# A CONDITION of CLF EXISTENCE for AFFINE SYSTEMS*

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Abstract: an equivalent condition of Control Lyapunov Function (CLF) existence is developed. The proposed condition summarizes the well known result of Sontag [15] and passivity theorem of Byrns, Isidory and Willems [3]. This condition also provides feedback equivalence to a passive system with not necessary well defined relative degree and normal form [3]. An example of computer simulation completes the work.

## 1. INTRODUCTION

The main approach to test asymptotic stability property of nonlinear dynamical systems is Lyapunov functions method [2,8,10,11,12]. It is known, that under some conditions for class of systems the existence of Lyapunov function is equivalent to asymptotic stability with respect to set [7,10, 12]. Unfortunately, the practical applications of this elegant theory are frequently difficult due to absence of clear methods of Lyapunov function construction for given system. It seems, that task becomes simpler, if we can in some way modify the right hand side of differential equation of the system, i.e. the control signal is available. In such situation one can synthesize control law to assign the desired Lyapunov function for given system. But another question arises: what kind of technique should we choose to assign this Lyapunov function? Then CLF theory gives answer on this question [1,15,16]. In classical work [15] a condition was obtained, which is necessary and sufficient that this Lyapunov function candidate can be assigned by some almost smooth control (continuous and smooth everywhere except the origin) to the indicated system. An example of such control law was also presented in [15]. This condition imposes that time derivative of Lyapunov function candidate should be negative definite on some set. This is always true for class of minimum phase systems [3], but this requirement is not natural for classes of dissipative systems [18], and, for example, for such wide class of dynamical systems as are passive systems, which in general allow not positive definite time derivative of Lyapunov function or so-called storage function. Passive systems stabilization theorem was proposed in [3] and it is based on detectability property of the system [3,13,14]. The mixing of these two results is the goal of this work: to propose condition of CLF existence, that possesses for time derivative of Lyapunov function candidate to be not positive definite on some set (opposite to passive system stabilization theorem, where not positive definiteness is stated for all state space).

## 2. DEFINITIONS AND FORMULATIONS

Let us consider the following model of affine nonlinear dynamic system

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\mathbf{G}(\mathbf{x})\mathbf{u}\,,\ \mathbf{y}=\mathbf{h}(\mathbf{x})\,, \tag{1}$$

where $\mathbf{x}\in R^n$ is state space vector; $\mathbf{u}\in R^m$ is input vector; $\mathbf{y}\in R^p$ is output vector; $\mathbf{f}$, $\mathbf{h}$ and columns of $\mathbf{G}$ are locally Lipschitz continuous vector fields, $\mathbf{h}(0)=0$, $\mathbf{f}(0)=0$. Euclidean norm will be denoted as $|\mathbf{x}|$, $\mathbf{u}(t)$ is measurable and locally essentially bounded function $\mathbf{u}:\mathcal{I}\to R^m$, where $\mathcal{I}$ is a subinterval of $R$, which contains the origin; if interval $\mathcal{I}$ does not specified, then $\mathcal{I}=R_{\geq 0}$. For initial state $\mathbf{x}_0$ and input $\mathbf{u}$ let $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$, if all other arguments of solution are clear from context; $\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})=\mathbf{h}(\mathbf{x}(t,\mathbf{x}_0,\mathbf{u}))$), which is defined on some finite time interval $[0,T)$.

Let $L_{\mathbf{f}}V(\mathbf{x})=\partial V(\mathbf{x})/\partial\mathbf{x}\,\mathbf{f}(\mathbf{x})$ denotes Lie derivative of differentiable function $V$ with respect vector field $\mathbf{f}$; and $L_{\mathbf{G}}V(\mathbf{x})=\partial V(\mathbf{x})/\partial\mathbf{x}\,\mathbf{G}(\mathbf{x})$ is covector of Lie derivatives of function $V$ with respect to columns of matrix function $\mathbf{G}(\mathbf{x})$; $L_{\mathbf{G}}\mathbf{h}(\mathbf{x})=\partial\mathbf{h}(\mathbf{x})/\partial\mathbf{x}\,\mathbf{G}(\mathbf{x})$ is a matrix. As usually, continuous function $\sigma:R_{\geq 0}\to R_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0)=0$; additionally it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. Function $V:R^n\to R_{\geq 0}$ is called positive definite and radially unbounded if for some functions $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ inequality $\alpha_1(|\mathbf{x}|)\leq V(\mathbf{x})\leq\alpha_2(|\mathbf{x}|)$ holds.

