Available online at www.sciencedirect.com

ScienceDirect

IFAC-PapersOnLine 48-11 (2015) 557–562

Universal controllers of V.A. Yakubovich: a systematic approach to LQR problems with uncertain external signals ⋆

Anton V. Proskurnikov *,**

* St.-Petersburg State University, Institute for Problems of Mechanical Engineering RAS, and ITMO University, Russia
** ENTEG Institute at the University of Groningen, The Netherlands
E-mail: avp1982@gmail.com

Abstract: Any practical control system is affected by external signals. Some of them are "informative" (e.g. reference signals), and the others (disturbances, measurement noises etc.) are undesired as they can visibly deteriorate the system behavior. General approaches to disturbance rejection and reference tracking have been elaborated in the framework of geometric control theory, among them are disturbance decoupling control and internal model principles. These approaches, however, assume the state and control of the system to be unconstrained. To take such constraints into account, one usually has to consider optimization problems where the performance index penalizes in some way the system process. Optimization problems in presence of uncertain signals have been addressed in the context of robust control and stochastic control. Standard methods like usually provide only *suboptimality* of the process; the optimal value can be found for either "worst-case" signal (like in minimax H^∞- and L^1-optimization approaches) or "on average", assuming the external signals to be stochastic with known spectral density. In a series of his papers published in 1992-2012, Vladimir A. Yakubovich promoted the approach of *universal controllers* for linear-quadratic regulation (LQR) problems under uncertain signals. The term "universal" emphasizes that the controller renders the solution of the closed-loop system optimal *for any* external signal. Although this is not possible for arbitrary signals, in some special classes of signals the universal controllers not only exist but may even be chosen *linear*. The existence of linear optimal controllers has been proved for two important classes of uncertain signals, that is, polyharmonic signals with known spectrum and signals with fast decreasing spectral density. In this paper we extend the results of V.A. Yakubovich to more general classes of systems and quadratic performance indices, arising in problems of optimal oscillation damping, reference tracking and model matching.

Keywords: LQR problem, universal controller, disturbance rejection, tracking, model matching

1. INTRODUCTION

The possibility to cope with uncertainty is a key benefit of the feedback control over the open-loop control techniques. Even measuring the full system state, one often is not able to predict its trajectory due to the instability, which makes open-loop control inapplicable. Instead, a feedback loop often may be designed, delivering the desired process (uncertain unless the initial conditions are exactly known) without its direct calculation. A commonly known example is the LQR problem where the optimal process is fetched by a simple linear controller, which does not depend on the initial conditions, but only on the coefficients of the plant and the quadratic performance index.

The initial condition, however, is only one source of uncertainty in the plant dynamics. In the present paper, we will be interested in control under uncertain *exogenous signals*, which may be both "undesirable" (like disturbances and measurement noises) and "informative" (references signals to trace). A natural question arises if a *universal* controller exists, meeting the control goal under any uncertain signals from a given class. For many practically important problem the answer to this question is known to be positive and the corresponding universal controller is rather simple.

Consider, for instance a linear plant, obeying the equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t) \\ y(t) &= C_y x(t) + D_y d(t) \\ z(t) &= C_z x(t) + D_z d(t). \end{aligned} \tag{1}$$

Here $x(t), u(t), y(t), z(t)$ stand respectively for the state, control, measured and controlled outputs; $d(t)$ is some exogenous signal, whose different components may include disturbances, noises, reference signals etc.

A typical problem of universal controller synthesis is the *disturbance decoupling problem*: to design a control law $y(\cdot) \mapsto u(\cdot)$ which makes $z(\cdot)$ independent of $d(\cdot)$, e.g. $z(t) \to 0$ as $t \to \infty$ for any $d(\cdot)$ (Wonham [1974],

⋆ The paper was partially supported by RFBR, grants 13-08-01014 and 14-08-01015, and St. Petersburg State University, grant 6.38.230.2015. Theorems 6 and 13 are obtained at Institute for Problems of Mechanical Engineering RAS and supported solely by Russian Scientific Foundation (RSF), grant 14-29-00142.

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2015.09.245

Morse and Wonham [1970], Stoorvogel and van der Woude [1991], Weiland and Willems [1989]). In Russian literature, this problem has been studied since the pioneering work Schipanov [1939] under the name of *(absolute) invariance* problem (Slita and Ushakov [2008], Solnechnyi [2014], Proskurnikov and Yakubovich [2003b], Yakubovich et al. [2011]). Depending on the algebraic properties of the plant, the decoupling problem may require the *feedforward* control scheme with measurement of the full signal $d(t)$ Wonham [1974], Proskurnikov and Yakubovich [2003b, 2004, 2006a]; in this case, its approximate solutions (that does not use $d(t)$) may be found by using high-gain techniques (Proskurnikov and Yakubovich [2003a], Weiland and Willems [1989]). A more general setup is to provide the *tracking* of some components of $d(t)$: $z(t)-Kd(t)\to 0$ or, more generally, *reference model* matching Morse [1973], Proskurnikov and Yakubovich [2007].

