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A new extension of the infinite-dimensional KYP lemma in the coercive case. ⋆

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Abstract: The Kalman-Yakubovich-Popov (KYP) Lemma, giving the frequency-domain criteria for solvability of LQR problems and feasibility of related LMIs and Lur'e-Riccati equations, is undoubtedly one of the cornerstone results in modern control theory. Numerous applications of the KYP lemma in nonlinear and robust control motivated its extension to systems with distributed parameters. Unlike the finite-dimensional case, where the KYP lemma admits elegant proofs by means of algebraic techniques and methods from convex analysis, a prerequisite for its infinite-dimensional extensions is feasibility of the corresponding LQR problem, resulting on assumptions of exponential or L^2-stabilizability. This property is restrictive and not easily verifiable for general infinite-dimensional systems. Being quite natural in the problems of absolute *stability*, it seems unnecessary in other applications of the KYP lemma which require only solvability of the operator inequality, but not positivity of its solution, such as e.g. instability or dichotomy criteria. In the present paper we show, that in the "strict" or *coercive* case the KYP lemma retains its validity, replacing stabilizability assumption with some technical assumption which holds, for instance, whenever the linear system is exponentially dichotomic.

Keywords: KYP lemma, infinite-dimensional systems, dissipation inequalities

1. INTRODUCTION

It will not be an exaggeration to say that the Kalman-Yakubovich-Popov (KYP) Lemma plays an exceptional role in automatic control, being one of its keystone results. Since the publication of two seminal papers Yakubovich [1962] and Kalman [1963], the KYP Lemma and its extensions have penetrated most of areas of the modern control theory, see recent surveys Barabanov et al. [1996], Gusev and Likhtarnikov [2006], Matveev [2006] and references therein for applications and principal historic milestones.

Initially, the development of the KYP lemma was promoted by the problems of *absolute stability* for nonlinear systems, representable as a feedback interconnection of a known linear part and an uncertain sector-bounded nonlinearity. Such systems are often referred to as *Lur'e-type* systems in honor of A.I. Lurie, who promoted a class of Lyapunov functions to examine their stability Lur'e [1957]. These Lyapunov functions are obtained via summation of a quadratic form (which is unknown and to be found) and integral of the nonlinearity; their existence boils down to solvability of special *linear matrix inequalities*, which in its turn comes to a frequency-domain condition, employing the transfer function of the linear part and slopes of the sector, restricting the nonlinearity Yakubovich [1962], Kalman [1963]. The history of absolute stability and related applications of the KYP lemma (such as criteria for instability, oscillations etc,) may be found in e.g. Yakubovich [1967], Barabanov [2000], Yakubovich [1998], Liberzon [2006], Popov [1973], Gelig et al. [2004], Leonov [1996] and references therein.

Besides the LMIs feasibility, the KYP lemma also gives conditions for solvability of the *Lur'e-Riccati* equations, giving the optimal feedback in LQR problems (see e.g. Matveev [2006]). Another important application of the KYP lemma is the conditions for passivity, feedback passivity and the Willems dissipativity of linear systems under quadratic supply rates Fradkov [2003], Willems [1972].

The numerous applications of the classical KYP lemma was a strong motivation to extend it for wider classes of systems, such as periodic systems Yakubovich [1986, 1988, 1990], Yakubovich et al. [2007], general non-stationary systems Savkin [1992], Fabbri et al. [2003], descriptor systems Camlibel and Frasca [2009], behavioral systems (van der Geest and Trentelman [1997]). However, the most complicated are extensions of the KYP lemma to various classes of distributed-parameters systems that has been studying since 1970s Brusin [1976], Yakubovich [1974, 1975], Antonov et al. [1975], Likhtarnikov and Yakubovich [1976, 1977]. The finite-dimensional KYP lemma can be proved in several ways, using algebraic tools Yakubovich [1973], Popov [1973], elegant methods of convex analysis and optimization Balakrishnan and Vandenberghe [2003], Rantzer [1996], Iwasaki and Hara [2005] or optimal control methods (Willems [1971],Matveev [2006]). In the infinite-dimensional case the first and second approaches

