment of agents on plane:

$$\ddot{\xi}_j + a\xi_j = u_j, \quad \xi_j = [x_j, y_j]^\top \in \mathbb{R}^2. \tag{17}$$

As was already mentioned, all results are valid for the space of any dimensionality because the control protocols are coordinatewise decoupled. In all test examples consideration is given to the multiagent system with the following parameters: needed is to arrange uniformly a group of $N = 5$ agents like (17) over a segment with the ends $\xi_0 = [-2,\ 3{,}1]^\top$ and $\xi_6 = [2,\ 2{,}1]^\top$.

The first example shows motion of agents (17), $a = 2$, under the action of the control protocol (3). In Fig. 1 the agents tend to arrange themselves equidistantly according to Theorem 1.

The following examples demonstrate applicability of algorithms (15) and (16) for the agents without negative velocity feedback, $a = 0$. The following constants were used for modeling: $p = 10$

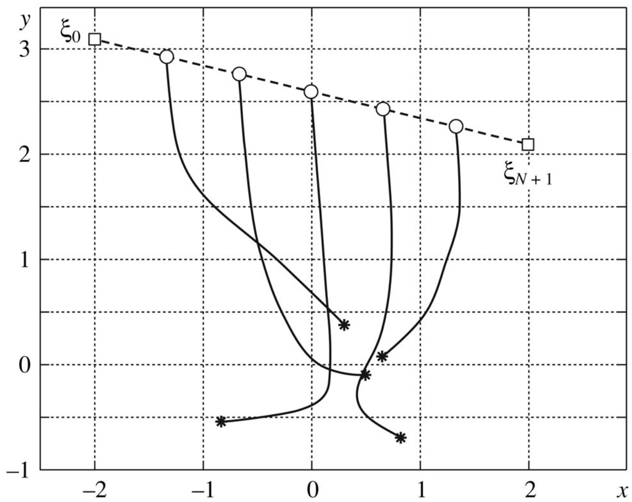

Fig. 1. Protocol (3) for the agents with damping velocity feedback ($a = 2$).

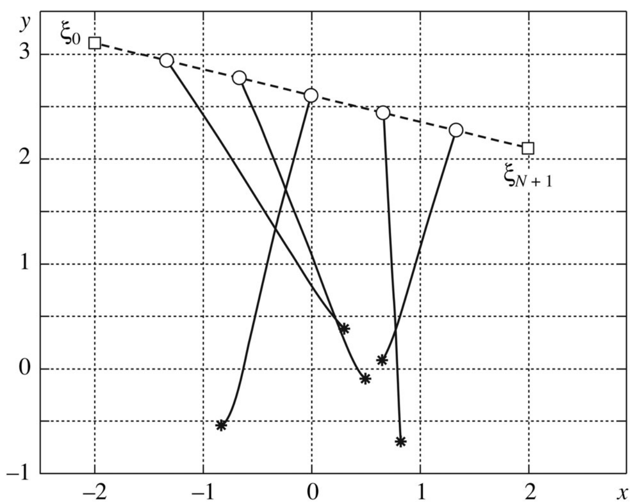

Fig. 2. Trajectories of frictionless agents ($a = 0$) under the control protocol (15).

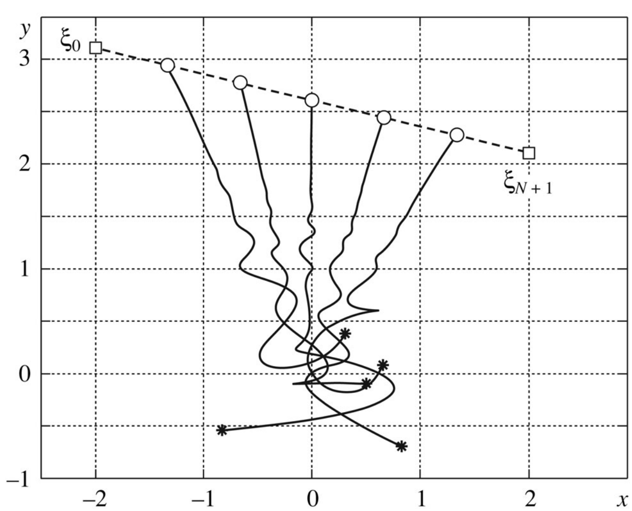

Fig. 3. Trajectories of frictionless agents ($a = 0$) under the control protocol (16).

and $\gamma = 1.4$ (the latter parameter is used only in algorithm (16)). Figure 2 depicts system behavior under the protocol (15) using relative velocities, and Fig. 3 shows the system trajectories under protocol (16). Both protocols enable uniform deployment as is stated in Theorems 2 and 3. It also deserves noting that algorithm (15) provides "smoother" trajectories and higher rate of convergence. Such behavior is attributed to the introduction of an additional dynamic system, the differentiating low-frequency filter retarding dynamics of the entire system.

5. CONCLUSIONS

The present paper considered a special problem of constructing a static formation of agents, the uniform deployment over a segment with fixed ends. In distinction to the previous publications, consideration was given to the case where the agent equations have second order, at that the agent can change its position with respect to the neighbors and the absolute, or also the relative, velocity. Additionally, consideration was also given to a more general case where measurement even of the relative speeds is impossible, and the estimates generated by the differentiating low-frequency filer are used instead. The theoretical results are supported by computer-aided modeling. In future the present authors plan to consider design of planar or spatial formation problems of mobile agents.

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APPENDIX

To prove stability of the closed-loop systems obtained by using protocols (3), (15), and (16) for agents like (11), we use the Polyak–Tsypkin stability criterion [34] based on the notion of "generalized frequency variable" and established later independently by S. Hara [33].