Notice that the aforementioned approaches result provide the stability of the closed-loop system, but assume the state and control variable to be unconstrained. In the face of limited control resources one has to consider optimization problems, where the existence of universal controllers is a non-trivial problem. Note the crucial difference between optimal universal controllers (OUCs) and classical approaches from robust control and stochastic control: the OUR provides optimality of the solution for *any* external signal from a given class, whereas classical control methods provide it either "on average" (as in stochastic control) or for the "worst-case" signal only (as done by H^∞-optimal controllers). Although existence of OUCs in an optimization problem seems to be a sort of "exception", V.A. Yakubovich proved it for LQR problems, where uncertain exogenous signal is either polyharmonic with known spectrum or has a fast decreasing spectral density.

In the present paper, we extend the results of V.A. Yakubovich to general plant (1) and quadratic performance index, depending on $[x(\cdot),u(\cdot),d(\cdot)]$. Our result encompasses all classes of LQR problems, addressed in the previous papers by V.A. Yakubovich and the author, namely, oscillation damping (Yakubovich [1997], Lindquist and Yakubovich [1997]), optimal tracking (Lindquist and Yakubovich [1999], Proskurnikov and Yakubovich [2006b,c, 2008]) and optimal model-matching problems (Proskurnikov and Yakubovich [2011, 2012]). Unlike the mentioned papers, we address a situation where the measurements, available to the controller, are restricted by the linear output $y(\cdot)$, including only some (possibly, none) components of the exogenous signal. As will be discussed, the same results can also be obtained for LTI plants with time delays and other infinite-dimensional systems. Below we confine ourselves to continuous-time dynamics; the discrete-time case may be considered analogously.

2. LQR PROBLEM UNDER POLYHARMONIC EXTERNAL SIGNALS

Throughout the section, we consider linear plant (1), where $x(t)\in\mathbb{R}^n, u(t)\in\mathbb{R}^m, d(t)\in\mathbb{R}^l, y(t)\in\mathbb{R}^k, z(t)\in\mathbb{R}^p$ stand for, respectively, state, control, exogenous signal, observed and controlled outputs. Including, if necessary, and observer-based stabilizing controller into the block (1), one may assume [1] that the system is stable, i.e. the matrix A is *Hurwitz*: $\det(\lambda I-A)\neq 0$ as $Re\,\lambda\geq 0$.

We assume that the signal $d(t)$ is polyharmonic with known spectrum $\omega_1,\ldots,\omega_N$, yet unknown amplitudes:

$$d(t)=\sum_{j=1}^N d_j e^{\imath\omega_j t}. \qquad (2)$$

In other words, $d(t)$ is an output of some *known* exogenous autonomous block with simple imaginary eigenvalues, whose initial conditions are uncertain. Such condition on the exogenous signal is the standard prerequisite in the *servomechanism problem* Davison and Goldenberg [1975], Davison and Copeland [1985] which is similar to tracking and disturbance decoupling problems, requiring to find a controller $y(\cdot)\mapsto u(\cdot)$ which provides the condition $z(t)-Kd(t)\to 0$ as $t\to\infty$ for all signals from the class just described. This problem gave rise to the celebrated paradigms of the *internal model control* (Francis and Wonham [1976]) and the *repetitive control* (Hara et al. [1988]). Although the mentioned approaches are widely used in control system design, they all have a common drawback, applying no restrictions on the control signal. Whereas the output oscillations are damped, the other variables of the system oscillate; a natural question thus arises how to optimize the "energy" of these oscillations.

To formalize the problem setup, we introduce the following quadratic performance index:

$$J[x(\cdot),u(\cdot),d(\cdot)]=\overline{\lim_{T\to\infty}}\,\frac{1}{T}\int_0^T F[x(t),u(t),d(t)]\,dt. \qquad (3)$$

Here F is a quadratic form, usually non-negative definite. Treating $F[x(t),u(t),d(t)]$ as "energy" of the process, its average value may be thought of as the average "power" of oscillations. The typical examples of the form F are:

(1) $F(x,u,d)=F_0(x,u)\geq 0$ corresponds to the *oscillation damping* Lindquist and Yakubovich [1997];
(2) $F(x,u,d)=F_0(x,u)+|R^{1/2}(z-Kd)|^2$, $z:=C_zx+D_zd$ corresponds to the optimal *tracking* Lindquist and Yakubovich [1999];
(3) $F(x,u,d)=F_0(x,u)+|R^{1/2}(z-Kd-Lx)|^2$ corresponds to the general model-matching problem [2].

In the examples (1)-(3) F_0 is a quadratic form

$$F_0(x,u)=x^*F_{xx}x+2x^*F_{xu}u+u^*F_{uu}u,$$

penalizing the energy of oscillations in the control and state, additionally, in the examples 2 and 3 the cost function also penalize the reference tracking error in $z(t)$.

The problem in question is to optimize the performance index J in the class of all *bounded* processes $[x(\cdot),u(\cdot)]$:

$$J\to\inf \text{ subject to (1) and } \sup_{t\geq 0}(|x(t)|+|u(t)|)<\infty. \qquad (4)$$

[1] Measuring the output $y(\cdot)$, the system can be stabilized under the condition that the triple (A,B,C_y) is stabilizable and detectable. Upon its failing, the plant has unstable dynamics, having no effect on the output $y(\cdot)$ and thus not suppressible by any control strategy, based on output measurements. Thus no controller is able to guarantee that the solutions of the closed-loop system are bounded.