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are unworkable but for some special situations Gusev [2005]. The main method, used for establishing the infinite-dimensional KYP lemma, is based on the linear-quadratic optimization Pandolfi [1997], Balakrishnan [1995], Louis and Wexler [1991], Curtain [1996], which naturally requires the assumption of *stabilizability*, providing the existence of L^2-bounded solutions under any initial conditions. In the case of strict or *coercive* frequency-domain inequalities the stabilizability is the only extra assumption for solvability of the operator inequalities, whereas additional restrictive assumptions are needed in the non-coercive case Yakubovich [1975], Likhtarnikov and Yakubovich [1977].

In the present paper we consider the situation where the stabilizability condition may fail, so the classical optimization approach is not applicable. We show, however, that in the coercive case the frequency-domain inequality still guarantees the solvability of linear operator inequalities, provided that some technical assumption holds that is much wider than stabilizability. This condition is valid, for instance, if the linear system has no spectrum on the imaginary axis (or the unit circle in the discrete-time case). The motivation to relax stabilizability assumptions is twofold. Firstly, its direct verification for general distributed-parameter systems may be quite troublesome. Secondly, many applications of the KYP lemma, including e.g. criteria for the instability Barabanov [1984] and dichotomy Barabanov [1982], Likhtarnikov and Yakubovich [1998], deal with nonlinear systems of Lur'e type, whose linear part is not necessarily stable.

2. PRELIMINARIES AND NOTATIONS

Throughout the paper we assume all Hilbert spaces complex. Given two vectors x_1, x_2 from some Hilbert space X, their scalar product is denoted by $x_1^* x_2$. As usual, for an operator $K : X \to Y$ we denote its adjoint with $K^* : Y \to X$. By a Hermitian form we always mean bounded one, unless otherwise stated. We call such a form $\mathcal{F}(x) = x^* Px$ *strict positive definite* (written $\mathcal{F}(x) \succ 0$) if $\varepsilon > 0$ exists such that $\mathcal{F}(x) \geq \varepsilon |x|^2$ for any x, the same notation applies to the self-adjoint operator $P = P^*$.

The Hilbert space l^2 consists of complex-valued sequences $(a_n)_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

Henceforth S^1 stands for the unit sphere on complex plane $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

3. THE KYP LEMMA FOR STABILIZABLE DISCRETE-TIME SYSTEMS.

In this section we recall the basic formulation of the infinite-dimensional KYP lemma for stabilizable systems. We confine ourselves only to the case of discrete-time systems with *bounded* operators and *coercive* inequalities. The infinite-dimensional KYP lemma for the discrete-time case Antonov et al. [1975] will be the base for our subsequent considerations.

Consider a pair of complex Hilbert spaces $X = \{x\}, U = \{u\}$ and a Hermitian form $F : X \times U \to \mathbb{R}$ as follows

$$F(x, u) = x^* F_{xx} x + 2\, Re\, x^* F_{xu} u + u^* F_{uu} u. \quad (1)$$

Here $F_{xx} = F_{xx}^* : X \to X$, $F_{uu} = F_{uu}^* : U \to U$ and $F_{xu} : U \to X$ are bounded linear operators.

Consider a linear system with the state $x(t) \in X$ and control $u(t) \in U$, obeying the discrete-time equation

$$x(t + 1) = Ax(t) + Bu(t),\ t = 0, 1, 2 \dots . \quad (2)$$

The coercive KYP lemma establishes conditions, providing the existence of a Hermitian form $V(x) = x^* Hx$ with $H = H^*$ bounded, such that for some $\varepsilon > 0$ the *dissipation inequality* holds along any solution $[x(\cdot), u(\cdot)]$ of (2):

$$V(x(t+1)) - V(x(t)) + F(x(t), u(t)) \geq \varepsilon(|x(t)|^2 + |u(t)|^2).$$

Getting rid of the plant dynamics, the latter condition shapes into the following linear operator inequality in H:

$$(Ax + Bu)^* H(Ax + Bu) - x^* Hx + F(x, u) \succ 0. \quad (3)$$

Definition 1. The system (2) is *stabilizable* if there exists a linear bounded operator $h : X \to U$ such that $A + Bh$ is a stable operator (i.e. spectrum lies inside the unit disc).

