ON ROBUSTNESS PROPERTY OF DYNAMICAL SYSTEMS FEEDBACK CONNECTION WITH RESPECT TO MULTIPLICATIVE DISTURBANCES

Bobtsov A.A. [†], **Efimov D.V.** [‡]

[†] *Department of Control Systems and Informatics*
St.-Petersburg State Institute of Fine Mechanics and Optics (Technical University)
Sablinskay, 14, St.-Petersburg, 197107 Russia
bobtsov@mail.ru

[‡] *Control of Complex Systems Laboratory*
Institute of Problem of Mechanical Engineering,
Bolshoi av., 61, V.O. , St-Petersburg, 199178 Russia
efde@mail.rcom.ru

Abstract: Feedback connection of (strict) passive systems with multiplicative disturbance on the input of one of systems is considered. Series of conditions are proposed, which provide (strict) passivity property for overall system. These conditions enlarge result from Hill, and Moylan (1977), where passivity property of such connection was established. Applications of proposed results to backstepping control and adaptive output control are investigated.

Keywords: disturbance variables, passivity, robustness, backstepping, adaptation.

1. INTRODUCTION

The investigation of influence of external disturbances on stability properties of nonlinear dynamical systems and their feedback connections was performed in many ways in passivity theory (Byrnes, *et al.*, 1991; Hill, and Moylan, 1977; Hill, and Moylan, 1980; Hill, and Moylan, 1991; Polushin, *et al.*, 2000; Willems, 1972), input-to-state stability theory (Sontag, 1989; Sontag, 1998), hyperstability theory by Popov (1973) and small-gain approach (Jiang, *et al.*, 1994). The most of above cited methods were developed for general nonlinear systems and do not rely on specific form of system equations dependence on external disturbances. However, the constructive and applied part of these theories was obtained with supposition, that disturbance is additive and it influences through the same channel as a control input. This work is devoted to multiplicative form of equations dependence on disturbance signal. The common structure of dynamical systems feedback connection is presented in Fig. 1.

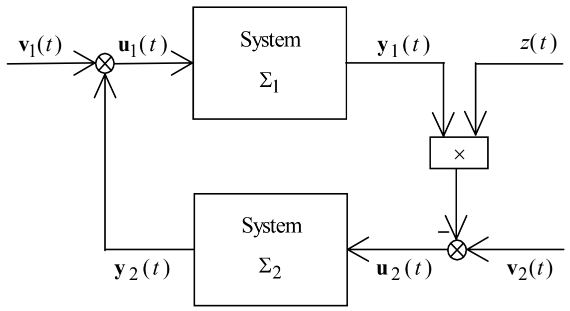

Fig. 1. Feedback connection of systems

Models of systems presented in Fig.1 can be taken as follows:

$$\Sigma_1: \quad \begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{f}_1(\mathbf{x}_1) + \mathbf{G}_1(\mathbf{x}_1)\mathbf{u}_1; \\ \mathbf{y}_1 &= \mathbf{h}_1(\mathbf{x}_1); \end{aligned}$$

$$\Sigma_2: \quad \begin{aligned} \dot{\mathbf{x}}_2 &= \mathbf{f}_2(\mathbf{x}_2) + \mathbf{G}_2(\mathbf{x}_2)\mathbf{u}_2; \\ \mathbf{y}_2 &= \mathbf{h}_2(\mathbf{x}_2), \end{aligned}$$

where $\mathbf{x}_1 \in R^{n_1}$, $\mathbf{x}_2 \in R^{n_2}$ are state space vectors, $\mathbf{y}_1 \in R^{m_2}$, $\mathbf{y}_2 \in R^{m_1}$ are outputs, $\mathbf{u}_1 \in R^{m_1}$, $\mathbf{u}_2 \in R^{m_2}$ are inputs of the systems, $\mathbf{v}_1 \in R^{m_1}$, $\mathbf{v}_2 \in R^{m_2}$ are additive disturbances, Lebesgue measurable and essentially bounded functions of time; $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{h}_1$, $\mathbf{h}_2$ are corresponding dimensions continuous and locally Lipschitz functions, $\mathbf{f}_1(0)=0$, $\mathbf{f}_2(0)=0$, $\mathbf{h}_1(0)=0$ and $\mathbf{h}_2(0)=0$; $z: R_{\ge 0} \to R_{\ge 0}$ is continuous and locally Lipschitz function (multiplicative disturbance) with $0 < z_{\min} \le z(t) \le z_{\max} < +\infty$ (constants $z_{\min}$ and $z_{\max}$ are unknown) and

$$\mathbf{u}_1(t) = \mathbf{y}_2(t) + \mathbf{v}_1(t),$$
$$\mathbf{u}_2(t) = -z(t)\mathbf{y}_1(t) + \mathbf{v}_2(t).$$

