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Asymptotic Properties of Nonlinear Singularly Perturbed Volterra Equations ⋆

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Abstract: In this paper we examine asymptotic behavior of dynamics systems in the *Lur'e form*, that can be decomposed into feedback interconnection of a linear part and a time-varying nonlinearity. The linear part obeys a singularly perturbed integro-differential Volterra equation of the convolutional type, whereas the nonlinearity is sector-bounded. For such a system we propose frequency-domain criteria of the stability and the "gradient-like" behavior, i.e. the attraction of any solution to one of equilibria points. Those criteria, based on the V.M. Popov's method of a priori integral indices, are uniform with respect to the small parameter.

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1. INTRODUCTION

Since the pioneering works by Lur'e [1957] and Popov [1961, 1973], the theory of absolute stability and related frequency-domain techniques have been widely used in analysis and control of nonlinear systems. The general idea of these techniques is to decompose a nonlinear system in question into feedback interconnection of a known linear time-invariant block and a "nonlinearity", that can be uncertain but satisfies sectorial or other quadratic constraints. Such a decomposition (obviously, non-unique) is referred to as the *Lur'e* form of a nonlinear system. The global stability and other important properties can be guaranteed by efficiently verifiably frequency-domain conditions, employing the transfer function of the linear part and the quadratic functional, constraining the nonlinearity. Two basic approaches to obtaining such frequency-domain criteria are the Kalman-Yakubovich-Popov (KYP) lemma, providing the existence of a quadratic Lyapunov function (see e.g. Yakubovich et al. [2004]), and the extensions of Popov's method for proving stability Popov [1961], known also as the method of *a priori integral indices* Rasvan [2006], see e.g. Yakubovich [2002], Leonov et al. [1996] for historical survey and recent results. Unlike the KYP lemma, applicable mainly to finite-dimensional systems in the state space form, the Popov's approach allows to study infinite-dimensional Lur'e systems as well.

In this paper, we apply the Popov method to investigate the stability of *singularly perturbed* nonlinear integro-differential Volterra equations and get frequency-domain conditions for their stability that are *uniform* with respect to the small parameter. Singularly perturbed equations are used to describe a vast range of physical and mechanical systems, operating in different "time scales", that is, having both slow and fast processes Fridrichs [1955], Dyke [1964], Cole [1968], Imanaliev [1974], Tang et al. [2014]. The singular perturbation means that the equation has a small parameter at the higher derivative; the order of the original unperturbed system (obtained as the parameter vanishes) is less than the order of the perturbed one. A glaring illustration of singularly perturbed equation is the equation of the electron generator Caprioli [1963] of the third order, turning into the Van der Pol equation for zero value of the parameter. Since the asymptotic properties of singularly perturbed equations may differ from these of unperturbed ones, the problems of stability and oscillations for various singularly perturbed integro-differential equations became the subject for special research Imanaliev [1972, 1974], Lizama and Prado [2006a,b], Parand and Rad [2011], Hussain and Al-saif [2013].

Below we examine the global stability of the singularly perturbed systems of Volterra equations under two classes of nonlinearities: time-varying sector-bounded maps and static functions with bounded derivative; those classes also require different assumptions about the linear part of the system. Besides the global stability, we also examine more complicated types of the asymptotic behavior. For instance, *phase synchronization systems* Leonov [2006], such as e.g. various phase-locked loops Margaris [2004], Leonov and Kuznetsov [2014], involve periodic nonlinearities and typically have infinite sequence of equilibria points. An

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important problem concerned with them is the *gradient-like* behavior, i.e. convergence of any solution to one of the equilibria. In this paper we obtain frequency-domain criteria of the gradient-like behavior, extending the results from Perkin et al. [2012] to singularly perturbed phase synchronization systems.