[2] The model matching problem requires to penalize the error $z(t)-z_m(t)$, where $z_m(t)$ is an output of some LTI *model*, feeding $d(t)$ as an input; including the model's state into the plant, one may assume that $z_m(t)=Kd(t)+Lx(t)$

Henceforth, we assume that the form F, given by

$$F(x,u,d) = \begin{bmatrix} x \\ u \\ d \end{bmatrix}^* \mathcal{F} \begin{bmatrix} x \\ u \\ d \end{bmatrix} = F_0(x,u) + \\ +2d^* F_{dx} x + 2d^* F_{du} u, \quad (5)$$

satisfies the following frequency-domain inequality

$$\tilde{u}^* \Pi(\imath\omega) \tilde{u} = \tilde{F}_0[(\imath\omega I - A)^{-1} B\tilde{u}, \tilde{u}] \geq \varepsilon |\tilde{u}|^2 \quad \forall \omega \in \mathbb{R}. \quad (6)$$

Here $\varepsilon > 0$ is independent of $\tilde{u} \in \mathbb{C}^m$ and $\tilde{F}_0$ stands for the Hermitian extension of F to the complex domain.

Condition (6) always holds if $F_0(x,u)$ is positively definite, which condition holds for many LQR problems. This condition cannot be fully discarded, furthermore, as discussed in Lindquist and Yakubovich [1999],Yakubovich [1997], if $\tilde{u}\Pi(i\omega_0)\tilde{u} < 0$ for some $\omega_0 \in \mathbb{R}$ and $\tilde{u} \in \mathbb{C}^m$, the problem (4) is ill-posed and $\inf J = -\infty$.

It may seem that our extremal problem is just a slight modification of the classic infinite-horizon LQR problem and may be solved via the Kalman-Yakubovich-Popov lemma and the Lur'e-Riccati equations Anderson and Moore [1989], Matveev [2006]. Thanks to the KYP lemma Matveev [2006], (6) implies that the *Lur'e equations*

$$2x^* H(Ax + Bu) + F_0(x,u) = |F_{uu}(u - hx)|^2 \quad \forall x, u \quad (7)$$

are solvable in $H = H^* \in \mathbb{R}^{n\times n}$, $h \in \mathbb{R}^{m\times n}$, moreover, $A+Bh$ is a Hurwitz matrix. After some manipulations, the following result may be proved (Lindquist and Yakubovich [1997, 1999], Proskurnikov and Yakubovich [2012]).

Lemma 1. Let (6) holds and h be a solution of (7). Then for any bounded solution $[x(\cdot), u(\cdot)]$ one has

$$J = J_{min} + \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T |F_{uu}(u(t) - hx(t) - \eta(t))|^2 dt, \quad (8)$$

where J_{min} is the optimal value, depending only on $d(\cdot)$ and $\eta(t) = \sum_{j=1}^N \eta_j e^{\imath\omega_j t}$, where $\eta_j = L_j d_j$ and L_j may be explicitly found. The solution $[x,u]$ is optimal if and only if $u(t) = hx(t) + \eta(t) + \varepsilon(t)$, where $T^{-1}\int_0^T |\varepsilon(t)|^2 dt \to 0$.

It may seem that Lemma 1 gives a solution of our problem and the controller $u(t) = hx(t) + \eta(t)$ is optimal. However, this solution is unsatisfactory for several reasons. First of all, the obtaining controller appears to be *non-robust* against small errors in the spectrum: replacing ω_j with $\omega_j' \approx \omega_j$ results in the jump of the cost functional Yakubovich [1995], Lindquist and Yakubovich [1999]. Secondly, it requires the identification procedure for the amplitudes d_j even if $d(t)$ is directly measured, since the linear filter $d(\cdot) \mapsto \eta(\cdot)$ appears to be anti-causal.

In the next section we show that the aforementioned flaws may be overcome and, moreover, an *optimal universal* controller exists under some technical assumptions which does not depend on $d(\cdot)$, however, provides an optimal process for any signal $d(t)$ with known spectrum (2). Unlike the controller from Lemma 1, it may be found without solving the Lur'e equations.

3. DESIGN OF OPTIMAL UNIVERSAL CONTROLLER

The result of Lemma 1, though it gives an inappropriate solution of the LQR problem, has the following important corollary. We say two functions $\xi_1(t), \xi_2(t) \in \mathbb{R}^s$ are *equivalent* if $\frac{1}{T}\int_0^T |\xi_1(t) - \xi_2(t)|^2 dt \to 0$ as $T \to \infty$.

Corollary 2. The solution $\xi(t) := [x(t)^\top, u(t)^\top, d(t)^\top]^\top$ is optimal if and only if it is equivalent to the function $\xi_*(t) = \sum_{j=1}^N \xi_j e^{\imath\omega_j t}$, where ξ_j linearly depend on d_j.