In this context disturbance z possesses at the least two interpretations. At first, it can reflect an error of measurement channel or specificity of the link, which connects systems Σ_1 and Σ_2. Such interpretation is clear and it is important in some real world applications (Kim, *et al.*, 1998). Another interpretation deals with time scales in the systems. Usually it is supposed that both subsystems Σ_1 and Σ_2 operate in the same time scale. But if, for example, system Σ_1 is a natural one while Σ_2 is handmade and its dynamics is calculated using computers, which are simultaneously solving different tasks, then time scales can be different due to computer should share its machine cycle times between various subtasks, that results to delays or speeding-ups of dynamics calculation of system Σ_2. Of course this problem is in the center of attention of specialists of computer systems, but in general in such situation it is not possible to guarantee exact coincidence of time scales. If system Σ_2 is a controller, then to handle this problem it is possible to use discrete time or sampled time equations for model of system Σ_2 instead of continuous. But such changes is not natural if system Σ_1 is described with continuous time differential equations and causes additional difficulties in analysis of overall system stability property (Nešić, *et al.*, 1999; Teel, *et al.*, 1998). If both systems Σ_1 and Σ_2 form a part of some network like Internet, then shifts in time scales are normal and they are caused by time delays in communications channels and waiting of responses on other connected systems. In such context connection between different time scales of system Σ_1, Σ_2 and signal $z(t)$ presence can be explained as follows. Let system Σ_1 be working in time $t \geq 0$, while Σ_2 operates in time $\tau \geq 0$ with different time scale (here for simplicity of consideration $\mathbf{v}_1(t) \equiv \mathbf{v}_2(t) \equiv 0$, $t \geq 0$):

$$\frac{d\,\mathbf{x}_1(t)}{d\,t} = \mathbf{f}_1\big(\mathbf{x}_1(t)\big) + \mathbf{G}_1\big(\mathbf{x}_1(t)\big)\mathbf{u}_1(t);$$

$$\mathbf{y}_1(t) = \mathbf{h}_1\big(\mathbf{x}_1(t)\big);$$

$$\frac{d\,\mathbf{x}_2(\tau)}{d\,\tau} = \mathbf{f}_2\big(\mathbf{x}_2(\tau)\big) + \mathbf{G}_2\big(\mathbf{x}_2(\tau)\big)\mathbf{u}_2(\tau);$$

$$\mathbf{y}_2(\tau) = \mathbf{h}_2\big(\mathbf{x}_2(\tau)\big);$$

$$\mathbf{u}_1(t) = \mathbf{y}_2(\tau),\ \mathbf{u}_2(\tau) = -\mathbf{y}_1(t),$$

and there exists an one-to-one transformation

$$\tau = T(t),$$

which connect different time scales in systems Σ_1 and Σ_2. Additionally suppose that function T is a solution of the differential equation:

$$\frac{\partial T(t)}{\partial t} = z(t),$$

using properties of signal z it is possible to conclude, that T is strictly increasing function of t.

To analyze properties of the system dynamics we should transform it to equal time scale. In this case equations of system Σ_2 can be rewritten in the following form:

$$\frac{d\,\mathbf{x}_2(t)}{d\,t} = \tilde{\mathbf{f}}_2\big(\mathbf{x}_2(t),t\big) + \mathbf{G}_2\big(\mathbf{x}_2(t)\big)\mathbf{u}_2(t);$$

$$\mathbf{y}_2(t) = \mathbf{h}_2\big(\mathbf{x}_2(t)\big);$$

$$\mathbf{u}_1(t) = \mathbf{y}_2(t),\ \mathbf{u}_2(t) = -z(t)\mathbf{y}_1(t),$$

where $\tilde{\mathbf{f}}_2(\mathbf{x}_2,t) = z(t)\mathbf{f}_2(\mathbf{x}_2)$. If vector field $\mathbf{f}_2$ is asymptotically stable in the origin or just stable, then $\tilde{\mathbf{f}}_2$ admits the same property. Therefore, investigating stability properties of systems connection it is necessary to pay attention only multiplicative occurrence of signal $z(t)$ on the input of system Σ_2.

Assuming that the system is stable for $z(t) \equiv 1$, $t \geq 0$, in this work some situations are discovered where presence of disturbance z does not influence on stability property of the system. In Section 2 definitions, exact formulation of the problem and main results are formulated and an important application is pointed out. Implementation of proposed results to task of adaptive output control of nonlinear system in Section 3 is presented. Conclusion finishes the paper in Section 4.

2. MAIN RESULT

Due to continuity properties imposed on functions in right hand side of differential equations of the system its solution is well defined at the least locally on time interval $[0,T)$, $T < +\infty$. If $T < +\infty$, then such system is called forward complete (see (Angeli, and Sontag, 1999) for necessary and sufficient conditions of forward completeness property). In this work it is supposed that both systems Σ_1 and Σ_2 from structure scheme in Fig. 1 belong to class of passive or strictly passive systems. More precisely, system of form Σ_1 is called *strictly passive* with respect to input $\mathbf{u}_1$ and output $\mathbf{y}_1$ (Byrnes, *et al.*, 1991; Hill, and Moylan, 1980; Hill, and Moylan, 1991; Polushin, *et al.*, 2000) with differentiable storage function $V_1 : R^{n_1+1} \to R_{\geq 0}$, if

$$\alpha_1\big(|\mathbf{x}_1|\big) \leq V_1(t,\mathbf{x}_1) \leq \alpha_2\big(|\mathbf{x}_1|\big),$$

$$\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial \mathbf{x}_1}\left[\mathbf{f}_1(\mathbf{x}_1) + \mathbf{G}_1(\mathbf{x}_1)\mathbf{u}_1\right] \leq -a_1(\mathbf{x}_1) + \mathbf{y}_1^T\mathbf{u}_1,$$