Whereas the gradient-like behavior characterizes the system asymptotics, it says nothing about its transient behavior. Often the phase locking is preceded by "cycle slipping", when the solution leaves the basin of attraction of the nearest equilibrium and is attracted to another stationary point. During this process the absolute phase error is growing; its maximal amplitude depends on the initial conditions and is naturally estimated by the *number of slipped cycles*. Mathematically, an output $\sigma(t)$ of the phase synchronization system with Δ-periodic exogenous input slips $k \in \mathbf{N}\bigcup\{0\}$ cycles if there exists such a moment $\widehat{t} \ge 0$ that $|\sigma(\widehat{t})-\sigma(0)| = k\Delta$, however for all $t \ge 0$ one has $|\sigma(t)-\sigma(0)| < (k+1)\Delta$. The cycle slipping is considered to be undesirable behavior of a synchronization system, leading to e.g. demodulation errors. This motivates to find possibly close estimates for the number of slipped cycles. Such estimates, improving those from Perkin et al. [2014] and extending them to singularly perturbed systems, are also given in the present paper.

2. CRITERIA FOR GLOBAL STABILITY

Consider a set of integro-differential Volterra equations with a positive parameter μ:

$$\mu\dot{\sigma}_\mu(t) + \sigma_\mu(t) = \alpha(t) - \int_0^t g(t-\tau)\varphi(\tau,\sigma_\mu(\tau))d\tau \ (t>0).(1)$$

Here α, g: $\mathbf{R}_+ \to \mathbf{R}$, φ: $\mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$. For each μ the solution of (1) is defined by initial condition

$$\sigma_\mu(0) = \hat{\sigma}. \quad (2)$$

Suppose the following requirements are fulfilled:
a) $\alpha(t) \in \mathbf{C}[\mathbf{0},+\infty)$, $g(t) \in \mathbf{C^1}[\mathbf{0},+\infty)$;
b) for a certain positive λ, the estimate

$$|\alpha(t)| < Me^{-\lambda t} \quad (3)$$

and the inclusions

$$\dot{g}(t)e^{\lambda t},\ g(t)e^{\lambda t} \in L_2[0,+\infty). \quad (4)$$

are true.

Suppose also that the function $\varphi(t,\sigma)$ is continuous and satisfies the "sector condition"

$$0 \le \frac{\varphi(t,\sigma)}{\sigma} \le k \qquad (k>0, t\in\mathbf{R}, \sigma \ne 0). \quad (5)$$

Equation (1) is a singularly perturbed equation with respect to the equation

$$\sigma_0(t) = \alpha(t) - \int_0^t g(t-\tau)\varphi(\tau,\sigma_0(\tau))d\tau \qquad (t>0). \quad (6)$$

The transfer function of the linear part of (6) is as follows:

$$\chi(p) = \int_0^\infty g(t)e^{-pt}dt \qquad (p\in\mathbf{C}). \quad (7)$$

Theorem 1. Let the requirements a), b) and the inequalities (5) be fulfilled. Suppose the frequency inequality

$$\frac{1}{k} + Re\,\chi(i\omega) > 0 \quad (i^2=-1) \quad (8)$$

holds for all $\omega \ge 0$. Then there exists a number $\bar{\mu} > 0$, such that the following assertions are true:
1.

$$\int_0^\infty \varphi^2(t,\sigma_\mu(t))\,dt < C, \qquad \forall\mu\in[0,\bar{\mu}), \quad (9)$$

and C does not depend on μ;
2.

$$\sigma_\mu(t) \to 0 \qquad t\to+\infty \quad (10)$$

uniformly in μ for $\mu \in [0,\bar{\mu})$.