It may be shown that the stabilizability is equivalent to the following condition of *L^2-controllability*, which looks much more general: for any $a \in X$ there exists a solution $[x(t), u(t)]$ with $x(0) = a$ and $\sum_{t=0}^{\infty} |x(t)|^2 + |u(t)|^2 < \infty$.

Theorem 2. Assume the system (2) is stabilizable. Then two conditions are equivalent:

(a) there exists $\varepsilon_0 > 0$ such that for any $\lambda \in S^1$ and any pair $x \in X, u \in U$ with $\lambda x = Ax + Bu$ one has
$$F(x, u) \geq \varepsilon_0(|x|^2 + |u|^2). \quad (4)$$
(b) operator inequality (3) has a solution $H = H^*$.

One of the operators H from (b) (and in fact, the maximal one) gives a solution in the following LQR problem

$$\sum_{t=0}^{\infty} F(x(t), u(t)) \to \min, x(0) = a, |x(\cdot)| + |u(\cdot)| \in l^2, \ (\mathbf{P})$$

whose optimal value $a^* Ha$ is achieved by using the feedback $u(t) = hx(t)$, where $h = -(B^* HB + F_{uu})^{-1} B^* HA$.

Below we will use the following simple extension of Theorem 2, dealing with the forms depending on some *output* rather than the whole state vector. Let $Y = \{y\}$ be another Hilbert space and $C : X \to Y$ be a linear operator.

Corollary 3. Assume the system (2) is *output stabilizable* with respect to the output $y(t) = Cx(t)$, that is, $A+BKC$ is stable for some linear bounded operator $K : Y \to U$ (the feedback $u = Ky$ stabilizes the plant). For a Hermitian form $F_0 : Y \times U \to \mathbb{R}$, two statements are equivalent:

(a') there exists $\varepsilon_0 > 0$ such that for any $\lambda \in S^1$ and any pair $x \in X, u \in U$ with $\lambda x = Ax + Bu$ one has
$$F(x, u) := F_0(Cx, u) \geq \varepsilon_0(|Cx|^2 + |u|^2). \quad (5)$$
(b') operator inequality (3) has a solution $H = H^*$.

Proof. For A being stable, conditions (5) and (4) are equivalent since $|x| = |A_\lambda^{-1} Bu| \leq c|u|$ whenever $\lambda x = Ax + Bu$, where $c > 0$ is constant and $A_\lambda := \lambda I - A$. The general case reduces to the case of stable system by introducing a new input $v := u - Ky$ and replacing the triple $A, B, F_0(y, u)$ with respectively $A + BKC, B, F_0(y, v + Ky)$, which replacement preserves both (5) and (3). □

Conditions of full or output stabilizability cannot be fully discarded, as illustrated by the following example. Let $B = 0$ and S^1 contains some spectral points of A, none of them being eigenvalues. This is exemplified e.g. by the forward shift operator on l^2 that has eigenvalues only inside the unit circle, whereas S^1 is its continuous spectrum. Suppose that $F_{xx} = 0$ and $F_{uu} \succ 0$, then (4) obviously holds since $\lambda x = Ax$ implies that $x = 0$ whenever $\lambda \in S^1$. Were the inequality (3) solvable in H, one would have $A^*HA - H \succ 0$. This however contradicts to the existence of spectral points $\lambda \in S^1$: for any such point, there exists a sequence x_n such that $|x_n| = 1$ and $Ax_n - \lambda x_n \to 0$ which entails that $x_n(A^*HA - H)x_n \to 0$.