where $a_1(\mathbf{x}_1) \geq \alpha_3\big(|\mathbf{x}_1|\big)$, $\alpha_1,\alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$; and system Σ_1 is *passive* with respect to input $\mathbf{u}_1$ and output $\mathbf{y}_1$ (Byrnes, *et al.*, 1991; Polushin, *et al.*, 2000) with differentiable storage function V_1 if above inequalities hold for $a(\mathbf{x}_1) \geq 0$. It is said, that function $\rho : R_{\geq 0} \to R_{\geq 0}$ belongs to class $\mathcal{K}$, if it is strictly increasing and $\rho(0) = 0$; $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(s) \to \infty$ for $s \to \infty$ (radially unbounded); continuous function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is from class $\mathcal{KL}$, if it is from class $\mathcal{K}$ for the first argument for any fixed second, and strictly decreasing to zero by the second argument for any fixed first one. In work (Willems, 1972) passive and strict passive properties were introduced for general nonlinear dynamical systems with not necessary positive definite and differentiable storage functions. According to nonlinear version of Kalman-Yakubovich-Popov Lemma (Byrnes, *et al.*, 1991) for (strictly) passive system vector field $\mathbf{f}_1$ is (asymptotically) stable with Lyapunov function V_1 and

$$\mathbf{h}_1(\mathbf{x}_1) = \mathbf{G}_1^T(\mathbf{x}_1)\frac{\partial V_1}{\partial \mathbf{x}_1}^T.$$

The following result was proved in (Hill, and Moylan, 1977) (case without multiplicative disturbance z).

Theorem 1. *Let feedback connection of the form* $\Sigma:\ \mathbf{u}_1(t) = \mathbf{y}_2(t) + \mathbf{v}_1(t)$, $\mathbf{u}_2(t) = -\mathbf{y}_1(t) + \mathbf{v}_2(t)$ *be given, systems* Σ_i, $i = 1,2$ *are (strictly) passive with respect to inputs* $\mathbf{u}_i$ *and outputs* $\mathbf{y}_i$ *with differentiable storage functions* V_i. *Then overall system* Σ *is (strictly) passive with respect to input* $\mathbf{v} = col\big(\mathbf{v}_1^T, \mathbf{v}_2^T\big)$ *and output* $\mathbf{y} = col\big(\mathbf{y}_1^T, \mathbf{y}_2^T\big)$ *for storage function*

$$V(t,\mathbf{x}_1,\mathbf{x}_2) = V_1(t,\mathbf{x}_1) + V_2(t,\mathbf{x}_2).$$

If Σ_1 *(*Σ_2*) is strictly passive while* Σ_2 *(*Σ_1*) is simple passive, then* Σ *is passive with respect to input*

$\mathbf{v}$ *and output* $\mathbf{y}$, *and asymptotically stable with respect to part of variables* $\mathbf{x}_1$ *(*$\mathbf{x}_2$*) for zero disturbances* $\mathbf{v}(t) \equiv 0$, $t \geq 0$. ■

The goals of this paper is to present conditions under which presence of multiplicative disturbance does not annihilate passivity property of the system stated in Theorem 1. To do so further we will consider feedback connection

$$\tilde{\Sigma}: \mathbf{u}_1(t) = \mathbf{y}_2(t) + \mathbf{v}_1(t),\ \mathbf{u}_2(t) = -z(t)\mathbf{y}_1(t) + \mathbf{v}_2(t).$$

First of all note, that constant signal $z(t) = z$ does not influence on any properties of system $\tilde{\Sigma}$. Indeed, let system Σ_1 be (strictly) passive with respect to input $\mathbf{u}_1$ and output $\mathbf{y}_1$ with differentiable storage function V_1, while for system Σ_2 the same property holds with respect to $\mathbf{u}_2$, $\mathbf{y}_2$ and function V_2:

$$\dot{V}_i \leq -a_i(\mathbf{x}_i) + \mathbf{y}_i^T \mathbf{u}_i,\ i = 1,2.$$

Then system $\tilde{\Sigma}$ is (strictly) passive with respect to input $\tilde{\mathbf{v}} = col\left(\mathbf{v}_1^T, z^{-1}\mathbf{v}_2^T\right)$ and output $\mathbf{y}$ with storage function

$$\tilde{V}(t, \mathbf{x}_1, \mathbf{x}_2) = V_1(t, \mathbf{x}_1) + z^{-1} V_2(t, \mathbf{x}_2).$$

This conclusion can be obtained after time derivative of function $\tilde{V}$ consideration:

$$\begin{aligned}\dot{\tilde{V}} &\leq -a_1(\mathbf{x}_1) + \mathbf{y}_1^T \mathbf{u}_1 - z^{-1} a_2(\mathbf{x}_2) + z^{-1}\mathbf{y}_2^T \mathbf{u}_2 = \\ &= -a_1(\mathbf{x}_1) + \mathbf{y}_1^T(\mathbf{y}_2 + \mathbf{v}_1) - z^{-1} a_2(\mathbf{x}_2) + \\ &\quad + z^{-1}\mathbf{y}_2^T(-z\mathbf{y}_1 + \mathbf{v}_2) \leq -a_1(\mathbf{x}_1) + \mathbf{y}_1^T \mathbf{v}_1 - \\ &\quad - z^{-1} a_2(\mathbf{x}_2) + z^{-1}\mathbf{y}_2^T \mathbf{v}_2 \leq -a_1(\mathbf{x}_1) - \\ &\quad - z^{-1} a_2(\mathbf{x}_2) + \mathbf{y}^T \tilde{\mathbf{v}}.\end{aligned}$$

Therefore it is worth to pay especial attention to kind of signal $z(t)$ dependence on time argument t. Let function z be continuous and locally Lipschitz, then for almost all $t \geq 0$ there exists well defined $\dot{z}(t)$, which can be used to formulate a desired condition.

L e m m a 1. *Let systems* Σ_i, $i = 1,2$ *be (strictly) passive with respect to inputs* $\mathbf{u}_i$, *outputs* $\mathbf{y}_i$ *with differentiable storage functions* V_i *and one of the following conditions holds*

a) $\dot{z}(t) \leq 0$ *for almost all* $t \geq 0$;

b) $\dot{z}(t) \geq 0$ *for almost all* $t \geq 0$.

