Automation and Remote Control, Vol. 66, No. 2, 2005, pp. 249–264. Translated from Avtomatika i Telemekhanika, No. 2, 2005, pp. 92–107.

Oscillation Conditions of Nonlinear Systems with Static Feedback[1]

D. V. Efimov and A. L. Fradkov

Institute of Problems of Mechanical Engineering, Russian Academy of Sciences, St. Petersburg, Russia
Received June 2, 2004

Abstract—Conditions are proposed of the availability of the oscillation property in the sense of Yakubovich in a system with a nonlinear nominal portion enclosed by static nonlinear output feedback. A technique is worked out to calculate analytical estimates for the amplitude of oscillations in the system. The relation between estimates for the amplitude of oscillations and the index of excitability of the system with respect to the input is established. Examples are given of the computer modeling for systems of the 2nd and the 3rd order, including Van der Pol and Lorentz systems confirming the applicability of the suggested solutions.

1. INTRODUCTION

Most of the works on the analysis or synthesis of nonlinear dynamic systems deal to some extent or another with the issue involving the stability of solutions of a closed system. The results of these works establish the proximity of the solutions of a system to prescribed limiting trajectories (equilibrium states, limiting cycles), which define the desired behavior of the system. Studying and estimating the deviation of a system from the desired limiting motion, it is possible to obtain the qualitative and the quantitative information on the behavior of the system [1–3].

In recent years, interest has grown in problems in which the random or oscillatory limiting mode of motion of a system, which is not fixed beforehand, can be taken to act as a desired mode. This leads to the necessity of developing methods of and approaches to the analysis and synthesis of complex irregular oscillatory motions. The classical solutions in this domain, which rest on the use of properties of the orbital stability, the stability in the sense of Zhurkovskiy, and the stability with respect to a portion of variables often give no way of obtaining the complete quantitative and qualitative characteristic of the behavior of a system for complex irregular oscillations [4, 5]. One of the possible approaches taken to overcome the arising difficulties relies on the notion of an index of excitability of a nonlinear system [6, 7] and makes it possible, using the excitability index, to estimate the amplitude of oscillations in a system excited by a bounded control.

An important and practically useful approach to the study of complex oscillatory modes of motion relies on the oscillation concept introduced by V.A. Yakubovich in 1973 [8]. In the framework of this approach, frequency conditions of oscillation are defined for the class of Lurie systems, each consisting of a nominal linear portion and a nonlinear output feedback path, in which case the Lyapunov function used for the investigation of a system is set up by relying on the quadratic form of variables of the state of a linear portion [4, 8, 9].

However, in the study of many physical and mechanical processes, the decomposition of a system not into a linear and a nonlinear portion but into two nonlinear portions appears to be more natural (for example, complex mechanical systems with the energy function that performs the role of the

[1] This work was supported by the Russian Foundation for Basic Research, project no. 02-01-00765, the Fund of assistance of the native science, and Program no. 19 of the Presidium of the Russian Academy of Sciences, project no. 1.4.

Lyapunov function for a portion of the system). The development of the methods of analysis and synthesis for this class of systems is a new and perspective problem of the modern theory of control.

This work contains one of the possible approaches to a solution of the problem of the detection and excitation of irregular oscillations for the class of nonlinear dynamic systems made up by way of the connection of nonlinear subsystems. Useful auxiliary definitions and properties are set out in Section 2 (two preliminary results and proofs are taken out to the Appendix). The basic definitions and conditions of the existence of oscillations are stated in Section 3. Section 4 contains a solution of the problem for synthesis of static output feedback, which ensures the origin of oscillations in a closed system with prescribed constraints on the amplitude of oscillations. The final conclusions and remarks are collected in Section 5. The examples based on the computer modeling accompany the fundamental results of the work.

2. PRELIMINARY RESULTS

Let a model of the dynamic system be represented in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}); \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in R^n$ is the state vector of the system; $\mathbf{u} \in R^m$ is the input vector; $\mathbf{y} \in R^p$ is the output vector; and $\mathbf{f}$ and $\mathbf{h}$ are locally Lipshitzian continuous functions on R^n, $\mathbf{h}(0) = 0$ and $\mathbf{f}(0) = 0$. The solution $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)$ of system (1) with the initial conditions $\mathbf{x}_0 \in R^n$ and the input $\mathbf{u}$ is defined as a minimum in the local way for $t \le T$, $T \ge 0$, and $\mathbf{y}(\mathbf{x}_0, \mathbf{u}, t) = \mathbf{h}(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t))$ (from here on, the symbols $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are used if all the remaining arguments are understood from the context). If $T = +\infty$, then the system displays the property of the continuability of solutions.

It is said that the function $\rho : R_{\ge 0} \to R_{\ge 0}$ belongs to the class $\mathcal{K}$ if it strictly increases and $\rho(0) = 0$; $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(s) \to \infty$ for $s \to \infty$ (the property of radial unboundedness); and the Lebesgue measurable function $\mathbf{x} : R_{\ge 0} \to R^n$ is bounded almost everywhere if

$$\|\mathbf{x}\| = ess\ \sup\{|\mathbf{x}(t)|, t \ge 0\} < +\infty,$$

where $|\cdot|$ denotes the norm of the vector in the Euclidean space, $R_{\ge 0} = \{\tau \in R : \tau \ge 0\}$. The following property with small changes was introduced in [10, 11].