Assume that it is required to consider stability of a high-order linear system given by

$$\phi\left(\frac{d}{dt}\right)x(t) = Ax(t), \tag{A.1}$$

where $\phi(s)$ is a scalar polynomial and A is an $N \times N$ matrix. Having denoted by $D(s) := \det(sI - A)$ the characteristic polynomial of the matrix A, one can readily demonstrate that system (A.1) is stable if and only if $G(s) := D(\phi(s))$ is a Hurwitz polynomial. In a more general case, let $\phi(s) = \psi(s)/\rho(s)$ be a rational function which is analytical on the closed right half-plane $\bar{\mathbb{C}}_+ := \{s : \operatorname{Re} s \geqslant 0\}$, that is, ρ is a Hurwitz polynomial. Then, system (A.1) takes on the form

$$\psi\left(\frac{d}{dt}\right)x(t) = \rho\left(\frac{d}{dt}\right)Ax(t)$$

and is stable if and only if the rational function $G(s) = D(\phi(s))$ has no zeros in $\bar{\mathbb{C}}_+$. Despite the fact that this property can be verified directly without using the structure $G(s)$, for high-degree polynomials such procedure may prove to be burdensome. The notion of Ω-*domain* [34] can be used as an alternative. By definition, the Ω-domain of the function $\phi(s)$ is the set of points λ on the complex plane for which the function $\phi(s) - \lambda$ has no zeros in the closed right half-plane:

$$\Omega = \left\{\lambda \in \mathbb{C} : \phi(s) - \lambda \neq 0, \operatorname{Re} s \geqslant 0\right\}.$$

The following Theorem A.1 [34] simplifies the task of examining for stability system (A.1) by decomposing it into two simpler subproblems such as (a) calculation of the eigenvalues of the matrix A and (b) determination of the Ω-domain of the function $\phi(s)$.

Theorem A.1. *The characteristic function $G(s) = D(\phi(s))$ has no zeros in $\bar{\mathbb{C}}_+$ if and only if all zeros of $D(s)$, that is, the eigenvalues λ_k of the matrix A, lie in the Ω-domain of the function $\phi(\cdot)$. The estimate*

$$|x(t)| \leqslant Ce^{\alpha t}, \quad \alpha := \max\{\operatorname{Re} s : \phi(s) = \lambda_k \quad \textit{for some} \quad k = 1, \ldots, N\}$$

is valid for any solution (A.1).

There is no need in the precise determination of the Ω-domain for proving convergence of the algorithms of uniform deployment. For the details of the corresponding algorithm the reader is referred to [34]. Since A of form (5) is a Hurwitz matrix, it suffices to prove that the corresponding Ω-domain includes all negative real numbers. Now we pass to proving the main results.

Proof of Theorem 1. Denote by $x = [x_1, x_2, \ldots, x_N]^\top$. Then, the closed-loop system (11), (3) can be easily rearranged in

$$\phi\left(\frac{d}{dt}\right)x(t) = Ax(t) + b, \quad \phi(s) = s^2 + as, \tag{A.2}$$

where A and b are given, respectively, by (5) and (6). Stability of the equilibrium position $x_* = -A^{-1}b$ is equivalent to stability of the self-sufficient system (A.1). By assumption $a > 0$; therefore, for $\lambda < 0$ equation $\phi(s) = \lambda$ has no unstable roots because $\phi(s) - \lambda$ is a Hurwitz polynomial. Whence stability follows immediately: all eigenvalues of the matrix A are real and negative and given by (7). According to Theorem A.1, the index of the exponent μ in (13) has the form $\mu = \max_k H(a, -\lambda_k)$ which results in $\mu = H(a, \hat{\lambda})$ because the function $H(a, \cdot)$ is nondecreasing, which proves Theorem 1.

Proof of Theorem 2. Let $a = 0$. Set down the closed-loop system (11), (15) as

$$s^2 x = (ps + 1)(Ax + b), \quad s := \frac{d}{dt},$$

which represents (A.2) with a rational function $\phi(s) = s^2/(ps + 1)$. Stability of the equilibrium $x_* = -A^{-1}b$ is equivalent to the stability of the autonomous system (A.1), which follows from Theorem A.1 because for $\lambda < 0$ equation $\phi(s) = \lambda$ has no unstable roots because for $p > 0$ and $\lambda < 0$ $s^2 - p\lambda s - \lambda$ is a Hurwitz polynomial. In particular, the Ω-domain includes all eigenvalues λ_k. The formula of the rate of convergence follows obviously from Theorem A.1, which proves Theorem 2.

Proof of Theorem 3. Similar to the previous proofs, system (11), (15) is rearranged in

$$s^2 x = q(s)(Ax + b), \quad s := \frac{d}{dt}, \quad q(s) := 1 + \frac{ps}{s + \gamma},$$

which is equivalent to system (A.2) with the rational function $\phi(s) = s^2(s + \gamma)/(s(p + 1) + \gamma)$. To verify stability, it suffices to demonstrate that the equation $\phi(s) - \lambda$ has no unstable roots for $\lambda < 0$. Stability follows from the Routh–Hurwitz criterion stating that $s^3 + as^2 + bs + c$ is a Hurwitz polynomial if and only if $a, b, c > 0$ and $ab > c$. In particular, the polynomial $s^2(s + \gamma) - \lambda(p + 1)s - \lambda\gamma$ is Hurwitz for $\gamma, p > 0$ and $\lambda < 0$. Consequently, the equilibrium position is exponentially stable, and the formula for the rate of convergence follows from Theorem A.1, which proves Theorem 3.

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