Proof. Let us rewrite the equation (1) in the form

$$\sigma_\mu(t) = \alpha_\mu(t) - \int_0^t g_\mu(t-\tau)\varphi(\tau,\sigma_\mu(\tau))d\tau \quad (t>0) \quad (11)$$

with

$$\begin{aligned} \alpha_\mu(t) &= \hat{\sigma}e^{-\frac{t}{\mu}} + \frac{1}{\mu}\int_0^t \alpha(\tau)e^{-\frac{1}{\mu}(t-\tau)}d\tau; \\ g_\mu(t) &= \frac{1}{\mu}\int_0^t e^{-\frac{1}{\mu}(t-\tau)}g(\tau)d\tau. \end{aligned} \quad (12)$$

Let $\eta_\mu(t) = \varphi(t,\sigma_\mu(t))$. Suppose $\bar{\mu} < \lambda^{-1}$. Then it is easy to establish that

$$|\alpha_\mu(t)| \le M_1e^{-\lambda t} \quad (M_1>0) \quad \forall\mu\in(0,\bar{\mu}). \quad (13)$$

Let is prove that $\forall\mu\in(0,\bar{\mu})$

$$\int_0^\infty g_\mu^2 e^{2\lambda_1 t}dt < C_1 \quad \lambda_1\in(0,\lambda), \quad (14)$$

where C_1 does not depend on μ. Indeed

$$g_\mu(t) = g(t) - g(0)e^{-\frac{t}{\mu}} - \int_0^t e^{-\frac{1}{\mu}(t-\tau)}\dot{g}(\tau)\,d\tau. \quad (15)$$

Let

$$M_2 = \int_0^\infty \dot{g}^2 e^{2\lambda t}dt.$$

Then

$$\begin{aligned} &\int_0^\infty \Big(\int_0^t e^{-\frac{1}{\mu}(t-\tau)}\dot{g}(\tau)d\tau\Big)^2 e^{2\lambda_1 t}dt \le \\ &\le M_2\int_0^\infty e^{-2(\lambda-\lambda_1)t}\Big(\int_0^t e^{2(\frac{1}{\mu}-\lambda)\tau}d\tau\Big)\,dt \end{aligned} \quad (16)$$

and

$$\int_0^\infty \Big(\int_0^t e^{-\frac{1}{\mu}(t-\tau)}\dot{g}(\tau)d\tau\Big)^2 e^{2\lambda_1 t}dt \le \frac{\mu M_2}{4(1-\mu\lambda_1)(\lambda-\lambda_1)} \tag{17}$$

which implies (14).

We can apply the standard scheme of Popov method of a priori integral indices now. Let

$$\eta_{\mu T} = \begin{cases} \eta_\mu(t) & \text{for } t \in [0,T], \\ 0 & \text{for } t<0,\, t>T. \end{cases}$$

$$\zeta_{\mu T}(t) = -\int_0^t g_\mu(t-\tau)\eta_{\mu T}(\tau)d\tau.$$

Let us use the designation

$$[f]^r(t) = f(t)e^{rt}.$$

Then

$$[\zeta_{\mu T}]^r(t) = -\int_0^t [g_\mu]^r(t-\tau)[\eta_{\mu T}]^r(\tau)d\tau.$$

For any $T>0$ it is true that

$$[\sigma_\mu]^r(t) = [\alpha_\mu]^r(t) + [\zeta_{\mu T}]^r(t) \quad \text{for } t\in[0,T]. \tag{18}$$

It is also true that

$$[\eta_{\mu T}]^{\lambda_0}(t), [\zeta_{\mu T}]^{\lambda_0}(t) \in L_2[0,+\infty]\bigcap L_1[0,+\infty], \quad \forall T>0, \forall\lambda_0<\lambda_1. \tag{19}$$

Consider the set of the functionals

$$J_{\mu T} = \int_0^\infty \Big\{[\zeta_{\mu T}]^{\lambda_0}(t)[\eta_{\mu T}]^{\lambda_0}(t) - (\frac{1}{k}-\delta_1)([\eta_{\mu T}]^{\lambda_0}(t))^2\Big\}dt, \tag{20}$$

where $\delta_1>0$ is a positive constant which will be chosen later. Let

$$\mathcal{F}[f](i\omega) = \int_0^\infty f(i\omega)e^{-i\omega t}dt \qquad (i^2=-1).$$