Since frequency-domain inequality (5) is a special case of non-strict frequency-domain conditions, the output stabilizability in Corollary 3 can be replaced with infinite-dimensional version of controllability.

Lemma 4. Corollary 3 remains valid without output stabilizability, assuming that A^{-1} is bounded and both pairs (A, B) and $(A^{-1}, A^{-1}B)$ are stabilizable (in other words, one may stabilize both original system and that with the *inverted time* $t = \dots, -2, -1, 0$).

Lemma 7 can be proved analogously to non-coercive version of the KYP lemma in Yakubovich [1975], its proof is however very technical and cannot be given here due to space limitations. Note that in the finite-dimensional case sumultaneous stabilizability and anti-stabilizability of the system entail its Kalman controllability.

As a consequence of a general result [Iwasaki and Hara 2005, Theorem 2] it follows that the case of *finite-dimensional* spaces $\dim X < \infty, \dim U < \infty$ the frequency-domain inequality, shaping into

$$F(x, u) > 0 \quad \forall (x, u) \neq 0, \lambda \in S^1 : \lambda x = Ax + Bu$$

is necessary and sufficient for the solvability of linear matrix inequality (3) *without* additional assumptions. Although generic MIMO system is stabilizable (and even controllable), this result is of high interest since it cannot be proved neither by algebraic nor by standard optimization techniques and, as far as the author is aware, still awaiting its control-theoretic interpretation. Below we give an independent prove of this fact and also extend it to the infinite-dimensional case. We also demonstrate that the coercive frequency-domain inequality is closely related to the solvability of the "relaxed" linear-quadratic optimization problem which, unlike the usual LQR problem, is feasible independently of the system stabilizability.

4. KYP LEMMA FOR BOUNDED OPERATORS: BEYOND STABILIZABILITY

In this section, we show that the stabilizability assumption in Theorem 2 may be significantly relaxed. Without this assumption, there is no straightforward relation between the KYP lemma and linear-quadratic optimization, which has been up to now a basic tool to prove the KYP lemma in infinite-dimensional case. However, such a relation may be restored by passing to an auxiliary optimization problem, which we call a *stabilizable relaxation*.

If the optimization problem (**P**) is infeasible because of the non-stabilizable plant (2), consider a one-parametrized family of problems $\mathbf{P}_\gamma$, dealing with an augmented system

$$x(t + 1) = Ax(t) + Bu(t) + v(t) \tag{6}$$

Here the control vector is $(u(t), v(t)) \in U \times X$, and the system is obviously stabilizable. So an admissible process in the following problem exists for any $a \in X$:

$$\begin{gathered} \sum_{t=0}^{\infty} F(x(t), u(t)) + \gamma |v(t)|^2 \to \min, \\ x(0) = a, |x(\cdot)| + |u(\cdot)| + |v(\cdot)| \in l^2, \end{gathered} \tag{$\mathbf{P}_\gamma$}$$

The step from original linear quadratic problem $\mathbf{P}$ to $\mathbf{P}_\gamma$ is a standard trick of Lagrangian relaxation: the constraint $x(t + 1) = Ax(t) + Bu(t)$ is replaced with an additional term in the cost function, penalizing the norm of the "discrepancy" $v(t) = x(t+1) - Ax(t) - Bu(t)$. If the original system is stabilizable and condition (a) holds, it may be shown that the optimal value in any problem $\mathbf{P}_\gamma$ coincides with that from $\mathbf{P}$ and the optimal process given by the controller $u(t) = hx(t), v(t) = 0$ provided that $\gamma > 0$ is sufficiently large and hence the optimal value $V_\gamma(a) = a^*H_\gamma a$ gives a solution $H = H_\gamma$ for (3). We show that those properties infact *do not* require the stabilizability and hold under much less restrictive conditions.