D e f i n i t i o n 1 [15]. *A differentiable positive definite and radially unbounded function $V:R^n\to R_{\geq 0}$ is called a CLF for (1), if for each $|\mathbf{x}|\neq 0$ there exists $\mathbf{u}\in R^m$ such, that:*

$$\partial V(\mathbf{x})/\partial\mathbf{x}[\mathbf{f}(\mathbf{x})+\mathbf{G}(\mathbf{x})\mathbf{u}]<0\,.$$

*Function $V$ is said to satisfy the small control property (SCP) for system (1) if*

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$$\lim_{|\mathbf{x}|\to 0}\sup\frac{L_{\mathbf{f}}V(\mathbf{x})}{|L_{\mathbf{G}}V(\mathbf{x})|}\le 0 . \qquad \blacksquare$$

Theorem 1 [15]. *A differentiable positive definite and radially unbounded function $V:R^n\to R_{\ge 0}$ is a CLF for (1), if for all $|\mathbf{x}|\ne 0$ we have*

$$L_{\mathbf{G}}V(\mathbf{x})\equiv 0 \;\Rightarrow\; L_{\mathbf{f}}V(\mathbf{x})<0 . \tag{2}$$

*Additionally, control law*

$$\mathbf{u}=-\kappa\left(L_{\mathbf{f}}V(\mathbf{x}),|L_{\mathbf{G}}V(\mathbf{x})|\right)L_{\mathbf{G}}V(\mathbf{x})^T , \tag{3}$$

$$\kappa(s,r)=\begin{cases}\left(s+\sqrt{s^2+r^4}\right)r^{-2} \text{ if } r\ne 0;\\ 0 \text{ if } r=0,\end{cases}$$

*globally asymptotically stabilizes system (1). Control (3) is continuous (almost smooth, if all $\mathbf{f}$, $\mathbf{G}$ and $V$ are smooth) if function $V$ possesses SCP property.* $\blacksquare$

Condition (2) supposes that time derivative of function $V$ with respect to system (1) is negative definite on set $\mathcal{Z}/\{0\}$, where

$$\mathcal{Z}=\{\mathbf{x}: L_{\mathbf{G}}V(\mathbf{x})\equiv 0\}.$$

This requirement fails in general for class of passive systems, i.e. such systems, that for all $\mathbf{x}\in R^n$ and $\mathbf{u}\in R^m$ time derivative of differentiable positive definite and radially unbounded function $W$ satisfies

$$\dot{W}\le \mathbf{y}^T\mathbf{u},\ \mathbf{y}=L_{\mathbf{G}}W(\mathbf{x})^T , \tag{4}$$

in this case function $W$ is also called storage function and function $\omega(\mathbf{y},\mathbf{u})=\mathbf{y}^T\mathbf{u}$ is called supply rate function. For passive systems always

$$L_{\mathbf{f}}W(\mathbf{x})\le 0 \text{ for all } \mathbf{x}\in R^n , \tag{5}$$

that in common case contradicts (2). On practice there exist classes of systems for which it is much easy to propose storage function $W$, than a CLF $V$. For example, for Hamiltonian systems it is possible to construct storage function $W$ basing on Hamiltonian function. Asymptotically stabilizing control law for passive systems is presented in the following theorem.

Theorem 2 [3]. *Let system (1) is passive with differentiable positive definite radially unbounded storage function $W$ and it is zero-state detectable (ZSD), i.e. for all $t\ge 0$ and $\mathbf{x}_0\in R^n$ is holding*:

$$\mathbf{y}(t)\equiv 0,\ \mathbf{u}(t)\equiv 0 \;\Rightarrow\; \lim_{t\to+\infty}\mathbf{x}(t)=0 . \tag{6}$$

*Then control*

$$\mathbf{u}=-\varphi(\mathbf{y}),\ \mathbf{y}^T\varphi(\mathbf{y})>0 \text{ for } |\mathbf{y}|\ne 0 \tag{7}$$

*globally asymptotically stabilizes system (1).* $\blacksquare$

In paper [3] this result was proven for just continuous storage function $W$. Additionally, it is worth to note, that passivity property can be introduced with not necessary positive definite and radially unbounded storage function. A sufficient condition of ZSD property was also presented in [3,13].