Then by means of Parseval equality

$$J_{\mu T} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big\{\mathcal{F}\Big[[\zeta_{\mu T}]^{\lambda_0}\Big]\mathcal{F}^*\Big[[\eta_{\mu T}]^{\lambda_0}\Big] - (\frac{1}{k}-\delta_1)\Big|\mathcal{F}\Big[[\eta_{\mu T}]^{\lambda_0}\Big]\Big|^2\Big\}d\omega.$$

Notice that

$$\mathcal{F}\Big[[\zeta_{\mu T}]^{\lambda_0}\Big] = -\mathcal{F}\Big[[g_\mu]^{\lambda_0}\Big]\mathcal{F}\Big[[\eta_{\mu T}]^{\lambda_0}\Big],$$
$$\mathcal{F}\Big[[g_\mu]^{\lambda_0}\Big] = \frac{\chi(-\lambda_0+i\omega)}{1-\lambda_0\mu+i\omega\mu}.$$

Then

$$J_{\mu T} = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \Big\{Re\,\frac{\chi(-\lambda_0+i\omega)}{1-\lambda_0\mu+i\omega\mu} + (\frac{1}{k}-\delta_1)\Big\}\,\Big|\mathcal{F}\Big[[\eta_{\mu T}]^{\lambda_0})\Big]\Big|^2 dt. \tag{21}$$

Consider

$$R(\omega) = Re\,\frac{\chi(-\lambda_0+i\omega)}{1-\lambda_0\mu+i\omega\mu} + \frac{1}{k}. \tag{22}$$

$$R(\omega) = \frac{1}{k} + \frac{1-\lambda_0\mu}{(1-\lambda_0\mu)^2+\omega^2\mu^2}Re\,\chi(-\lambda_0+i\omega) + \frac{\omega\mu}{(1-\lambda_0\mu)^2+\omega^2\mu^2}Im\,\chi(-\lambda_0+i\omega). \tag{23}$$

In virtue of relation (4) and the frequency inequality (8) there exists such $\lambda_0\in(0,\lambda_1)$, that

$$\frac{1}{k} + \inf_{\omega\in[0,+\infty)}\{Re\,\chi(-\tau+i\omega)\} > 0, \quad \forall\tau\in[0,\lambda_0). \tag{24}$$

It also follows from (4) that the function $|\omega\, Im\,\chi(-\lambda_0+i\omega)|$ is bounded. Let

$$N = \sup_{\omega\in[0,+\infty)} |\omega\, Im\,\chi(-\lambda_0+i\omega)|$$

and

$$L = \inf_{\omega\in[0,+\infty)} Re\,\chi(-\lambda_0+i\omega).$$

Let a positive number $\bar{\mu}$ be so small that

$$\delta_1 = \frac{1}{k} - \frac{L_0}{1-\lambda_0\bar{\mu}} - \frac{\bar{\mu}N}{(1-\lambda_0\bar{\mu})^2} > 0$$

with $L_0=0$ for $L>0$ and $L_0=-L$ for $L<0$. Then for all $\mu\in[0,\bar{\mu})$ we have

$$R(\omega) > \delta_1. \tag{25}$$

It follows from (21) and (25) that for all $\mu\in[0,\bar{\mu})$ and $T>0$ the inequality

$$J_{\mu T} \le 0 \tag{26}$$

is valid.

We have

$$J_{\mu T} = \int_0^T \Big(\sigma_\mu(t)\eta_\mu(t) - \frac{1}{k}\eta_\mu^2(t)\Big)e^{2\lambda_0 t}dt - \int_0^T \alpha_\mu(t)\eta_\mu(t)e^{2\lambda_0 t}dt + \delta_1\int_0^T \eta_\mu^2(t)e^{2\lambda_0 t}dt. \tag{27}$$

It follows from (5) that

$$\sigma_\mu\eta_\mu - \frac{1}{k}\eta_\mu^2 \ge 0.$$

On the other hand

$$\alpha_\mu(t)\eta_\mu \le \varepsilon\alpha_\mu^2 + \frac{1}{4\varepsilon}\eta_\mu^2,$$