The proof is obvious from Lemma 1 since $A + Bh$ is a Hurwitz matrix and $\eta(t) = \sum_{j=1}^N L_j d_j e^{\imath\omega_j t}$. Corollary 2 suggests the way to find the optimal process without solving the Lur'e equation (7). Let $\xi_j = [x_j^\top, u_j^\top, d_j^\top]^\top$. Being equivalent to some solution of (1), $\xi_*(t)$ is also a solution and hence that $\imath\omega_j x_j = Ax_j + Bu_j + Ed_j$. Thus

$$\xi_j = P_j u_j + Q_j d_j,\ P_j = \begin{bmatrix} A_{\imath\omega_j}^{-1} B \\ I_m \\ 0 \end{bmatrix},\ Q_j = \begin{bmatrix} A_{\imath\omega_j}^{-1} E \\ 0 \\ I_l \end{bmatrix}, \quad (9)$$

where $A_{\imath\omega} := \imath\omega I_n - A$. A straightforward computation shows that $F(\xi_j) = u_j^* \Pi(\imath\omega_j) u_j + 2\,Re\, u_j^* P_j \mathcal{F} Q_j d_j$, where $\Pi(\imath\omega) = \Pi(\imath\omega)^* > 0$ is the matrix from (6), and

$$J[\xi_*(\cdot)] = \sum_{j=1}^N F(\xi_j) = \sum_{j=1}^N |\Pi(\imath\omega_j)^{1/2}(u_j - R_j d_j)|^2 + J_0, \quad (10)$$

where $J_0 = J_0(d_j, R_j)$ and matrices R_j are defined by

$$R_j = -\Pi(\imath\omega_j)^{-1} P_j \mathcal{F} Q_j. \quad (11)$$

As a result, we get the following lemma.

Lemma 3. Let $\xi_*(t) = \sum_{j=1}^N \xi_j e^{\imath\omega_j t}$, where ξ_j are given by (23) with R_j is given by (11) and $u_j = R_j d_j$. Then any solution $\xi(\cdot)$, equivalent to $\xi_*(\cdot)$, is optimal, and vice versa.

Lemma 3 suggests to find optimal universal controller in the class of *stabilizing* linear controller. The controller

$$N\left(\frac{d}{dt}\right) u(t) = M_x\left(\frac{d}{dt}\right) x(t) + S\left(\frac{d}{dt}\right) d(t), \quad (12)$$

where M, N, S are matrix polynomials and $\det N \not\equiv 0$, is said to *stabilize* the system (1) if

$$\det \begin{bmatrix} sI_n - A & -B \\ -M_x(s) & N(s) \end{bmatrix} \neq 0 \quad \forall s : Re\, s \geq 0;$$

A special case of a controller (12), accessing only output $y(t)$ instead of the full information $[x(t), d(t)]$, is

$$N\left(\frac{d}{dt}\right) u(t) = M\left(\frac{d}{dt}\right) y(t). \quad (13)$$

Controller (13) corresponds to (12) with $M_x(s) = M(s)C_y$ and $S(s) = M(s)D_y$. Lemma 3 implies that any controller (13), providing that the transfer matrix of the closed-loop system $W_{ud}(s)$ satisfies the *interpolation constraints*:

$$W_{ud}(\imath\omega_j) = R_j \quad \forall j = 1, 2, \ldots, N, \quad (14)$$

giving thus rise to the following problem.

Problem 4. Find all stabilizing controllers (13), such that the transfer matrix satisfies interpolation conditions (14).

To solve this problem, we use the following parametrization of stabilizing controllers (12). This parametrization, similar to that from Youla et al. [1976], was proposed in Yakubovich [1995], Proskurnikov and Yakubovich [2004].

Lemma 5. Let A be a Hurwitz matrix. Any controller (12) with the matrix polynomials $M_x(s)$ and $N(s)$ defined by

$$M_x(s) = \delta(s) r(s), N(s) = M_x(s) A_s^{-1} B + \rho(s) I_m, \quad (15)$$

where $A_s := sI - A$, $\delta(s) := \det A_s$, $\varrho(s)$ is a scalar Hurwitz polynomial, $r(s)$ is a matrix polynomial and $\det N \not\equiv 0$, stabilizes (1). For controller (15) one has

$$W_{ud}(s) = [S(s) + M_x(s)A_s^{-1}E]/\rho(s).$$

Any other controller $\tilde{N}u = \tilde{M}_x y + \tilde{S}d$ is equivalent to (15), that is, Hurwitz matrix polynomials $H, \tilde{H}$ exists such that

$$H^{-1}[N, M_x, S] = \tilde{H}^{-1}[\tilde{N}, \tilde{M}_x, \tilde{S}].$$

It can be easily noticed that equivalent controllers are both stabilizing or not and transfer matrices under such controllers coincide. Returning to controllers (13), Lemma 5 entails that any such controller with

$$M(s) = \delta(s)r_0(s), N(s) = M(s)C_y A_s^{-1}B + \rho(s)I_m \quad (16)$$

is stabilizing, provided that r_0 is a matrix polynomial and ρ is a scalar Hurwitz polynomial. Furthermore, there are no other stabilizing controllers (13) up to the equivalence.

Controller (16) provides that $W_{ud}(s) = M(s)[C_y A_s^{-1}E + D_y]/\rho(s)$, and hence constraints (14) shape into

$$\delta(\imath\omega_j)r_0(\imath\omega_j)[D_y + C_y A_{\imath\omega_j}^{-1}E] = \rho(\imath\omega_j)R_j. \quad (17)$$

It is obvious that this equations can be solved, given that

$$rk[D_y + C_y A_{\imath\omega_j}^{-1}E] = l = \dim d(t) \quad \forall j = 1, \ldots, N. \quad (18)$$

This yields in the following criterion of OUC existence.