## 3. MAIN RESULTS

Let us compare conditions and control laws of theorems 1 and 2. First of all note, that according to (4) control (3) is a form of (7), indeed for all $|\mathbf{y}|\ne 0$ (let further as in (4) $\mathbf{y}=L_{\mathbf{G}}V(\mathbf{x})^T$) inequality

$$\mathbf{y}^T\kappa\left(L_{\mathbf{f}}V(\mathbf{x}),|L_{\mathbf{G}}V(\mathbf{x})|\right)L_{\mathbf{G}}V(\mathbf{x})^T>0$$

is true. Condition (2) in theorem 1 corresponds conditions ZSD and (4) in theorem 2. Condition (4) possesses (5), but in (2) negative definiteness of $L_{\mathbf{f}}V(\mathbf{x})$ is required on set $\mathcal{Z}$ (set $\mathcal{Z}$ is called zero dynamic set [3], i.e. the set of all trajectories of the system with $\mathbf{y}(t)\equiv 0$ for all $t\ge 0$). The last fact means that in theorem 1 for all $\mathbf{x}\in\mathcal{Z}/\{0\}$ limit relation

$$\lim_{t\to+\infty}\mathbf{x}(t)=0 \tag{8}$$

is satisfied. Opposite to (6) in theorem 2 attractiveness of the origin is guarantied only for vanishing input. But by construction, control (3) equals to zero on set $\mathcal{Z}$, hence from (2) follows (6). According to (5) the converse statement is in general false for storage function $W$ of passive system. So, ZSD property and condition (2) reflect the same property of the system (1): equality (8) is satisfied on set $\mathcal{Z}$ with vanishing control. It is possible to say, that formulation of this property in a theorem in form ZSD can be easier verified, than in form (2). In other words, condition (2) possesses the knowledge of Lyapunov function of the system on set $\mathcal{Z}$ with vanishing inputs, but as remarked above, there are not any universal approaches of such Lyapunov function construction for general nonlinear system. In theorem 2 condition (5) is the most restrictive: passivity approach is working only for stable system with known Lyapunov function.

Hence, both theorems are very closely connected in their conditions and assume a prior knowledge of some kind Lyapunov function for system (1) without inputs. The shortage of the first theorem is requirement of negative definiteness of Lyapunov function candidate on set $\mathcal{Z}$ in (2). The fact, that condition (5) holds globally for all $\mathbf{x}\in R^n$ is the shortage of the second theorem. What will happen if we mix these theorems to build a new result?

Let us start with assumption, that there exists positive definite and radially unbounded storage function $U:R^n\to R_{\ge 0}$ with the following property:

$$L_{\mathbf{G}}U(\mathbf{x})\equiv 0 \;\Rightarrow\; L_{\mathbf{f}}U(\mathbf{x})\le 0 . \tag{9}$$

It is easy to see, that (9) generalizes conditions (2) and (5). According to (9) only Lyapunov stability of system (1) on set $\mathcal{Z}$ with vanishing input is needed. If system (1) has well defined relative degree in sense [3], then condition (2) is equivalent to globally minimum phase property of (1), i.e. on set $\mathcal{Z}$ system (1) is asymptotically Lyapunov stable (if system admits relative degree, then this set is submanifold). In this way condition (9) coincides with definition of globally weak minimum phase property [3], i.e. on zero dynamic set $\mathcal{Z}$ system (1) is Lyapunov stable. Set $S=\{\mathbf{x}: L_{\mathbf{f}}U(\mathbf{x})=0, L_{\mathbf{G}}U(\mathbf{x})=0\}$ describes uncontrolled subset in full state space, where behavior of the system is not defined. To ensure attractiveness of the origin on set $S$ we should introduce ZSD property analog for system (1) with output $\mathbf{y}=L_{\mathbf{G}}U(\mathbf{x})^T$ on this set.

Definition 2. *System (1) is called ZSD with differentiable positive definite and radially unbounded storage function $U$, if for all $t\geq 0$ with $\mathbf{x}(t)\in S$ and $\mathbf{u}(t)\equiv 0$ equality (8) is satisfied.* ■

Hence, if system (1) admits ZSD property in sense of definition 2, then it is sufficient to provide by control law the asymptotic stability property of set $S$. If storage function $U$ from definition 2 additionally possesses condition (9) for system (1), then stabilization of zero dynamic set $\mathcal{Z}$ by appropriate choosing control is enough (in this case zero dynamic set is defined as $\mathcal{Z}=\{\mathbf{x}: L_{\mathbf{G}}U(\mathbf{x})\equiv 0\}$).