It can be easily shown that if the Hermitian form $F_\gamma(x, u, v) = F(x, u) + \gamma|v|^2$ satisfies (3) for some H, the same is obviously true for $F(x, u)$ (by merely putting $v = 0$). Our goal is to disclose conditions ensuring that $F_\gamma(x, u, v)$ satisfies the frequency-domain condition (4).

We adopt an additional assumption.

Assumption 5. Let $P_\lambda = \{\lambda x - Ax - Bu : x \in X, u \in U\}$ stand for the image of the linear operator $[\lambda I - A, B] : X \times U \to X$. For any $\lambda \in S^1$ there exist maps $x_\lambda : P_\lambda \to X$, $u_\lambda : P_\lambda \to U$ and a constant $c_\lambda > 0$ such that $\lambda x_\lambda(v) = Ax_\lambda(v) + Bu_\lambda(v) + v \, \forall v \in P_\lambda$ and $|x_\lambda(v)|, |u_\lambda(v)| \leq c_\lambda |v|$.

The direct verification of Assumption 5 may be troublesome. However, it holds in the important case, described in the following Lemma.

Lemma 6. Let P_λ be closed in X for any $\lambda \in S^1$. Then Assumption 5 holds.

Proof. Since the linear bounded operator $[\lambda I - A, B] : X \times U \to P_\lambda$ is bounded and P_λ is a Hilbert space, it is an *open map*, i.e. the image of the unit ball from $X \times U$ is an open neighborhood of 0. This implies that for any $v \in X$ such that $|v| \leq \varepsilon$ there exist $x_\lambda(v), u_\lambda(v)$ such that $|x_\lambda(v)|, |u_\lambda(v)| \leq 1$. For v outside the ball $|v| \leq \varepsilon$ one can put $x_\lambda(v) = \varepsilon^{-1}|v|x_\lambda(\varepsilon v/|v|)$ so that $|x_\lambda(v)| \leq \varepsilon^{-1}|v|$. Analogously one prolong $u_\lambda(v)$. So Assumption 5 holds with $c_\lambda = \max(\varepsilon^{-1}, 1)$. □

In fact, under assumptions of Lemma 6 x_λ, u_λ can always be chosen linear, but we do not need this property.

Assumption 5 obviously holds for the finite-dimensional systems $\dim X, \dim U < \infty$. An important situation where the condition from Lemma 6 holds independently of B is the *dichotomy*: the spectrum of A does not intersect S^1, so all the solutions are either exponentially stable or exponentially unstable. In this case $(\lambda I - A)X = X$ for any λ. More generally, it holds for $A = A_0 + K$, where A_0 is dichotomic and K is compact.

The following theorem is our first main result and extends the coercive KYP lemma in the case of bounded operators to the case, where the system may be non-stabilizable.

Theorem 7. Theorem 2 retains its validity, relaxing the stabilizability to Assumption 5.

Proof. The implication (b)⇒ (a) is independent of stabilizability and is verified by substitution of x, u such that $\lambda x = Ax + Bu$ into (3). To prove the inverse, we are going to apply Theorem 2 to the the augmented system (6) and the Hermitian form $F_\gamma(x, u, v)$. For $\lambda \in S^1$, the equality $\lambda x = Ax + Bu + v$ entails that $x = x_\lambda(v) + \delta x$, $u = u_\lambda(v) + \delta u$ where $\lambda\,\delta x = A\,\delta x + B\,\delta u$ and hence $F(\delta x, \delta u) \geq \varepsilon(|\delta x|^2 + |\delta u|^2)$. Therefore, for each $\lambda \in S^1$ there exist $\gamma(\lambda)$ and $\varepsilon(\lambda)$ such that $F_{\gamma(\lambda)}(x, u, v) \geq 2\varepsilon(\lambda)(|x|^2 + |u|^2 + |v|^2)$. The latter inequality implies that $F_{\gamma(\lambda)}(x, u, v) \geq \varepsilon(\lambda)(|x|^2 + |u|^2 + |v|^2)$ whenever $\lambda' x = Ax + Bu + v$ and λ' belongs to sufficiently small open ball $J_\lambda \ni \lambda$. Since S^1 is a compact set, there exists finite sequence $\lambda_1, \ldots, \lambda_n$ such that J_{λ_i} cover S^1. Taking $\gamma = \max_j \gamma_{\lambda_i}$, one proves condition (a) for the augmented form $F_\gamma(x, u, v)$ and hence (3) holds for the augmented system (6) and F_γ:

$$(Ax+Bu+v)^* H_\gamma (Ax+Bu+v) - x^* H_\gamma x + F_\gamma(x, u, v) \succ 0.$$

Substituting $v = 0$, one establishes (3) with $H = H_\gamma$. □

The claim of Corollary 3 may be extended in the same way, but for the only difference. To use Corollary 3 for the relaxed form $F_0(Cx, u) + \gamma|v|^2$, one has to provide *output* stabilizability of the system (6). This follows e.g. from the *detectability*: $A + KC$ is a stable operator for some $K : Y \to X$, which property we assume to hold. On the other hand, as implied by Lemma 4, it is sufficient to verify the both stabilizability and "anti-stabilizability" of (6), which is easily shown to hold if A^{-1} is bounded.

Theorem 8. Corollary 3 remains valid without the stabilizability assumption, provided that 1) either (A, C) is detectable or A^{-1} is bounded; 2) Assumption 5 holds. In the latter Assumption the condition $|x_\lambda(v)| \leq c_\lambda |v|$ is relaxable to $|Cx_\lambda(v)| \leq c_\lambda |v|$.

5. CONTINUOUS-TIME DYNAMICS: UNBOUNDED OPERATOR CASE

In this section, we develop analogue of the discrete-time Theorem 7 for the case of continuous time systems. The crucial difference between sampled continuous and time models is that the latter ones are usually obey equations with *unbounded* operators, to which numerous PDE and integral equations may be reduced.

We start with the problem formulation. Consider a continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(t) \in X, u(t) \in U. \qquad (7)$$

Here $A : D(A) \to X$ stands for some *unbounded* operator, generating a C_0-semigroup $S(t)$ on X. A (weak) solution of (7) may be defined by the Cauchy formula

$$x(t) = S(t)x(0) + \int_0^t S(t - \varkappa)Bu(\varkappa)d\varkappa.$$

For simplicity, we confine ourselves to the case where B is a bounded operator. For a Hermitian form $F(x, u)$, we are interested in the solvability of the following strict operator "dissipation inequality" with respect to $H = H^*$ and $\varepsilon > 0$

$$x(T)^* Hx(T) - x^*(0)Hx(0) + \int_0^T F(x(t), u(t))dt \geq$$
$$\geq \varepsilon \int_0^T (|x(t)|^2 + |u(t)|^2)dt \quad \forall T \geq 0,$$

or, equivalently, continuous-time analog [1] of (3):

$$2x^* H(Ax + Bu) + F(x, u) \succ 0. \qquad (8)$$

The following assumption is a continuous-time analog of Assumption 5, replacing S^1 with the imaginary axis $\imath\mathbb{R}$.

Assumption 9. Let $P_{\imath\omega} = (\imath\omega I - A)D(A) + BU$ be the image of the operator $[\imath\omega - A, B] : D(A) \times U \to X$. Suppose that for any $\omega \in \mathbb{R}$ there exist maps $x_\omega : P_{\imath\omega} \to P_{\imath\omega}$, $u_\omega : P_{\imath\omega} \to U$ and a constant $c_\omega > 0$ such that $\imath\omega x_\omega(v) = Ax_\omega(v) + Bu_\omega(v) + v\ \forall v \in P_{\imath\omega}$ and $|x_\omega(v)|, |u_\omega(v)| \leq c_\omega |v|$.

Obviously, Lemma 6 remains valid, replacing S^1 with $\imath\mathbb{R}$.

The next theorem is our second main result, establishing the KYP lemma for non-stabilizable plants (7).