Definition 1. System (1) is said to be passive with a continuous function $V : R^n \to R_{\ge 0}$ and $\beta : R^n \to R_{\ge 0}$ if for $\forall \mathbf{x}_0 \in R^n$ and the Lebesgue measurable input $\mathbf{u} : R_{\ge 0} \to R^m$, $t \ge 0$, the following inequality is met:

$$V(\mathbf{x}(t)) \le V(\mathbf{x}_0) + \int_0^t \omega(\mathbf{x}(\tau), \mathbf{y}(\tau), \mathbf{u}(\tau)) d\tau, \quad \omega(\mathbf{y}, \mathbf{u}) = \mathbf{y}^{\mathrm{T}}\mathbf{u} - \beta(\mathbf{x}). \tag{2}$$

The functions ω and V are called the functions of expenditure and reserve for system (1), respectively, and the function β specifies the rate of dissipation in the system.

If $\beta(\mathbf{x}) \ge \widehat{\beta}(|\mathbf{x}|)$, $\widehat{\beta} \in \mathcal{K}$, then such a system is called strictly passive. If inequality (2) can be written in the form of the equality, then it is said that system (1) is endowed with the property of passivity with the known dissipation rate β. In the case of the continuous differentiability of the reserve function, inequality (2) can be rewritten in a simpler form

$$\dot{V}(\mathbf{x}, \mathbf{u}) = L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x}) \le \omega(\mathbf{x}, \mathbf{u}, \mathbf{y}).$$

The following two properties are used in the work for the formulation of basic results in Sections 3 and 4.

Definition 2. System (1) is called h-dissipative if for this system there exists a continuously differentiable reserve function V and functions $\sigma \in \mathcal{K}$ and α, $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, such that for $\forall \mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$, we have

$$\underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|), \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x}) \leq -\alpha(|\mathbf{y}|) + \sigma(|\mathbf{u}|).$$

The h-dissipativity property was introduced in [12] with small changes. In this case, the expenditure function is $\omega(\mathbf{y}, \mathbf{u}) = -\alpha(|\mathbf{y}|) + \sigma(|\mathbf{u}|)$. The $\mathbf{y}$-strictly passive systems serve as an important example of this kind of systems [11]. The passive systems (1) can be endowed with the h-dissipativity property with the use of appropriate static output feedback.

Definition 3 ([13]). System (1) is said to be stable (displays the stability) from the input-output to the state vector (SIOSV) if for this system there exists a function W with the following properties.

(1) There exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ for which

$$\alpha_1(|\mathbf{x}|) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|) \quad \text{at} \quad \forall\, \mathbf{x} \in R^n;$$

(2) The function W is continuously differentiable and there exist functions $\alpha_3 \in \mathcal{K}_\infty$, $\sigma_1, \sigma_2 \in \mathcal{K}$ such that for $\forall\, \mathbf{x} \in R^n$ and $\forall\, \mathbf{u} \in R^m$ the following inequality is valid:

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})} W(\mathbf{x}) \leq -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|).$$

The function W with the above properties is called the SIOSV Lyapunov function. If in Definition 3 we set $\sigma_2(s) \equiv 0$, then it reduces to the formulation of the property of stability from the input to the state vector (SISV) [14]. In this case, for system (1) there exists a SISV Lyapunov function [15], i.e., a continuously differentiable function U, for which the following inequalities are fulfilled at all $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$:

$$\alpha_4(|\mathbf{x}|) \leq U(\mathbf{x}) \leq \alpha_5(|\mathbf{x}|), \quad \alpha_4, \alpha_5 \in \mathcal{K}_\infty;$$
$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})} U(\mathbf{x}) \leq -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|), \quad \alpha_6 \in \mathcal{K}_\infty, \quad \delta \in \mathcal{K}.$$

The relation of the properties introduced in Definitions 2 and 3 to the SISV property is defined by the result of Lemma A.1 (see the Appendix), which is proved in [16] for the case of more bounded requirements on the h-dissipative reserve function:

$$\alpha_7(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \alpha_8(|\mathbf{x}|), \quad \alpha_7, \alpha_8 \in \mathcal{K}_\infty.$$

The most common result in this direction were obtained in [2], in which for system (1) the equivalence between the SISV property and the properties of the SIOSV and the stability from the input to the output was proved (the former property is related to the h-dissipativity property; the more detailed information on this property can be found in [17]). Further, relying on the result of Lemma A.1 and the introduced properties, we will undertake to develop the conditions of existence of oscillations in the system of the form (1).

3. CONDITIONS OF EXISTENCE OF OSCILLATIONS

We will define the meaning of the term “oscillation” because of interest to us are primarily irregular, nonperiodic oscillations. There exist a few approaches to the definition and understanding of oscillatory processes for dynamic nonlinear systems [18]. The results of this work elaborate the approach suggested in [4, 8, 9] with modifications that are required to account for the generality of the form of the assignment and the high dimension of system (1).