Remark 1. For passive systems (1) (condition (4) is satisfied) the definition 2 of ZSD property is equivalent to definition of ZSD property (6) in theorem 2. Indeed, as pointed out in [3] (proof of proposition 3.4), that for passive system with differentiable positive definite and radially unbounded storage function $U$ the following implication holds:

$$\mathbf{y}(t)\equiv 0,\ \mathbf{u}(t)\equiv 0 \ \Rightarrow\ \mathbf{x}(t)\in S,\ \mathbf{u}(t)\equiv 0.$$

So, definition (2) expands applicability of ZSD property on class not necessary passive systems. ■

To stabilize ZSD system (1) with respect to set $\mathcal{Z}$ it is possible to use control law (3) from theorem 1 with minor modifications, which deal with continuity property of obtained control. Before we prove this result let us introduce the following property.

Definition 3. *A differentiable positive definite and radially unbounded storage function $U$ is called well posed for system (1), if there exists a continuous function $\rho: R_{\geq 0}\to R$, such, that*

$$\lim_{|L_{\mathbf{G}}U(\mathbf{x})|\to 0}\sup\frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|}\leq\rho(|\mathbf{x}|). \quad ■$$

It is possible to note, that if $\rho(s)\leq 0$ for all $s\geq 0$, then system (1) admits SCP-like condition with respect to set $\mathcal{Z}=\{\mathbf{x}: L_{\mathbf{G}}U(\mathbf{x})\equiv 0\}$ (see definition 1, where SCP with respect to the origin was defined and paper [5], where SCP property with respect to set was used).

Theorem 3. *Let system (1) with some well posed storage function $U: R^n\to R_{\geq 0}$ is ZSD and admits condition (9). Then:*

1. *Control law*

$$\mathbf{u}=-\kappa\left(L_{\mathbf{f}}U(\mathbf{x}), |L_{\mathbf{G}}U(\mathbf{x})|\right)L_{\mathbf{G}}U(\mathbf{x})^T, \qquad (10)$$

*where function $\kappa$ is taken from (3), globally asymptotically stabilizes system (1). Control (10) is continuous (almost smooth, if all $\mathbf{f}$, $\mathbf{G}$ and $U$ are smooth) if function $\rho(s)\leq 0$ for all $s\geq 0$.*

2. *Control law ($\eta(s)s>0$ for all $s\neq 0$)*

$$\mathbf{u}=-\pi(\mathbf{x})\frac{L_{\mathbf{G}}U(\mathbf{x})^T}{|L_{\mathbf{G}}U(\mathbf{x})|}, \qquad (11)$$

$$\pi(\mathbf{x})=\eta(|L_{\mathbf{G}}U(\mathbf{x})|)+\max_{\mathbf{x}}\left\{0, \rho(\mathbf{x}), \frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|}\right\},$$

*globally asymptotically stabilizes system (1). Control (11) is continuous if function $\rho(s)\leq 0$ for all $s\geq 0$.*

Proof. Let us prove all statements of the theorem consecutively.

1. First of all note, that function $\kappa(s,r)$ from (3) is continuous while $r=0$ only if $s\leq 0$ [5,15]. Condition (9) assumes situation, then functions $L_{\mathbf{f}}U(\mathbf{x}(t))$ and $L_{\mathbf{G}}U(\mathbf{x}(t))$ reach their zero level in the same time $t'$. Hence, it is possible a situation, then $L_{\mathbf{f}}U(\mathbf{x}(t))>0$ for some nearest $t<t'$ and function $\kappa$ loses continuity property while $t\to t'$. In such situation for each $\mathbf{x}_0\in R^n$ we associate control (10) with the following set valued map

$$\mathbf{U}(\mathbf{x})=\begin{cases}-\kappa(a(\mathbf{x}), |\mathbf{b}(\mathbf{x})|)\mathbf{b}(\mathbf{x})^T, \text{ if } |\mathbf{b}(\mathbf{x})|\neq 0;\\ [-2\Theta, 2\Theta]^m, \text{ if } |\mathbf{b}(\mathbf{x})|=0,\end{cases}$$