Then overall system $\tilde{\Sigma}$ *is (strictly) passive with respect to output* $\mathbf{y}$

a) *and input* $\mathbf{v}_a = col\left(z\mathbf{v}_1^T, \mathbf{v}_2^T\right)$ *for storage function* $V_a(t, \mathbf{x}_1, \mathbf{x}_2) = z(t) V_1(t, \mathbf{x}_1) + V_2(t, \mathbf{x}_2)$;

b) *input* $\mathbf{v}_b = col\left(\mathbf{v}_1^T, z^{-1}\mathbf{v}_2^T\right)$ *for storage function* $V_b(t, \mathbf{x}_1, \mathbf{x}_2) = V_1(t, \mathbf{x}_1) + z(t)^{-1} V_2(t, \mathbf{x}_2)$.

If systems Σ_1 *(*Σ_2*) is strictly passive while* Σ_2 *(*Σ_1*) is passive then system* $\tilde{\Sigma}$ *is passive with respect to output* $\mathbf{y}$, *pare* $\mathbf{v}_a$, V_a *or* $\mathbf{v}_b$, V_b *correspondingly and asymptotically stable with respect to part of variables* $\mathbf{x}_1$ *(*$\mathbf{x}_2$*) for vanishing input* $\mathbf{v}$.

P r o o f. Note, that both storage functions V_a and V_b are positive definite and radially unbounded due to property $0 < z_{\min} \leq z(t) \leq z_{\max} < +\infty$ and the fact that both V_1 and V_2 have the same properties. Their time derivatives take form:

$$\begin{aligned}\dot{V}_a &= \dot{z}(t) V_1 + z(t)\dot{V}_1 + \dot{V}_2 \leq -z(t) a_1(\mathbf{x}_1) + \\ &\quad + z(t)\mathbf{y}_1^T \mathbf{u}_1 - a_2(\mathbf{x}_2) + \mathbf{y}_2^T \mathbf{u}_2 \leq \\ &\leq -z_{\min} a_1(\mathbf{x}_1) - a_2(\mathbf{x}_2) + z(t)\mathbf{y}_1^T \mathbf{v}_1 + \mathbf{y}_2^T \mathbf{v}_2;\end{aligned}$$

$$\begin{aligned}\dot{V}_b &= \dot{V}_1 - \frac{\dot{z}(t)}{z(t)^2} V_2 + z(t)^{-1}\dot{V}_2 \leq -a_1(\mathbf{x}_1) + \\ &\quad + \mathbf{y}_1^T \mathbf{u}_1 - z(t)^{-1} a_2(\mathbf{x}_2) + z(t)^{-1}\mathbf{y}_2^T \mathbf{u}_2 \leq \\ &\leq -a_1(\mathbf{x}_1) - z_{\max}^{-1} a_2(\mathbf{x}_2) + \mathbf{y}_1^T \mathbf{v}_1 + z(t)^{-1}\mathbf{y}_2^T \mathbf{v}_2.\end{aligned}$$

Passivity properties stated in the Lemma easily follow from previous inequalities. Claim about asymptotic stability of the system with respect to part of variables $\mathbf{x}_1$ or $\mathbf{x}_2$ for vanishing input $\mathbf{v}$ can be proved using results from (Rumyantsev, and Oziraner, 1987; Vorotnikov, 1998). ■

Thus, according to Lemma 1 uniform decreasing or increasing of time scales in both systems in Fig. 1 do not affect on stability properties proposed in Theorem 1 for feedback connection Σ. But even modeling of simple time delay on input of system Σ_2 requires a sign alternating of time derivative $\dot{z}$. It is clear, that if system $\tilde{\Sigma}$ is forward complete and signal $\dot{z}$ has constant sign at infinity then system again is stable. But more interesting situation includes a sign varying signal $\dot{z}$ without fixed asymptotic limit, but with sufficiently large time period between time instants of sign of $\dot{z}$ changing. Let

$$\mathcal{T}_1 = \{t : \dot{z}(t) \leq 0\},\ \mathcal{T}_2 = \{t : \dot{z}(t) \geq 0\}$$

define sets of time instants with positive or negative sign of $\dot{z}$. It is clear that these sets consist on connected subsets or subintervals:

$$\mathcal{T}_1 = \bigcup_j \left[t_j^1, t_{j+1}^1\right),\ \mathcal{T}_2 = \bigcup_j \left[t_j^2, t_{j+1}^2\right),\ j = 0,1,2,\ldots,$$

then

$$\tau_z = \min\left\{\inf_j \left(t_{j+1}^1 - t_j^1\right), \inf_j \left(t_{j+1}^2 - t_j^2\right)\right\}$$

characterizes minimum length of time interval with constant sign of $\dot{z}$.