Theorem 6. An optimal universal controller (13) exists, provided that the rank condition (18) holds. Such a controller may be found in the form (16), where a matrix polynomial r_0 and a scalar Hurwitz polynomial ρ satisfy (17). Any other universal controller (13) is equivalent to one of the just described controllers for appropriate r_0, ρ.

Remark 7. It may be easily shown that the optimal universal controller (13), if exists, is *robust* to small deviations in the spectrum $\omega_j' \approx \omega_j$. Indeed, since $A_{\imath\omega}^{-1}$ continuously depends on $\omega \in \mathbb{R}$, it is easily shown that $W_{du}(\imath\omega_j') \approx R_j$ and hence $J \approx J_0$ due to (10).

Remark 8. It can be easily seen that one is able to get r_0, ρ in a way that $\deg \rho > \deg \delta + \deg r_0$. In this case the closed-loop system is well-posed in the sense that the controller's transfer matrix $N^{-1}M$, as well as the closed-loop transfer matrices from d to x, u, are stable and proper.

Finally, one can compare the interpolation conditions (14) with those arising from the internal maximal principle. Suppose that d is a disturbance to be decoupled from the output z, so that $W_{zd}(\imath\omega_j) = 0 \, \forall j$, where $W_{zd}(s) = C_z A_s^{-1}BW_{ud}(s) + [D_z + C_z A_s^{-1}E]$ is a transfer function from d to z. In other words, (14) are replaced with

$$C_z A_{\imath\omega_j}^{-1}BW_{ud}(\imath\omega_j) = -(D_z + C_z A_{\imath\omega_j}^{-1}E), \; \forall j. \quad (19)$$

Along with (18), the solvability of (19) usually requires that $rk(C_z A_{\imath\omega_j}^{-1}B) = \dim z$, i.e. $\dim u \leq \dim z$, which is a much more restrictive condition. Optimal universal controller can exist even in situation of *underactuated* plant, which is not the case for usual internal model controller.

4. EXTENSIONS: LQR PROBLEMS FOR TIME-DELAY SYSTEMS

A natural question arises if the results obtained in Section III may be extended to infinite-dimensional systems, for instance, those obeying delay equations.

It appears that Lemma 1 and Corollary 2 remain valid for systems in Hilbert spaces. Namely, assume that x, u, d are no longer vectors but elements of the corresponding Hilbert spaces $\mathcal{X} = \{x\}$, $\mathcal{U} = \{u\}$ and $\mathcal{D} = \{d\}$. Let B, E stand for the bounded operators, and A is the infinitesimal generator of a stable C_0-semigroup $S(t)$, that is assumed to be *exponentially decaying* as $t \to \infty$. By definition, $x(t)$ is a solution of (1) with initial condition $x(0) = x_0$ if

$$x(t) = S(t)x_0 + \int_0^t S(t - \varkappa)[Bu(\varkappa) + Ed(\varkappa)]d\varkappa.$$

Assuming that $\mathcal{F} = \mathcal{F}^*$ from (5) to be bounded operator on $\mathcal{X} \times \mathcal{U} \times \mathcal{D}$, Lemma 1 and Corollary 2 retain their validity due to the KYP lemma for one-parameter semigroups (Likhtarnikov and Yakubovich [1977, 1998]), providing the Lurie-Riccati equations (7) are solvable in the class of bounded operators $H = H^*$ and h. The details can be found in Proskurnikov and Yakubovich [2006b] and here are omitted due to space limitations.

Lemma 3, giving the formulas for optimal process, remains valid in Hilbert spaces as well, giving a sufficient condition for the existence of linear OUC. This condition is however not constructive since parametrization of stabilizing controllers for infinite-dimensional systems is not available. Moreover, even the existence of one such controller (which is a prerequisite to assume that the block (1) is stable itself) is a self-standing problem; in general, stabilizing controller for a distributed parameter system is infinite-dimensional as well, so its practical implementation is highly non-trivial.

However, for the class of stationary *delay* systems, e.g. those obeying the following equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau) + Bu(t) + Ed(t) \\ y(t) &= C_y x(t) + D_y d(t) \\ z(t) &= C_z x(t) + D_z d(t), \end{aligned} \quad (20)$$

the stabilizing controllers may be also be found in the class of time-delay systems (with the same delay τ). In this case, the results of Section III, including Lemma 5 and the main Theorem 6, remain valid, defining everywhere A_s by $A_s := sI - A - A_1 e^{-\tau s}$ and replacing the word "polynomial" with "quasi-polynomial", that is function $f(s) = p(s) + q(s)e^{-s\tau}$, where p, q are polynomials. As for usual polynomials, f is called Hurwitz if $\det f(s) \neq 0$ as $Re\, s \geq 0$. Analogous extension is possible in the case of multiple delays Proskurnikov and Yakubovich [2006b], in which case the class of quasi-polynomials should also be extended to contain functions with multiple exponentials.

5. LQR PROBLEM FOR SPECIAL CLASS OF STOCHASTIC EXOGENOUS SIGNALS

A crucial restriction concerning the spectrum of the signal $d(t)$ is its finiteness, although formulas for the optimal process from Lemma 3 remain valid e.g. for $d(t)$ being *almost periodic* (Bohr [1956]). For the infinite spectrum, one has to satisfy infinitely many interpolation conditions $W_{ud}(\imath\omega_j) = R_j$ in order to design a linear optimal universal controller, which in general is hardly possible with a practically implementable finite-dimensional controller. It is obvious, however, that if the signal has a few dominating harmonics, the amplitudes of the others being small, one can construct an optimal controller for these dominating frequencies; since the influence of the remaining harmonics is weak, such a controller gives a suboptimal solution.