$a(\mathbf{x})=L_{\mathbf{f}}U(\mathbf{x})$, $\mathbf{b}(\mathbf{x})=L_{\mathbf{G}}U(\mathbf{x})$, where $\Theta=\rho\left(\max\{\alpha_1^{-1}\circ\alpha_2(|\mathbf{x}_0|), |\mathbf{x}_0|\}\right)$, functions $\alpha_1, \alpha_2\in\mathcal{K}_\infty$ are from function $U$ definition:

$$\alpha_1(|\mathbf{x}|)\leq U(\mathbf{x})\leq\alpha_2(|\mathbf{x}|),$$

and system (1), (10) takes form

$$\dot{\mathbf{x}}\in\mathbf{F}(\mathbf{x}),\ \mathbf{F}(\mathbf{x})=\mathbf{f}(\mathbf{x})+\mathbf{G}(\mathbf{x})\mathbf{U}(\mathbf{x}). \qquad (12)$$

If function $\mathbf{F}(\mathbf{x})$ is upper semicontinuous on $R^n$ with non-empty convex and compact values [6], then solution of (12) is absolutely continuous function $\varphi(\mathbf{x}_0, t)$ defined for all $t\leq T\leq +\infty$ and satisfies (12) for almost all $t\leq T$, $\varphi(\mathbf{x}_0, 0)=\mathbf{x}_0$. All desired properties of function $\mathbf{F}$ are followed by the same properties of function $\mathbf{U}(\mathbf{x})$, but function $\mathbf{U}(\mathbf{x})$ is upper semicontinuous on $R^n$ with non-empty convex and compact values. Indeed, function $\mathbf{U}$ is always continuous except the neighborhood of set $\mathcal{Z}$, but according to previous discussion and conditions of the theorem

$$\lim_{|L_{\mathbf{G}}U(\mathbf{x})|\to 0}\sup\frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|}\leq\rho(|\mathbf{x}|)$$

and

$$\lim_{|L_{\mathbf{G}}U(\mathbf{x})|\to 0}\sup\ \mathbf{U}(\mathbf{x})\leq 2\rho(|\mathbf{x}|).$$

Now let us consider function

$$W(\mathbf{x})=\sup_{\mathbf{v}\in\mathbf{U}(\mathbf{x})}\{a(\mathbf{x})+\mathbf{b}(\mathbf{x})\mathbf{v}\}= =\begin{cases}-\sqrt{a(\mathbf{x})^2+|\mathbf{b}(\mathbf{x})|^4}\le 0, \text{if } |\mathbf{b}(\mathbf{x})|\ne 0;\\ a(\mathbf{x})\le 0, \text{if } |\mathbf{b}(\mathbf{x})|=0.\end{cases}$$

So, function $W$ is negative semidefinite, hence form [6], all solutions of (12) are bounded

$$|\mathbf{x}(t)|\le\alpha_1^{-1}\circ\alpha_2(|\mathbf{x}_0|)$$

and $T=+\infty$. From theorem 14 in [4] it also follows, that $\mathbf{x}(t)$ approaches the largest weakly-invariant set in $\{\mathbf{x}:W(\mathbf{x})=0\}$, which obviously belongs to set $\mathcal{Z}$. Further, conclusion of the first part of the theorem follows from ZSD property.

Let $\rho(s)\le 0$ for all $s\ge 0$, then as was discussed above there exists only one way to lose continuity property of control. But if inequality

$$\lim_{|L_\mathbf{G}U(\mathbf{x})|\to 0}\sup\frac{L_\mathbf{f}U(\mathbf{x})}{|L_\mathbf{G}U(\mathbf{x})|}\le 0$$

holds, then continuity property is saved (in such case $\Theta=0$). In work [5] was also proved continuity property of control (10) if SCP property with respect to set $\mathcal{Z}$ holds. The substantiation of almost smoothness property of such control was presented in [15]. Substituting control (10) in expression of time derivative of storage function $U$ we obtain:

$$\dot U=a(\mathbf{x})-\mathbf{b}(\mathbf{x})\kappa(a(\mathbf{x}),|\mathbf{b}(\mathbf{x})|)\mathbf{b}(\mathbf{x})^T= =\begin{cases}-\sqrt{a(\mathbf{x})^2+|\mathbf{b}(\mathbf{x})|^4}, \text{if } |\mathbf{b}(\mathbf{x})|\ne 0;\\ a(\mathbf{x}), \text{if } |\mathbf{b}(\mathbf{x})|\ne 0.\end{cases}$$

The last inequality and condition (9) ensure for system (1), (10) asymptotic stability property with respect to set $\mathcal{Z}$. But on this set control (10) vanishes and ZSD property holds.