Let $(4\varepsilon)^{-1}<\delta_1$. Then it follows from (26) and (27) that for all $T>0$

$$\int_0^T \eta_\mu^2(t)e^{2\lambda_0 t}dt \le C_0\int_0^T \alpha_\mu^2(t)e^{2\lambda_0 t}dt \;(C_0 = \frac{4\varepsilon}{4\varepsilon\delta_1-1}). \tag{28}$$

So the first conclusion of the theorem is proved. In order to prove the second conclusion it is sufficient to evaluate the function

$$\zeta_\mu(t) = \int_0^t g_\mu(t-\tau)\eta_\mu(\tau)\,d\tau.$$

We have

$$\zeta_\mu(t) \le e^{-\lambda_0 t}\sqrt{\int_0^t g_\mu^2(\tau)e^{2\lambda_0\tau}d\tau}\sqrt{\int_0^t \eta_\mu^2(\tau)e^{2\lambda_0\tau}d\tau}. \quad (29)$$

It follows from (13), (14) and (28) that

$$|\sigma_\mu(t)| \le C_2 e^{-\lambda_0 t},$$

where C_2 does not depend on μ. Theorem 1 is proved.

Consider now the special case of (1) when the nonlinear function does not depend on t explicitly:

$$\mu\dot\sigma_\mu(t) + \sigma_\mu(t) = \alpha(t) - \int_0^t g(t-\tau)\varphi(\sigma_\mu(\tau))\,d\tau \ (t>0)(30)$$

with $\varphi : \mathbf{R} \to \mathbf{R}$

Theorem 2. Suppose the following conditions are fulfilled:
1. The requirements a), b) on the functions $\alpha(t)$, $g(t)$ are valid
2.

$$g(t) \in C^2(0,+\infty)$$

and

$$\ddot g(t)e^{\lambda t} \in L_2[0,+\infty).$$

3.

$$\varphi(\sigma) \in C^1(0,+\infty)$$

and the set of inequalities

$$|\varphi(\sigma)| < m \quad (m>0);$$
$$0 < \frac{\varphi(\sigma)}{\sigma} < k \quad (\sigma \ne 0);$$
$$\alpha_1 \le \frac{d\varphi}{d\sigma} \le \alpha_2,$$

with $\alpha_1 \le 0$, $\alpha_2 \ge k$.
4. There exist positive numbers ϑ and τ such that for all $\omega \ge 0$ the inequality

$$\frac{1}{k} + Re\,\{(1+i\omega\vartheta)\chi(i\omega)+ \\ +\tau\omega^2(\alpha_1\chi(i\omega)+1)^*(\alpha_2\chi(i\omega)+1)\} \ge 0.$$

holds.
Then there exists $\bar\mu > 0$ such that for all $\mu \in [0,\bar\mu)$ the limit relation

$$\sigma_\mu(t) \to 0 \ \text{as} \ t \to +\infty.$$

is true.

3. GRADIENT-LIKE BEHAVIOR

Consider a distributed parameter synchronization system which is described by the integro-differential Volterra equation with a small parameter at the higher derivative:

$$\mu\ddot\sigma_\mu(t) + \dot\sigma_\mu(t) = \alpha(t) + \rho\varphi(\sigma_\mu(t-h)) - \\ -\int_0^t \gamma(t-\tau)\varphi((\sigma_\mu(\tau))\,d\tau \quad (t \ge 0). \quad (31)$$

Here $\mu > 0$, $h \ge 0$, $\rho \in \mathbf{R}$, $\gamma, \alpha : [0,+\infty) \to \mathbf{R}$; $\varphi : \mathbf{R} \to \mathbf{R}$. The map φ is assumed C^1-smooth and Δ-periodic with two simple isolated roots on $[0,\Delta)$. The kernel function $\gamma(\cdot)$ is piece-wise continuous, the function $\alpha(\cdot)$ is continuous. For each μ the solution of (31) is defined by specifying initial condition

$$\sigma_\mu(t)|_{t\in[-h,0]} = \sigma^0(t), \quad (32)$$

where $\sigma^0(\cdot)$ is continuous and $\sigma(0+0) = \sigma^0(0)$.