Promoting and extending this idea, V.A. Yakubovich proved the existence of suboptimal controllers (the value of the cost functional may be arbitrarily close to the optimal one) for some class of stochastic signals with continuous spectrum, whose spectral density is fast decreasing Yakubovich [1995, 1997], Proskurnikov and Yakubovich [2008, 2012]. Although such signals are formally expressed by stochastic integrals, they are "almost" deterministic: their dynamics in the past allow to predict the future ones with arbitrary precision (Gikhman and Skorokhod [2004]).

Consider a stochastic signal

$$d(t) = \int_{-\infty}^{+\infty} e^{\imath\omega t}\hat{d}(\omega)d\nu_\omega, \quad \hat{d}(\omega)\hat{d}(\omega)^* \le \sigma(\omega)I_l. \tag{21}$$

Here ν_ω stands for the stochastic process with uncorrelated increments, $\mathbb{E}d\nu_\omega = 0$ and $\mathbb{E}d\nu_{\omega_1}d\nu_{\omega_2} = \delta(\omega_1 - \omega_2)I_l d\omega_1 d\omega_2$; the spectral density $\hat{d}(\imath\omega)$ is generally uncertain, the only available information about it is its upper bound $\sigma(\omega)$, assumed to be *fast decreasing* as follows:

$$\int_{-\infty}^{+\infty} \frac{\ln\sigma(\omega)}{1+\omega^2}d\omega = -\infty, \int_{-\infty}^{+\infty} (1+|\omega|^M)\sigma(\omega)d\omega < +\infty \,\forall M \ge 0. \tag{22}$$

Solutions of (1) and other the differential equations with stochastic processes (21) may be understood, for instance, in the sense of *generalized stochastic processes* theory Gelfand [1955]. For instance, the process $\xi(t) := [x(t)^\top, u(t)^\top, d(t)^\top]^\top$, given by integral

$$\xi(t) = \int_{-\infty}^{+\infty} e^{\imath\omega t}\hat{\xi}(\omega)d\nu_\omega, \tag{23}$$

satisfies (1) if $\hat{\xi}(\omega) = [\hat{x}(\omega)^\top, \hat{u}(\omega)^\top, \hat{d}(\omega)^\top]^\top$, where

$$\hat{\xi}(\omega) = P_\omega\hat{u}(\omega) + Q_\omega\hat{d}(\omega),$$
$$P_\omega = \begin{bmatrix} A_{\imath\omega}^{-1}B \\ I_m \\ 0 \end{bmatrix}, Q_\omega = \begin{bmatrix} A_{\imath\omega}^{-1}E \\ 0 \\ I_l \end{bmatrix}. \tag{24}$$

For the quadratic performance index analogous to (3)

$$I[x(\cdot), u(\cdot), d(\cdot)] = \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\int_0^T F[x(t), u(t), d(t)]\,dt \tag{25}$$

we consider the following optimization problem:

$$I[x(\cdot), u(\cdot), d(\cdot)] \to \inf, \quad \sup_{T\ge 0} \mathbb{E}(|x(T)|^2 + |u(T)|^2) < \infty. \tag{26}$$

The following lemma is an analogue of Lemma 1.

Lemma 9. Let (6) holds and h be a solution of (7). Then for any bounded (in the sense of (26)) solution one has

$$I = I_{min} + \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\int_0^T |F_{uu}(u - hx - \eta(t))|^2 dt, \tag{27}$$

where J_{min} is the optimal value, depending only on $\hat{d}(\cdot)$ and $\eta(t) = \int_{-\infty}^{+\infty} e^{\imath\omega t}\hat{\eta}(\omega)d\nu_\omega$, where $\hat{\eta}(\omega) = L(\imath\omega)\hat{d}(\omega)$ and $L(\imath\omega)$ is a known bounded function. The solution is optimal if and only if $u(t) = hx(t) + \eta(t) + \varepsilon(t)$, where $T^{-1}\mathbb{E}\int_0^T |\varepsilon(t)|^2 dt \to 0$.

The "controller" $u = hx + \eta$ obtained from Lemma 9 is unsatisfactory, like the feedback in Lemma 9. Even when d is directly measured, the filter $d(\cdot) \mapsto \eta(\cdot)$ with transfer matrix $L(\imath\omega)$ cannot be practically implemented, being anticipative (one may show that $L(s)$ is unstable rational matrix). So one has either to predict the future values of $d(t)$ or identify its spectral function $\hat{d}(\imath\omega)$. Both procedures are quite complicated and the robustness of the resulting solution against noises and estimation errors is an open question. Instead of using this straightforward solution, one can retrace the arguments from Section III.

We say two stochastic functions $\xi_1(t), \xi_2(t) \in \mathbb{R}^s$ are *equivalent* if $\frac{1}{T}\mathbb{E}\int_0^T |\xi_1(t) - \xi_2(t)|^2 dt \to 0$ as $T \to \infty$.