2. If $\rho(s)\le 0$ for all $s\ge 0$, then control (11) is continuous and locally Lipschitz (continuous function $\pi(\mathbf{x})=0$ for all $\mathbf{x}\in\mathcal{Z}$). If it is not the case, i.e. system does not admit SCP with respect to set $\mathcal{Z}$, then on set $\mathcal{Z}$ control (11) loses continuity property. Let us first assume, that this SCP property holds, then time derivative of function $U$ for system (1), (11) can be written as follows:

$$\begin{aligned}\dot U&=L_\mathbf{f}U(\mathbf{x})-L_\mathbf{G}U(\mathbf{x})\pi(\mathbf{x})\frac{L_\mathbf{G}U(\mathbf{x})^T}{|L_\mathbf{G}U(\mathbf{x})|}=\\ &=L_\mathbf{f}U(\mathbf{x})-\pi(\mathbf{x})|L_\mathbf{G}U(\mathbf{x})|=L_\mathbf{f}U(\mathbf{x})-\\ &-\left(\eta(|L_\mathbf{G}U(\mathbf{x})|)+\max_\mathbf{x}\left\{0,\rho(\mathbf{x}),\frac{L_\mathbf{f}U(\mathbf{x})}{|L_\mathbf{G}U(\mathbf{x})|}\right\}\right)\times\\ &\times|L_\mathbf{G}U(\mathbf{x})|\le 0 \text{ for all } |L_\mathbf{G}U(\mathbf{x})|\ne 0.\end{aligned}$$

Thus, asymptotic stability property with respect to set $\mathcal{Z}$ is obtained [7,12], further, statement of part 2 of the theorem follows from conditions (9) and ZSD. If system does not possess SCP property, then proof relies on the same arguments as in part 1 of the theorem. Indeed, control (11) can be associated with the following set valued map:

$$\mathbf{U}(\mathbf{x})=\begin{cases}-\pi(\mathbf{x})\frac{L_\mathbf{G}U(\mathbf{x})^T}{|L_\mathbf{G}U(\mathbf{x})|}, \text{if } |L_\mathbf{G}U(\mathbf{x})|\ne 0;\\ \mathcal{E}=\{\mathbf{e}\in R^m, |\mathbf{e}|=1\}, \text{if } |L_\mathbf{G}U(\mathbf{x})|=0.\end{cases}$$

In papers [9,16] control laws were proposed, which are closely connected with control (11), but they were formulated basing on CLF existence. ■

R e m a r k 2. In work [5] a CLF formulation for output asymptotic stabilization of system (1) was given. In that paper control (10) was used as a asymptotic stabilizer for (1). Continuity property of this control law was guarantied if SCP property with respect to zero dynamic set holds. It seems, that storage function $U$ admits all conditions from [5] to be a CLF for output $\mathbf{y}=L_\mathbf{G}U(\mathbf{x})^T$ except the one, which deals with the time invariance property of stabilized set $\mathcal{Z}$. But this property is followed from ZSD and SCP conditions. Thus, result of part 2 of the theorem is not new with the exception of additional properties imposing, that is needed on stabilized zero dynamic set to provide convergence to the origin of all trajectories of the system. ■

Note also, that function $\eta$ in (11) always can be chosen bounded, hence control (11) is bounded, if term $\max_\mathbf{x}\left\{0,|\rho(\mathbf{x})|,\frac{|L_\mathbf{f}U(\mathbf{x})|}{|L_\mathbf{G}U(\mathbf{x})|}\right\}$ is, but it depends on properties of vector field $\mathbf{f}$.

In general, function $U$ in the theorem 3 is not a CLF for the system (1) in the sense of definition 1. Indeed, condition (2) does not follow from (9), and time derivative of $U$ is negative semidefinite. But if we can propose such storage function for a system, then there exists a feedback control (in contrast to open loop control law) that globally asymptotic stabilizes system (1). Hence, according to converse CLF result [15] system admits a CLF. This statement can be formulated as follows.