We assume for definiteness that

$$\int_0^\Delta \varphi(\sigma)\,d\sigma \le 0. \quad (33)$$

We assume also that the linear part of (31) is stable:

$$|\alpha(t)| + |\gamma(t)| \le Me^{-rt} \quad (M, r > 0). \quad (34)$$

Let

$$\alpha_1 = \inf_{\sigma\in[0,\Delta)} \frac{d\varphi}{d\sigma}, \quad \alpha_2 = \sup_{\sigma\in[0,\Delta)} \frac{d\varphi}{d\sigma}.$$

Then

$$\alpha_1 \le \frac{d\varphi}{d\sigma} \le \alpha_2, \quad \forall\sigma \in \mathbf{R}. \quad (35)$$

Notice that $\alpha_1 < 0 < \alpha_2$. Equation (31) is a singularly perturbed equation with respect to in integro-differential Volterra equation ($\mu = 0$)

$$\dot\sigma(t) = \alpha(t) + \rho\varphi(\sigma(t-h)) - \int_0^t \gamma(t-\tau)\varphi((\sigma(\tau))\,d\tau. \quad (36)$$

Equation (36) as well as equation (31) has a denumerable set of equilibriums.

In paper (Perkin et al. [2012]) some sufficient frequency-algebraic conditions for gradient-like behavior of equation (36) were obtained. They were formulated in terms of the transform function of its linear part which is defined by the formula

$$K(p) = -\rho e^{-hp} + \int_0^t \gamma(t)e^{-pt}\,dt \quad (p \in \mathbf{C}). \quad (37)$$

In this paper we extend these frequency-algebraic criteria to singularly perturbed equation (31).

Equation (31) can be reduced to integro-differential Volterra equation

$$\dot{\sigma_\mu}(t) = \alpha_\mu(t) - \int_0^t \gamma_\mu(t-\tau)\varphi(\sigma_\mu(\tau))d\tau \quad (t>0), \quad (38)$$

where

$$\alpha_\mu(t) = \dot\sigma(0)e^{\frac{-t}{\mu}} + \frac{1}{\mu}\int_0^t e^{\frac{\lambda-t}{\mu}}\alpha(\lambda)d\lambda + \frac{\rho}{\mu}J_0, \quad (39)$$

$$J_0 = \left\{ \begin{array}{l} \int_{-h}^{t-h} e^{\frac{\lambda+h-t}{\mu}}\varphi(\sigma(\lambda))d\lambda, \ t \le h, \\ \int_{-h}^{0} e^{\frac{\lambda+h-t}{\mu}}\varphi(\sigma(\lambda))d\lambda, \ t > h, \end{array} \right\}.$$

$$\gamma_\mu(t) = \frac{1}{\mu}\int_0^t e^{\frac{\lambda-t}{\mu}}\gamma(\lambda)d\lambda - \frac{\rho}{\mu}\left\{\begin{array}{ll} e^{\frac{h-t}{\mu}}, & t \geq h, \\ 0, & t < h \end{array}\right\}. \quad (40)$$

The transfer function for equation (31) is as follows

$$K_\mu(p) = \frac{K(p)}{1+\mu p}. \quad (41)$$

Let us introduce the function

$$\Phi(\sigma) = \sqrt{(1-\alpha_1^{-1}\varphi'(\sigma))(1-\alpha_2^{-1}\varphi'(\sigma))},$$

and constants

$$\nu := \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta |\varphi(\sigma)|d\sigma},$$

$$\nu_0 := \frac{\int_0^\Delta \varphi(\sigma)d\sigma}{\int_0^\Delta \Phi(\sigma)|\varphi(\sigma)|d\sigma}.$$