T h e o r e m 2. *Let systems* Σ_i, $i = 1,2$ *be strictly passive with respect to inputs* $\mathbf{u}_i$, *outputs* $\mathbf{y}_i$ *with differentiable storage functions* V_i. *Then for* $\mathbf{v}(t) \equiv 0$, $t \geq 0$ *there exists a* $\tau_Z > 0$, *such, that for any signal* $z(t)$ *with* $\tau_z \geq \tau_Z$ *the origin of overall system* $\tilde{\Sigma}$ *is globally attractive and for any* $z(t)$ *and initial conditions* $\mathbf{x}(0)$ *(*$\mathbf{x} = col\left(\mathbf{x}_1^T, \mathbf{x}_2^T\right)$*) there exists a constant* $0 < X_{z(t),\mathbf{x}(0)} < +\infty$, *such, that*

$$\left|\mathbf{x}(t)\right| \leq X_{z(t),\mathbf{x}(0)} \text{ for all } t \geq 0.$$

P r o o f. First, let us base forward completeness property of system $\tilde{\Sigma}$. By default the system solution is defined on some time interval $\left[0, T\right)$, which can be presented as concatenation of subintervals $[0,T)=\bigcup_j [t_j,t_{j+1})$, $j=0,1.2,...$, such, that $[t_{2k},t_{2k+1})\in\mathcal{T}_1$, $[t_{2k+1},t_{2k+2})\in\mathcal{T}_2$, $k=0,1.2,...$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ were defined above. In other words, on each even subinterval $\dot z(t)\le 0$ and on each odd $\dot z(t)\ge 0$ (opposite case can be analyzed in the same way). Then according to Lemma 1 system $\tilde\Sigma$ is stable on these subintervals for vanishing input $\mathbf{v}$ and there exist some functions σ_1, σ_2 from class $\mathcal{K}$, such, that for $k=0,1.2,...$

$$t\in[t_{2k},t_{2k+1})\in\mathcal{T}_1 \Rightarrow |\mathbf{x}(t)|\le\sigma_1(|\mathbf{x}(t_{2k})|);$$
$$t\in[t_{2k+1},t_{2k+2})\in\mathcal{T}_2 \Rightarrow |\mathbf{x}(t)|\le\sigma_2(|\mathbf{x}(t_{2k+1})|).$$

Suppose, that system is not forward complete, then there exists $\bar T=\arg\inf_t\{|\mathbf{x}(t)|=+\infty\}$, but such $\bar T$ can not belong to set of subintervals $\mathcal{T}_1$ or $\mathcal{T}_2$, due to $|\mathbf{x}(\bar T)|\le\sigma_i(|\mathbf{x}(t_{2k+i-1})|)$, $i=1,2$ that contradicts minimality of $\bar T$, thus, $\tilde\Sigma$ is forward complete.

Now let us suppose that $k=0,1.2,...,K<+\infty$, then applying Lemma 1 for system $\tilde\Sigma$ at the last subinterval one can obtain passivity of the system (and, hence, stability for zero input $\mathbf{v}$) and asymptotic convergence of $\mathbf{x}(t)$ to the origin for strictly passive systems Σ_1 and Σ_2; or if only system Σ_1 (Σ_2) possesses strict passivity property, then $\mathbf{x}_1(t)$ ($\mathbf{x}_2(t)$) should converges to zero. So, let us further consider case $K=+\infty$ with infinite sign changes of $\dot z$.

Let $k\ge 0$ be arbitrary and consider two consecutive intervals $[t_{2k},t_{2k+1})\in\mathcal{T}_1$ and $[t_{2k+1},t_{2k+2})\in\mathcal{T}_2$, systems Σ_1 and Σ_2 are strictly passive. Then according to Lemma 1 overall system is also strictly passive on each of mentioned subintervals, i.e. there exist functions β_1, β_2 from class $\mathcal{KL}$, such, that

$$t\in[t_{2k},t_{2k+1})\Rightarrow|\mathbf{x}(t)|\le\beta_1(|\mathbf{x}(t_{2k})|,t-t_{2k});$$
$$t\in[t_{2k+1},t_{2k+2})\Rightarrow|\mathbf{x}(t)|\le\beta_2(|\mathbf{x}(t_{2k+1})|,t-t_{2k+1}).$$

Substituting the first estimate in the last one for $t=t_{2k+1}$ it is possible to receive

$$t\in[t_{2k+1},t_{2k+2})\in\mathcal{T}_2\Rightarrow$$
$$|\mathbf{x}(t)|\le\beta_2(\beta_1(|\mathbf{x}(t_{2k})|,t_{2k+1}-t_{2k}),t-t_{2k+1}).$$

If inequality

$$\beta_2(\beta_1(|\mathbf{x}(t_{2k})|,t_{2k+1}-t_{2k}),t_{2k+2}-t_{2k+1})<|\mathbf{x}(t_{2k})|$$

is satisfied, then $\mathbf{x}(t)$ is bounded and decreases on each pare of subintervals as considered. By properties of signal $z(t)$ for any $k=0,1.2,...$

$$t_{k+1}-t_k\ge\tau_z,$$

hence, there exists a $\tau_Z>0$, such, that for any $\tau_z\ge\tau_Z$ series of inequalities holds:

$$|\mathbf{x}(t_{2k+2})|\le$$
$$\le\beta_2(\beta_1(|\mathbf{x}(t_{2k})|,t_{2k+1}-t_{2k}),t_{2k+2}-t_{2k+1})\le$$
$$\le\beta_2(\beta_1(|\mathbf{x}(t_{2k})|,\tau_z),\tau_z)<|\mathbf{x}(t_{2k})|$$

and solution of the system is bounded and converges to the origin. ■

Proposed result claims that there exists a "slow" enough signal $z(t)$ ("slow" in the sense, that time derivative of this signal changes its sign with large enough time period), such, that stability properties of system Σ are saved for $\tilde\Sigma$, thus some stability properties of feedback connection Σ are *invariant* under proper multiplicative disturbance on the input of system Σ_2.