C o r o l l a r y 1. *Let system (1) possesses well posed storage function $U:R^n\to R_{\ge 0}$ with $\rho(s)\le 0$ for all $s\ge 0$, it is ZSD and admits condition (9).Then system has a CLF with SCP. The converse statement is also true.*

P r o o f. If system possesses well posed storage function, ZSD property and condition (9), then according to theorem 3 it can be stabilized by continuous state feedback in forms (10) or (11). Controlled system is continuous and locally Lipschitz, hence from converse Lyapunov theorems [10,17] there exists a Lyapunov function for controlled system, which can be viewed as CLF candidate. The converse follows by simply observation, that as remarked above condition (2) provides ZSD property and condition (9). Additionally, condition (2) means, that

$$\lim_{|L_{\mathbf{G}}V(\mathbf{x})|\to 0} \sup \frac{L_{\mathbf{f}}V(\mathbf{x})}{|L_{\mathbf{G}}V(\mathbf{x})|} \le 0 .$$

This limit can tend to infinity, but it always stays negative. Thus CLF with SCP is well posed storage function with $\rho(s) \le 0$ for all $s \ge 0$ in sense of definition 3. ■

From the corollary also follows, that if system can be asymptotically stabilized by continuous feedback, then it is "weakly minimum phase" (the inverted commas stress the fact, that relative degree [3] is not necessary). Statement of the corollary can be developed to the case then function $\rho$ takes positive values using result from [17]. To do so, it is enough to prove locally Lipschitz property for differential inclusion (12), then theorems 1 and 2 in [17] provide for asymptotically stable inclusion (12) the existence of a Lyapunov function, which can be considered as CLF too.

The main result of theorem 3 can be reformulated using different point of view. In work [3] was claimed, that system (1) is locally feedback equivalent to a passive system with $C^2$ (double differentiable) storage function $U$, which is positive definite, if and only if system has relative degree $\{1,...,1\}$ at $\mathbf{x}=0$ (origin is also regular point for (1), i.e. rank $\{L_{\mathbf{G}}\mathbf{h}(\mathbf{x})\}$ is constant in neighborhood of the origin) and it is weakly minimum phase. The global version of this result holds, if system has globally defined normal form and it is globally weak minimum phase. Let us drop rather restrictive assumption about existence of globally defined normal form and relative degree and propose another conditions of feedback equivalence to passive systems.

L e m m a 1. *Let system (1) has well posed (with $\rho(s) \le 0$ for all $s \ge 0$) storage function $U : R^n \to R_{\ge 0}$, which admits condition (9). Then control*

$$\mathbf{u} = -\max_{\mathbf{x}} \left\{ 0, \frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|} \right\} \frac{L_{\mathbf{G}}U(\mathbf{x})^T}{|L_{\mathbf{G}}U(\mathbf{x})|} + \mathbf{v} \tag{13}$$

*provides passivity property for the system with respect to the output $\mathbf{y} = L_{\mathbf{G}}U(\mathbf{x})^T$ and new input $\mathbf{v}$ with storage function $U$.*

P r o o f. Note, that function $\tilde{\pi}(\mathbf{x}) = \max_{\mathbf{x}} \left\{ 0, \frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|} \right\}$ tends to zero, while norm of output function $\mathbf{y}$ goes to zero, hence control (13) is continuous. Let us substitute control (13) in (1) and compute time derivative of function $U$:

$$\dot{U} = L_{\mathbf{f}}U(\mathbf{x}) - \max_{\mathbf{x}} \left\{ 0, \rho(\mathbf{x}), \frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|} \right\} \times \\ \times |L_{\mathbf{G}}U(\mathbf{x})| + L_{\mathbf{G}}U(\mathbf{x})\mathbf{v} \le \mathbf{y}^T \mathbf{v}.$$ ■

It is worth to note, that control (11) includes in itself passivity feedback equivalence control (13). Additional output feedback with ZSD property provides, according to theorem 2, asymptotic stabilization of the system.