Theorem 3. Suppose there exist positive ε, δ, τ, ϑ, and $a \in [0,1]$ such that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality

$$\begin{array}{l} Re\{\vartheta K(i\omega) - \tau(K(i\omega)+\alpha_1^{-1}i\omega)^*(K(i\omega)+\alpha_2^{-1}i\omega)\} \\ -\varepsilon|K(i\omega)|^2 - \delta \geq 0 \quad (i^2=-1). \end{array} \quad (42)$$

holds;

2) the quadratic form

$$W(\xi,\eta,\zeta) = \varepsilon\xi^2 + \delta\eta^2 + \tau\zeta^2 + a\vartheta\nu\xi\eta + (1-a)\vartheta\nu_0\eta\zeta \quad (43)$$

is positive definite.

Then there exists $\mu_0 > 0$ such that for all $\mu \in (0,\mu_0)$ the following relations are true:

$$\begin{array}{c} \dot\sigma_\mu(t),\, \varphi(\sigma_\mu(t)) \in L_2[0,\infty); \\ \lim_{t\to\infty}\dot\sigma_\mu(t) = 0; \\ \lim_{t\to\infty}\sigma_\mu(t) = q, \quad \text{where} \quad \varphi(q)=0. \end{array} \quad (44)$$

4. ESTIMATES FOR CYCLE SLIPPING

Let us obtain certain estimates for the number of slipped cycles for the unperturbed equation (36). We start with the following technical lemma which is a cornerstone point in estimating the number of slipped cycles.

Lemma 4. Suppose there exist positive ϑ, ε, δ, τ such that for all $\omega \geq 0$ the frequency-domain inequality holds:

$$\begin{array}{l} Re\{\vartheta K(i\omega) - \tau(K(i\omega)+\alpha_1^{-1}i\omega)^*(K(i\omega)+\alpha_2^{-1}i\omega)\} \\ -\varepsilon|K(i\omega)|^2 - \delta \geq 0 \quad (i^2=-1). \end{array}$$

Then the following quadratic functionals

$$\begin{array}{l} I_T = \int_0^T \{\vartheta\dot\sigma(t)\varphi(\sigma(t)) + \varepsilon\dot\sigma^2(t) + \delta\varphi^2(\sigma(t)) + \\ +\tau(\alpha_1^{-1}\dot\varphi(\sigma(t)) - \dot\sigma(t))(\alpha_2^{-1}\dot\varphi(\sigma(t)) - \dot\sigma(t))\}\, dt \end{array}$$

are uniformly bounded along the solution of (36):

$$I_T \leq Q, \quad (45)$$

where Q does not depend on T.

Lemma 4 was proved in Perkin et al. [2014] and it is clear from the proof that for concrete PSS the value of Q may be found explicitly. In the following assertion specific estimate for Q is presented.

Lemma 5. Let $|\alpha_1| = \alpha_2$ and $\varphi(\sigma(0)) = \varphi(\sigma(T)) = 0$. Then if the conditions of Lemma 4 are fulfilled the following estimate holds

$$\begin{array}{r} I_T \leq q := \frac{1}{r}(\vartheta Mm + 2(\varepsilon+\tau)Mm(\frac{M}{r}+\rho) + \\ +(\varepsilon+\tau)\frac{M^2}{2}), \end{array} \quad (46)$$

where

$$m = \sup|\varphi(\sigma)|.$$

The next two theorems give estimates for the number of slipped cycles. To start with, we introduce auxiliary functions

$$P(\varepsilon,\tau,\sigma) = \sqrt{\varepsilon + \tau\Phi^2(\sigma)},$$

$$r_{1j}(k,\vartheta,\varepsilon,\tau,x) = \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j\frac{x}{\vartheta k}}{\int_0^\Delta |\varphi(\sigma)|P(\varepsilon,\tau,\sigma)d\sigma} \quad (j=1,2),$$

Theorem 6. Suppose there exist positive ϑ, ε, δ, τ and natural k such that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality (42) holds;

2)

$$4\delta > \vartheta^2(r_{1j}(k,\vartheta,\varepsilon,\tau,Q))^2 \quad (j=1,2), \quad (47)$$

where Q is given by (45).