The most important application of Theorem 2 is feedback stabilization using integrator backstepping method (Byrnes, *et al.*, 1991; Krstić, *et al.*, 1995). Indeed, classical statement of the solved by this approach task is as follows. Let a system be given:

$$\dot{\mathbf{x}}_1=\mathbf{f}(\mathbf{x}_1)+\mathbf{G}(\mathbf{x}_1)\mathbf{x}_2;$$
$$\dot{\mathbf{x}}_2=\mathbf{u},$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ form state space vector, $\mathbf{u}$ is control; functions in right hand side are smooth and ensure existence of solution, $\mathbf{f}(0)=0$. It is supposed, that $\mathbf{x}_1$-subsystem is already stabilized by some feedback and it is asymptotically stable for $\mathbf{x}_2=0$ with known Lyapunov function V_1. It is necessary to design a new feedback, that provides asymptotic stability property for overall system with some new known Lyapunov function $\tilde V$, in other words, it is necessary to transfer control for $\mathbf{x}_1$-subsystem through integrator $\mathbf{x}_2$. The solution is

$$\mathbf{u}=-\phi(\mathbf{x}_2)-\frac{\partial V_1}{\partial\mathbf{x}_1}\mathbf{G}(\mathbf{x}_1),$$

with $\tilde V(\mathbf{x}_1,\mathbf{x}_2)=V_1(\mathbf{x}_1)+0.5\mathbf{x}_2^T\mathbf{x}_2$ is a Lyapunov function. In fact closed loop system can be considered as Σ with strict passive systems Σ_1, Σ_2 for

$$\mathbf{f}_2(\mathbf{x}_2)=-\phi(\mathbf{x}_2),\ \mathbf{G}_2(\mathbf{x}_2)=\mathbf{I},\ \mathbf{h}_2(\mathbf{x}_2)=\mathbf{x}_2,$$
$$\mathbf{h}_1(\mathbf{x}_1)=\frac{\partial V}{\partial\mathbf{x}_1}\mathbf{G}(\mathbf{x}_1),\ V_2(\mathbf{x}_2)=0.5\mathbf{x}_2^T\mathbf{x}_2,$$

where $\mathbf{I}$ is identity matrix of corresponding dimension. Therefore, Theorem 1 establishes conditions which provide for backstepping control robust stability property with respect to multiplicative disturbance, which can reflect influence of different time scales in subsystems Σ_1, Σ_2 or time delay presented in the channel, which connects output $\mathbf{y}_1$ and input $\mathbf{u}_2$. The closely connected task was solved in (Kanellakopoulos, 1997) for systems in strict feedback form, but in that work an input-to-state stability paradigm (Sontag, 1989) was applied to ensure robust properties of the system and additional modification was introduced in control law.

3. OUTPUT ADAPTIVE CONTROL

It is worth to stress that result of Theorem 2 does not work for case then one of systems Σ_1 or Σ_2 is simple passive. In this case according to Theorem 1 system Σ is asymptotically stable with respect to part of variables for vanishing input $\mathbf{v}$. This case reflects structure scheme of classical adaptive control system. Indeed, let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\omega(\mathbf{x},t)(\theta - \widehat{\theta}) \qquad (1)$$

be the model of undetermined plant, where $\mathbf{x}$ is its state vector, ω is known regressor function, θ is vector of unknown parameters of the plant, $\widehat{\theta}$ is vector of θ estimates adjusted by adaptive controller. It is assumed that in ideal case $\widehat{\theta} = \theta$ and plant is asymptotically stable with Lyapunov function:

$$\alpha_1(|\mathbf{x}|) \le V(t,\mathbf{x}) \le \alpha_2(|\mathbf{x}|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) \le -\alpha(|\mathbf{x}|),\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty,\ \alpha \in \mathcal{K}.$$

In this case adaptation algorithm takes form (Fomin, *et al.*, 1981; Fradkov, *et al.*, 1999; Krstić, *et al.*, 1995):

$$\dot{\widehat{\theta}} = -\gamma\,\omega(\mathbf{x},t)^T \mathbf{G}(\mathbf{x})^T \frac{\partial V}{\partial \mathbf{x}}^T,\ \gamma > 0. \qquad (2)$$

Under change of variables

$$\mathbf{x}_1 = \mathbf{x},\ \mathbf{x}_2 = \widehat{\theta} - \theta$$

and with substitution

$\mathbf{f}_1(\mathbf{x}_1) = \mathbf{f}(\mathbf{x})$, $\mathbf{G}_1(\mathbf{x}_1) = \mathbf{G}(\mathbf{x})\omega(\mathbf{x},t)$, $\mathbf{f}_2(\mathbf{x}_2) = 0$, $\mathbf{G}_2(\mathbf{x}_2) = \gamma\mathbf{I}$, $\mathbf{h}_1(\mathbf{x}_1,t) = \omega(\mathbf{x},t)^T \mathbf{G}(\mathbf{x})^T \frac{\partial V}{\partial \mathbf{x}}^T$, $\mathbf{h}_2(\mathbf{x}_2) = \mathbf{x}_2$, $V_1(t,\mathbf{x}_1) = V(t,\mathbf{x})$, $V_2(\mathbf{x}_2) = 0.5\gamma^{-1}\mathbf{x}_2^T\mathbf{x}_2$ system (1), (2) can be transformed to feedback connection Σ with strictly passive system Σ_1 and passive system Σ_2. Let only output $\mathbf{y}_1$ be measured by adaptive controller, then algorithm (2) can be rewritten as follows

$$\dot{\widehat{\theta}} = -\gamma\, z(t)\mathbf{y}_1,$$

where multiplicative disturbance z reflects measurement error of plant output signal $\mathbf{y}_1$.