Lemma 10. The solution $\xi(t) := [x(t)^\top, u(t)^\top, d(t)^\top]^\top$ is optimal if and only if it is equivalent to the function $\xi_*(t) = \int_{-\infty}^{+\infty} e^{\imath\omega t}\hat{\xi}_*(\omega)d\nu_\omega$, where $\hat{\xi}_*(\omega)$ satisfies (24), and the respective $\hat{u}_*(\omega)$ may be found from

$$\hat{u}_*(\omega) = R(\imath\omega)\hat{d}(\omega), \quad R(\imath\omega) = -\Pi(\imath\omega)^{-1}P_\omega\mathcal{F}Q_\omega.$$

Moreover, on any other stochastic process in the form $\xi(t) = \int_{-\infty}^{+\infty} e^{\imath\omega t}\hat{\xi}(\omega)d\nu_\omega$, satisfying (24), one has

$$I[\xi(\cdot)] = I_{\min} + \int_{-\infty}^{\infty} |\Pi(\imath\omega)^{1/2}(\hat{u}(\omega) - \hat{u}_*(\omega))|^2 d\omega. \tag{28}$$

The important difference between the stochastic and polyharmonic case is that one is not able to design a controller (13), providing that $W_{ud}(\imath\omega) = R(\imath\omega)$, since the latter rational matrix is typically not stable, and thus such a controller, if existed, would be anticipating. However, it is possible to make the second term in (28) as small as possible, as shown in Yakubovich [1997],Proskurnikov and Yakubovich [2006c]. Retracing the arguments from [Yakubovich 1997, Section 3], inequalities (28) and (21) entail that for some $c > 0$ the inequality holds

$$I[\xi(\cdot)] \le I_{\min} + c\int_{-\infty}^{\infty} |W_{ud}(\imath\omega) - R(\imath\omega)|^2\sigma(\omega)d\omega \tag{29}$$

whenever $\xi(t)$ is generated by a stabilizing controller (13) (by definition, we put $|W|^2 := Tr\,WW^*$). The second term may be rendered arbitrarily small under some technical condition, as follows from the following approximation lemma, extending results by M.G. Krein, Kolmogorov and Szegö on weighted approximation by polynomials on $\mathbb{R}$.

Lemma 11. Yakubovich [1997] Let $\sigma(\omega)$ be fast decreasing (22). For any function $X_0(\omega)$, polynomially growing as $\omega \to \pm\infty$ and $\varepsilon > 0$, there exists a *stable proper* rational function $X(\lambda)$ such that $\int_{-\infty}^{\infty} |X(\imath\omega) - X_0(\omega)|^2\sigma(\omega)d\omega < \varepsilon$.

Remark 12. Assume that $Z(\lambda)$ is such a stable proper rational matrix that $\det Z(\lambda)^*Z(\lambda) \not\equiv 0$. Then $X(\lambda)$ can be chosen as $X_1(\lambda)Z(\lambda)$, where X_1 is also stable proper. Indeed, let $Z(\imath\omega)Z(\imath\omega)^* \le \psi I$, then $|X_1M - X_0|^2 \le \psi|X_1 - X_0(Z^*Z)^{-1}Z^*|^2$. Since $X_0(Z^*Z)^{-1}Z^*$ also grows polynomially, the claim follows from Lemma 11.

We are now ready to formulate sufficient condition for existence of a unversal *suboptimal* controller (13), ensuring that $I \le I_{\min} + \varepsilon$ for *any* spectral density $\hat{d}$, satisfying (21).

Theorem 13. Let the following polynomial matrix

$$Z(s) = D_y + C_yA_s^{-1}E \tag{30}$$

be of the full rank, that is, $\det Z(\imath\omega)^* Z(\imath\omega) \not\equiv 0$. Then for any $\varepsilon > 0$ there exist universal controller (13) that renders the solution ε-suboptimal *for any* signal (21).

Proof. As follows from Lemma 5, we can confine ourselves to the controllers (16), which provide the transfer matrix $W_{ud}(s) = \rho(s)^{-1}\delta(s)r(s)Z(s)$. Accordingly to Remark 12, one can choose $X(s) = \rho(s)^{-1}\delta(s)r(s)$ in the class of stable proper rational matrices such that $\int_{-\infty}^{\infty} |X(\imath\omega)Z(\imath\omega) - R(\imath\omega)|^2\sigma(\omega)d\omega < \varepsilon$, given such a matrix, one can choose a Hurwitz polynomial $\rho(s) = \rho_0(s)\delta(s)$ in a way that $r(s) = \rho_0(s)X(s)$ is a matrix polynomial, and hence $I \leq I_{min} + c\varepsilon$ due to (29). □

REFERENCES

Anderson, B. and Moore, J. (1989). *Optimal Control: Linear Quadratic Methods.* Prentice-Hall, London.

Bohr, H. (1956). *Almost Periodic Functions.* Chelsea, N.Y.

Davison, E. and Copeland, B. (1985). Gain margin and time lag tolerance constraints applied to the stabilization problem and robust servomechanism problem. *IEEE Trans. Automat. Contr.*, AC-30, 229–239.

Davison, E. and Goldenberg, A. (1975). Robust control of a general servomechanism problem: The servo compensator. *Automatica*, 11, 461–471.

Francis, B. and Wonham, W. (1976). The internal model principle of control theory. *Automatica*, 12, 457–465.