Theorem 10. Let B and F be bounded, $F_{uu} \succ 0$ and Assumption 9 holds. Assume also there exist $k > 0$ such that $(kI - A)^{-1}$ and $(kI + A)^{-1}$ are bounded. Then two conditions are equivalent:

(c) there exists $\varepsilon_0 > 0$ such that for any $\omega \in \mathbb{R}$ and any pair $x \in D(A), u \in U$ with $\imath\omega x = Ax + Bu$ one has

$$F(x, u) \geq \varepsilon_0(|x|^2 + |u|^2). \qquad (9)$$

(c) condition (8) holds for some bounded $H = H^*$;

Proof The implication (d)⇒(c) is straightforward and requires no additional assumptions. The proof of the inverse implication employs the idea of *Cayley transform*, proposed in Yakubovich [1974] and reducing the continuous-time system (7) to the discrete-time system (2) with bounded operators. By rescaling the operators A,B if necessary, one may assume without loss of generality that $k = 1$ and hence bounded operators $\mathcal{C} := (A - I)^{-1}$ and $(A + I)^{-1}$ exists.

Define now $\mathcal{A} := (A + I)(A - I)^{-1}$, $\mathcal{B} = -\sqrt{2}(A - I)^{-1}B$ and let $z = \frac{1}{\sqrt{2}}[(A - I)x + Bu]$ (where $x \in D(A)$), so that $x = \sqrt{2}\mathcal{C}(z - Bu)$. We are going to apply Theorem 7 to the Hermitian form $\mathcal{F}(y, u) = F(\sqrt{2}(y - \mathcal{C}Bu, u)$, where $y = \mathcal{C}z$, and the discrete-time system

$$z(t + 1) = \mathcal{A}z(t) + \mathcal{B}u(t), \quad y(t) = \mathcal{C}z(t). \qquad (10)$$

To verify Assumption 5 and condition (a'), notice that the relation $\lambda z = \mathcal{A}z + \mathcal{B}u + v$, making some computations, shapes into

$$(\lambda + 1)x = (\lambda - 1)(Ax + Bu) + v.$$

When $\lambda \in S^1\{1\}$, the latter relation implies that $\imath\omega x = Ax + Bu + w$, where $w = (\lambda - 1)^{-1}v$ and ω varies over $\mathbb{R}$. For $\lambda = 1$, one has $x = v/2$ and u arbitrary. This immediately implies Assumption 5 for the system (10) and (taking $v = 0$ and using that $F_{uu} \succ 0$) condition (a'). By assumption, we have $\mathcal{A}^{-1} = (A - I)(A + I)^{-1}$, which is a bounded operator. Applying Theorem 8, we prove that (c)⇒(d) □

Examining the proof, one may easily show that the result of Theorem 10 remains also valid, replacing (c) with condition (c'):

(c) there exists $\varepsilon_0 > 0$ such that for any $\omega \in \mathbb{R}$ and any pair $x \in D(A), u \in U$ with $\imath\omega x = Ax + Bu$ one has

$$F_0(Cx, u) \geq \varepsilon_0(|Cx|^2 + |u|^2), \qquad (11)$$

[1] the Hermitian form in (8) is defined for $\forall x \in D(A), u \in U$, so $\succ$ stands for the positive definiteness in the incomplete space $D(A) \times U$.

where C is some bounded operator; in this case (8) holds for $F(x, u) = F_0(Cx, u)$.

6. CONCLUSION

In this paper, we extend the "coercive" Kalman-Yakubovich-Popov lemma, establishing the solvability of operator inequalities under the *strict* frequency-domain condition. Unlike the result, published in the literature, we do not assume the plant to be stabilizable. This not only simplifies the use of the KYP lemma when the direct verification of the stabilizability may be troublesome, but opens up a wide perspective of proving dichotomy, instability and other conditions for Lurie systems, where the stabilizability of linear part may fail. Such applications, and also extensions of the KYP lemma in the non-coercive case are subject of the ongoing research.

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