E x a m p l e. Let model of dynamical system is given:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 \frac{x_3^2}{1+x_3^2}; \\ \dot{x}_2 &= -x_1 - x_2; \\ \dot{x}_3 &= x_2 x_3 + x_3 u, \end{aligned} \tag{14}$$

$(x_1, x_2, x_3) \in R^3$ are state coordinates, $u \in R$ is control. For system (14) we consider function

$$U(x_1, x_2, x_3) = 0.5\left(x_1^2 + x_2^2 + \ln(1 + x_3^2)\right) \tag{15}$$

as a candidate of well posed storage function. Function (15) is differentiable, positive definite and radially unbounded, its time derivative for system (14) takes form:

$$\dot{U} = -x_2^2 + (x_2 + x_1^2 + u) x_3^2 (1 + x_3^2)^{-1}.$$

Thus, $a(x_1, x_2, x_3) = -x_2^2 + (x_1^2 + x_2) x_3^2 (1 + x_3^2)^{-1}$ and $b(x_1, x_2, x_3) = x_3^2 (1 + x_3^2)^{-1}$, from this condition (9) holds. Note also, that for any $(x_1, x_2, x_3) \in R^3$ the following inequality is true $\frac{a(x_1, x_2, x_3)}{|b(x_1, x_2, x_3)|} \le x_1^2 + |x_2|$, hence, storage function (15) is well posed for system (14). System (14) can be viewed as "globally weakly minimum phase" system without well defined relative degree at point $x_3 = 0$, it means, that results [3] can not be used. Due to only condition (9) is satisfied for function (15) theorem 1 can not be applied too (condition (2) fails due to for any $x_1 \ne 0$ and $x_2 = x_3 = 0$ equality $\dot{U} = 0$ holds). Of course, theorem 1 would be used in the case, then a CLF function could be proposed for system (14), the existence of a CLF for system (14) follows from corollary 1. Further, if $y \equiv 0$ ($y = x_3^2 (1 + x_3^2)^{-1}$) and $u \equiv 0$, then system (14) is reduced to asymptotically stable linear system and ZSD property also holds. Therefore, all conditions of theorem 3 are satisfied and control (11) for system (14) can be written as follows:

$$u = -\tanh(x_3) - x_1^2 - |x_2|. \tag{16}$$

Trajectories of system (14), (16) are shown on the figure. □

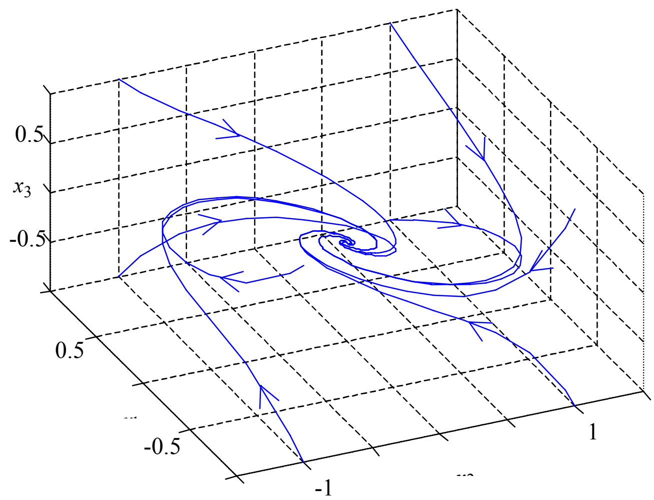

Fig. The trajectories of controlled system.

## 4. CONCLUSION

In this paper the condition of CLF existence for affine in control systems is presented. This condition is formulated in terms of existence of so-called well posed storage function. The development of ZSD property on class of not necessary passive systems is proposed. Two stabilizing control algorithms are considered. The connections between CLF, passivity and proposed approaches are discussed. Combining results from [5] and this it is possible to generalize theorem 3 on case of the set stabilization.

It is possible to say, that this paper gives an answer on the question: can we add something new in procedure of full state stabilization using CLF for output asymptotic stabilization proposed in [5]? According to the main result of this work, the answer is as follows: passing from condition (2) to (9) and ZSD we should go from SCP to rather restrictive SCP with respect set. Indeed, loosely speaking in SCP case function $\frac{L_{\mathbf{f}}U(\mathbf{x})}{|L_{\mathbf{G}}U(\mathbf{x})|}$ should be equal to zero only at the origin, but in the case of SCP with respect to set this function should vanish on the set $Z$ (which contains the origin). On other hand as was pointed out during proof of corollary 1, SCP with respect to set follows from condition (2) in the same way as condition (9) and ZSD.

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