Let us investigate conditions which allow to save for system $\widetilde{\Sigma}$ stability property of system Σ. This result will be based on the following property. System Σ_1 is called *observable* with respect to output $\mathbf{y}_1$ if the implication holds (Byrnes, *et al.*, 1991):

$$\mathbf{y}_1(t) \equiv 0,\ \mathbf{u}_1(t) \equiv 0,\ t \ge 0 \Rightarrow \mathbf{x}(t) \equiv 0,\ t \ge 0.$$

Theorem 3. *Let systems Σ_1 be strictly passive with respect to input $\mathbf{u}_1$, output $\mathbf{y}_1$ with differentiable storage functions V_1 and $\mathbf{G}_1(0) = 0$; system Σ_2 is passive with respect to input $\mathbf{u}_2$, output $\mathbf{y}_2$ with differentiable storage functions V_2. Systems Σ_1, Σ_2 are observable with respect to outputs $\mathbf{y}_1$, $\mathbf{y}_2$ correspondingly. Then for $\mathbf{v}(t) \equiv 0$, $t \ge 0$ system $\widetilde{\Sigma}$ is globally attractive with respect to variables $\mathbf{x}_1(t)$ and for any $z(t)$ and initial conditions $\mathbf{x}(0)$ there exists a constant $0 < X_{z(t),\mathbf{x}(0)} < +\infty$, such, that*

$$|\mathbf{x}(t)| \le X_{z(t),\mathbf{x}(0)} \textit{ for all } t \ge 0.$$

Proof. At first step again we will prove forward completeness property of system $\widetilde{\Sigma}$. Let us introduce two storage functions for the system:

$$W_1(t,\mathbf{x}_1,\mathbf{x}_2) = V_1(t,\mathbf{x}_1) + z_{\max}^{-1}V_2(t,\mathbf{x}_2);$$

$$W_2(t,\mathbf{x}_1,\mathbf{x}_2) = V_1(t,\mathbf{x}_1) + z_{\min}^{-1}V_2(t,\mathbf{x}_2).$$

Their time derivatives can be rewritten as follows:

$$\dot{W}_1 \le -a_1(\mathbf{x}_1) - z_{\max}^{-1}a_2(\mathbf{x}_2) + \mathbf{y}_1^T\mathbf{y}_2 - z_{\max}^{-1}z(t)\mathbf{y}_2^T\mathbf{y}_1;$$

$$\dot{W}_2 \le -a_1(\mathbf{x}_1) - z_{\min}^{-1}a_2(\mathbf{x}_2) + \mathbf{y}_1^T\mathbf{y}_2 - z_{\min}^{-1}z(t)\mathbf{y}_2^T\mathbf{y}_1.$$

Then

$$\dot{W}_1 \le -a_1(\mathbf{x}_1) - z_{\max}^{-1}a_2(\mathbf{x}_2) \text{ for } t \in \mathcal{T}_3; \qquad (3)$$

$$\dot{W}_2 \le -a_1(\mathbf{x}_1) - z_{\min}^{-1}a_2(\mathbf{x}_2) \text{ for } t \in \mathcal{T}_4, \qquad (4)$$

where

$$\mathcal{T}_3 = \{t : \mathbf{y}_1^T(t)\mathbf{y}_2(t) \le 0\},\ \mathcal{T}_4 = \{t : \mathbf{y}_1^T(t)\mathbf{y}_2(t) \ge 0\}.$$

Thus, as in proof of Theorem 2, on each time subinterval from set $\mathcal{T}_3$ or $\mathcal{T}_4$ solution of the system is bounded by initial conditions on the beginning of the subinterval. Therefore, trajectory can not escape to infinity in finite time and system $\widetilde{\Sigma}$ is forward complete.

Now we will prove boundedness of system trajectories. As before interval of solution definition $[0,+\infty)$ can be presented as concatenation $[0,+\infty) = \bigcup_j [t_j, t_{j+1})$, $j = 0,1.2,...$, where $[t_{2k}, t_{2k+1}) \in \mathcal{T}_3$, $[t_{2k+1}, t_{2k+2}) \in \mathcal{T}_4$, $k = 0,1.2,...$. At the last instants t_{2k}, t_{2k+1}, t_{2k+2} an equality is satisfied

$$\mathbf{y}_1^T(t)\mathbf{y}_2(t) = 0.$$

If both outputs equal to zero, then from observability property system is located in the equilibrium at the origin; if $\mathbf{y}_1(t) = 0$, then according to observability property and equality $\mathbf{G}_1(0) = 0$ all trajectories of the system belong to an invariant submanifold $\mathbf{x}_1 = 0$. Therefore switching under these conditions can be excluded from further consideration. If $\mathbf{y}_2(t) = 0$, then simply $W_1(t) = W_2(t)$. Let $k \ge 0$ be arbitrary and consider two consecutive subintervals $[t_{2k}, t_{2k+1}) \in \mathcal{T}_3$ and $[t_{2k+1}, t_{2k+2}) \in \mathcal{T}_4$. According to the previous discussion in time instant t_{2k+1} equalities

$$\mathbf{y}_2(t_{2k+1}) = 0,\ W_1(t_{2k+1}) = W_2(t_{2k+1})$$

are satisfied. Note also that by definitions equality $W_1(t) \le W_2(t)$ holds for all $t \ge 0$. So using (3), (4), the following properties are true:

$$t \in [t_{2k}, t_{2k+1}) \in \mathcal{T}_3 \Rightarrow W_1(t) \le W_1(t_{2k});$$

$$t \in [t_{2k+1}, t_{2k+2}) \in \mathcal{T}_4 \Rightarrow$$

$$W_1(t) \le W_2(t) \le W_2(t_{2k+1}) = W_1(t_{2k+1}) \le W_1(t_{2k}),$$

and function W_1 does not increase on these subintervals, due to k was chosen arbitrary it is possible to receive boundedness property of the system solution.