Gelfand, I. (1955). Generalized stochastic processes. *Dokl. Akad. Nauk SSSR*, 100, 853–856.

Gikhman, I. and Skorokhod, A. (2004). *Linear Multivariable Control: A Geometric Approach.* Springer.

Hara, S., Yamamoto, Y., Omata, T., and Nakano, M. (1988). Repetitive control system: A new type servo system for periodic exogenous signals. *IEEE Trans. Automat. Contr.*, 33(7), 659–667.

Likhtarnikov, A. and Yakubovich, V. (1977). The frequency theorem for continuous one-parameter semigroups. *Math. of the USSR-Izvestiya*, 11(4), 849–864.

Likhtarnikov, A. and Yakubovich, V. (1998). Dichotomy and stability of undetermined nonlinear systems in Hilbert spaces. *St. Petersburg Math. J.*, 9(6), 1129–1147.

Lindquist, A. and Yakubovich, V. (1997). Optimal damping of forced oscillations in discrete-time systems. *IEEE Trans. Autom. Control*, 42(6), 786–802.

Lindquist, A. and Yakubovich, V. (1999). Universal regulators for optimal tracking in discrete-time systems affected by harmonic disturbances. *IEEE Trans. Autom. Control*, 44(9), 1688–1704.

Matveev, A. (2006). Theory of optimal control in the works of V.A. Yakubovich. *Autom. Remote Contr.*, 67(10), 1645–1698.

Morse, A. (1973). Structure and design of linear model following systems. *IEEE Trans. Autom. Contr.*, 18(4), 346–354.

Morse, A. and Wonham, W. (1970). Decoupling and pole assignment by dynamic compensation. *SIAM J. Control Optim.*, 8(3), 317–337.

Proskurnikov, A. and Yakubovich, V. (2003a). Approximate solution to the problem of the invariance of a control system. *Doklady Math.*, 68(2), 998–1002.

Proskurnikov, A. and Yakubovich, V. (2003b). The problem of the invariance of a control system. *Doklady Math.*, 67(2), 291–295.

Proskurnikov, A. and Yakubovich, V. (2004). The problem of absolute invariance for control systems with delay. *Doklady Math.*, 70(1), 666–670.

Proskurnikov, A. and Yakubovich, V. (2006a). The problem of the invariance of a control system with respect to some of the output variables. *Doklady Math.*, 73(1), 142–146.

Proskurnikov, A. and Yakubovich, V. (2006b). Universal regulators for optimal tracking of polyharmonic signals in systems with delays. *Doklady Math.*, 73(1), 147–151.

Proskurnikov, A. and Yakubovich, V. (2006c). Universal regulators for optimal tracking of stochastic signals with an unknown spectral density. *Doklady Math.*, 74(1), 614–618.

Proskurnikov, A. and Yakubovich, V. (2007). Linear control systems with a reference model. *Doklady Math.*, 76(1), 634–637.

Proskurnikov, A. and Yakubovich, V. (2008). Optimal tracking of stochastic signals with unknown spectral density in discrete-time control systems. *Doklady Math.*, 78(1), 631–635.

Proskurnikov, A. and Yakubovich, V. (2011). Optimal model matching problem for stochastic signals with an unknown fast decreasing spectral density. *Doklady Math.*, 83(1), 126–130.

Proskurnikov, A. and Yakubovich, V. (2012). Universal controllers in model matching optimal control problems for unknown external signals. *J. Comput. Syst. Sci. Int.*, 51(2), 214–227.

Schipanov, G. (1939). Theory and methods of automatic controllers design (in russian). *Avtom. Telemekh.*, (1), 4–37.

Slita, O. and Ushakov, A. (2008). Providing the invariance of the continuous system output with respect to exogeneous signal and endogeneous parametric disturbances: an algebraic approach. *J. of Computer and Systems Sci. Int.*, 47(4), 518–526.

Solnechnyi, E. (2014). Using G.V. Shchipanov's ideas to provide perturbation-invariance of the output of a distributed linear plant. *Autom. Remote Contr.*, 75(1), 75–86.

Stoorvogel, A. and van der Woude, J. (1991). The disturbance decoupling problem with measurement feedback and stability for systems with direct feedthrough matrices. *Syst. & Contr. Lett.*, 17, 217–226.

Weiland, S. and Willems, J. (1989). Almost disturbance decoupling with internal stability. *IEEE Trans. Automat. Contr.*, 34, 277–286.

Wonham, W. (1974). *Linear Multivariable Control: A Geometric Approach.* Springer-Verlag, New York.

Yakubovich, V. (1995). Universal regulators in linear-quadratic optimization problems. In *Trends in Control. A European Perspective.* Springer.

Yakubovich, V. (1997). Universal controllers in stochastic problems of optimal control of linear stationary plants. *Autom. Remote Contr.*, 58(6), 1026–1036.

Yakubovich, V., Melnikov, A., Proskurnikov, A., and Luchin, R. (2011). Signal invariance and trajectory steering problem for an autonomous wheeled robot. In *Proc. of IEEE CDC-ECC*, 3344–3349.

Youla, D., Bongiorno, J., and Jabr, H. (1976). Modern Wiener-Hopf design of optimal controllers, part II. *IEEE Trans. Autom. Contr.*, 21, 319–338.