Then any solution of (36) slips less than k cycles, that is, the inequalities hold

$$|\sigma(0)-\sigma(t)| < k\Delta \quad \forall t \geq 0. \quad (48)$$

To proceed with the next result, we introduce the following functions for $j=1,2$

$$r_j(k,\vartheta,x) := \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j\frac{x}{\vartheta k}}{\int_0^\Delta |\varphi(\sigma)|d\sigma},$$

$$r_{0j}(k,\vartheta,x) := \frac{\int_0^\Delta \varphi(\sigma)d\sigma + (-1)^j\frac{x}{\vartheta k}}{\int_0^\Delta \Phi(\sigma)|\varphi(\sigma)|d\sigma},$$

and matrices $T_j(k,\vartheta,x) :=$

$$:= \left\| \begin{array}{ccc} \varepsilon & \frac{a\vartheta r_j(k,\vartheta,x)}{2} & 0 \\ \frac{a\vartheta r_j(k,\vartheta,x)}{2} & \delta & \frac{a_0\vartheta r_{0j}(k,\vartheta,x)}{2} \\ 0 & \frac{a_0\vartheta r_{0j}(k,\vartheta,x)}{2} & \tau \end{array} \right\|,$$

where $a \in [0,1]$ and $a_0 := 1-a$.

Theorem 7. Suppose there exist positive ϑ, ε, δ, τ, $a \in [0,1]$ and natural k satisfying the conditions as follows:

1) for all $\omega \geq 0$ the frequency-domain inequality (42) holds;

2) the matrices $T_j(k, \vartheta, Q)$ $(j = 1, 2)$ where the value of Q is defined by (45), are positive definite.
Then for the solution of (36) the inequality (48) holds.

The proofs of Theorems 6 and 7 are presented in (Perkin et al. [2014]).

Lemma 5 and Theorem 7 imply the following assertion.

Theorem 8. Let $|\alpha_1| = \alpha_2$ and $\sigma(0) = \sigma_0$ where $\varphi(\sigma_0) = 0$. Suppose there exist positive $\vartheta, \varepsilon, \delta, \tau, a \in [0, 1]$ and natural k such that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality(42) holds;

2) the matrices $T_j(k, \vartheta, q)$ $(j = 1, 2)$, with q defined by (46), are positive definite.
Then for any solution of (36) the estimate (48) holds.

The frequency-algebraic estimates of the number of slipped cycles, obtained for unperturbed equation can be extended to singular perturbed equation (31).

Let

$$q_0 = q + (\vartheta m + 2(\varepsilon + \tau)m(\frac{M}{r} + \rho))\rho m h + (\varepsilon + \tau)\rho^2 m^2 h,$$

where q is defined by (46).

Theorem 9. Let $\alpha_2 = |\alpha_1|$ and $\sigma(0) = \sigma_0$ where $\varphi(\sigma_0) = 0$. Suppose there exist positive $\vartheta, \varepsilon, \delta, \tau, a \in [0, 1]$ and natural k such that the following conditions are fulfilled:

1) for all $\omega \geq 0$ the frequency-domain inequality(42) holds;

2) the matrices $T_j(k, \vartheta, q_0)$ $(j = 1, 2)$ are positive definite.
Then there exists value μ_0 such that for all $\mu \in (0, \mu_0)$ the following assertion is true: for any solution of (31) the estimates hold as follows

$$|\sigma_\mu(0) - \sigma_\mu(t)| < k\Delta \quad \forall t \geq 0. \qquad (49)$$

5. CONCLUSION

The paper is devoted to asymptotic behavior of singularly perturbed infinite dimensional control systems described by integro-differential Volterra equations. Several frequency-domain stability criteria are extended from unperturbed to singularly perturbed systems. The criteria are uniform with respect to the small parameter.

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