Let us base convergence to zero of variable $\mathbf{x}_1$. Note, that from inequalities (3), (4) and $a_2(\mathbf{x}_2) \le 0$ property

$$\int_o^t a_1(\mathbf{x}_1(\tau))d\tau \le C < +\infty$$

holds for any $t \ge 0$ (using closely connected arguments as in Hespanha, *et al.* (2002)). Combining this fact with boundedness of $\mathbf{x}$, $\dot{\mathbf{x}}$ it is possible to obtain attractiveness property for variable $\mathbf{x}_1$. ■

The last Theorem does rely on existence of $\dot{z}$, hence requirements to signal $z(t)$ can be weakened to simply continuity and Lebesgue measurability. Unfortunately, observability properties imposed in the Theorem and condition $\mathbf{G}_1(0)=0$ restrict applicability of this result. For adaptive systems condition $\mathbf{G}_1(0)=0$ typically holds due to it is usually supposed, that $|\omega(\mathbf{x},t)| \le \rho(|\mathbf{x}|)$ for $\rho \in \mathcal{K}$ and $t \ge 0$. Additionally, adaptation algorithm, like introduced above, is always observable with respect to desired output $\widehat{\theta} - \theta$.

4. CONCLUSION

In this work conditions for feedback connection of dynamical systems are developed, which ensure robustness/invariance of stability properties with respect to multiplicative disturbances. Multiplicative disturbance presence is originated by different time scales phenomena in the system or external influences on measurement devices. It is supposed that systems in feedback connection are passive or strictly passive. Proposed conditions guarantee that multiplicative disturbance presence does not break passivity property of feedback connection of systems of this class. Main applications of presented results are backstepping control and adaptive output control.

5. ACKNOWLEDGEMENT

The paper is supported by Russian Science Support Foundation and by Program of Russian Academy of Science Presidium № 19.

REFERENCES

Angeli, D. and E.D. Sontag (1999). Forward completeness, unboundedness observability, and their Lyapunov characterizations. *Systems and Control Letters*, **38**, pp. 209 – 217.

Byrnes, C.I., A. Isidori and J.C. Willems (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Aut. Contr.*, **36**, 11, pp. 1228 – 1240.

Chen, G., J.L. Moiola and H.O. Wang (2000). Bifurcation control: theories, methods, and applications. *Intern. J. of Bifurcation and Chaos*, **10**, pp. 511 – 548.

Fomin, V.N., A.L. Fradkov and V.A. Yakubovich (1981). *An adaptive control of dynamic plants*. Moscow: Science. (in Russian)

Fradkov, A.L., I.V. Miroshnik and V.O. Nikiforov (1999). *Nonlinear and adaptive control of complex systems*. Kluwer Academic Publishers.

Hespanha, J.P., D. Liberzon and E.D. Sontag (2002). Nonlinear observability and invariance principle for switched systems. *Proc. of 41th IEEE CDC'02*, Las Vegas.

Hill, D. and P. Moylan (1977). Stability results for nonlinear feedback systems. *Automatica*, **13**, pp. 377 – 382.

Hill, D. and P. Moylan (1980). Dissipative dynamical systems: Basic input – output and state properties. *J. Franklin Inst.*, **309**, pp. 327 – 357.

Hill, D. and P. Moylan (1991). The stability of nonlinear dissipative systems. *IEEE Trans. Aut. Contr.*, **21**, pp. 708 – 711.

Jiang, Z.-P., A. Teel and L. Praly (1994). Small – gain theorem for ISS systems and applications. *Math. Control Signal Systems*, **7**, pp. 95 – 120.

Kanellakopoulos, I. (1997). Robust Nonlinear Control Design with Input and Measurement Disturbances. *Proc. American Control Conference*, Albuquerque, pp. 1283 – 1286.

Krstić, M., I. Kanellakopoulos and P.V. Kokotović (1995). *Nonlinear and adaptive control design*. New York Wiley.

Kim, Y.-W., G. Rizzoni and V. Utkin (1998). Automotive engine diagnostics and control via nonlinear estimation. *IEEE Control Systems*.

Nešić, D., A.R. Teel and P.V. Kokotović (1999). Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximation. *Systems and Control Letters,* **38**, pp. 259 – 270.

Popov, V. M. (1973). *Hyperstability of Control Systems*. Springer Verlag, Berlin. (Romanian version in 1966)

Polushin, I.G., A.L. Fradkov and D. Hill (2000). Passivity and passification of nonlinear systems. *Automation and Remote Control*, **3,** pp. 3 – 33.

Rumyantsev, V.V. and A.S. Oziraner (1987). *Stability and stabilization of motion with respect to part of variables*. Moscow: Science. (in Russian)

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Aut. Contr.*, **34**, pp. 435 – 443.

Sontag, E.D. (1998). Comments on integral variants of ISS. *Systems and Control Letters*, **34**, pp. 93 – 100.

Teel, A.R., D. Nešić and P.V. Kokotović (1998). A note on input-to-state stability of sampled data nonlinear systems. *Proc. of 37th IEEE CDC'98*, Tampa, pp. 2473 – 2479.

Vorotnikov, V.I. (1998). *Partial Stability and Control*. Boston e.t.c.: Birkhauser.

Willems, J.C. (1972). Dissipative dynamical systems – Part I: General theory. *Arch. Rational Mechanics and Analysis*, **45**, pp. 321 – 351.