Definition 4. The solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in R^n$ of system (1) is called the $\lfloor \pi^-, \pi^+ \rfloor$-oscillation with respect to the output $\psi = \eta(\mathbf{x})$ (where $\eta : R^n \to R$ is a continuous function) if the solution of system (1) is defined for all $t \geq 0$ and

$$\varliminf_{t\to+\infty} \psi(t) = \pi^-; \quad \varlimsup_{t\to+\infty} \psi(t) = \pi^+; \quad -\infty < \pi^- < \pi^+ < +\infty.$$

The solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ at $\mathbf{x}_0 \in R^n$ of system (1) is called oscillatory if for this solution there exists a certain output ψ such that this solution is the $\lfloor \pi^-, \ \pi^+ \rfloor$-oscillation with respect to the output ψ for the some $-\infty < \pi^- < \pi^+ < +\infty$. System (1) at $\mathbf{u}(t) \equiv 0$, $t \geq 0$ is called oscillatory if at almost all $\mathbf{x}_0 \in R^n$, the solutions $\mathbf{x}(\mathbf{x}_0, 0, t)$ of the system are oscillatory.

The oscillation property stated in Definition 4 is presented for the zero input and arbitrary initial conditions in system (1). The following property develops the property that is already introduced in the case of the nonzero input for the prescribed initial conditions [7].

Definition 5. Let $\mathbf{u} : R_{\geq 0} \to R^m$ be Lebesgue measurable and almost everywhere bounded time functions $t \geq 0$ and $\mathbf{x}_0 \in R^n$, such that $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)$ is defined for all $t \geq 0$. Then, the functions $\chi_\psi^-(\gamma)$ and $\chi_\psi^+(\gamma)$, called the lower and the upper index of excitability of system (1) at the point $\mathbf{x}_0$ with respect to the output $\psi = \eta(\mathbf{x})$ (where $\eta : R^n \to R$ is a certain continuous function), are specified for $0 \leq \gamma < +\infty$ in the following way:

$$\left(\chi_{\psi,\mathbf{x}_0}^-(\gamma), \chi_{\psi,\mathbf{x}_0}^+(\gamma)\right) = \arg\sup_{(a,b)\in\mathcal{E}(\gamma)} \{b - a\},$$

$$\mathcal{E}(\gamma) = \left\{ (a,b) = \begin{pmatrix} \chi_{\psi,\mathbf{x}_0}^-(\mathbf{u}) = \varliminf_{t\to+\infty} \eta(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)) \\ \chi_{\psi,\mathbf{x}_0}^+(\mathbf{u}) = \varlimsup_{t\to+\infty} \eta(\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)) \end{pmatrix} \right\}_{\|\mathbf{u}\|\leq\gamma}.$$

The lower and the upper index of excitability with respect to the output ψ for system (1), which displays the property of continuability of solutions, are defined as

$$\chi_\psi^-(\gamma) = \inf_{\mathbf{x}_0\in R^n} \chi_{\psi,\mathbf{x}_0}^-(\gamma), \quad \chi_\psi^+(\gamma) = \sup_{\mathbf{x}_0\in R^n} \chi_{\psi,\mathbf{x}_0}^+(\gamma)$$

(on the assumption that the introduced quantities are finite).

The obtained characteristic turns out to be closely related to the Cauchy gain investigated recently in [19] (for the problem of suppressing oscillations). Indeed, $\pi^+ - \pi^-$ or $\chi_{\psi,\mathbf{x}_0}^+(\mathbf{u}) - \chi_{\psi,\mathbf{x}_0}^-(\mathbf{u})$ are asymptotic amplitudes of the signal $\psi(t)$ in the sense of [19] for the cases of the input $\mathbf{u}$ equal to zero and different from zero, while $\chi_\psi^+(\gamma)$ serves to be the estimate of the Cauchy gain for system (1).

In a similar way, we can introduce the indices of excitability with respect to an arbitrary vector output $\boldsymbol{\psi} = \boldsymbol{\eta}(\mathbf{x})$. In this case, the number of indices will coincide with the dimension of the vector $\boldsymbol{\psi}$. The latter property was introduced only for fixed initial conditions in view of the fact that the indices of excitability for arbitrary initial conditions may have a complex form of the dependence on the norm of the output. For example, if $\chi_{\psi,\mathbf{x}_a}^+(\gamma) = +\infty$ for $\mathbf{x}_a \in A \subset R^n$ at a selected γ, then by analogy we can introduce for consideration the indices of excitability on the prescribed set R^n/A.

The fulfillment of the equalities $\pi^- = \chi_\psi^-(0)$ and $\pi^+ = \chi_\psi^+(0)$ also specifies the indices of excitability. For the nonzero input, a value of $\chi_\psi^+(\gamma) - \chi_\psi^-(\gamma)$ correspond to a maximum asymptotic amplitude of a signal ψ (on the prescribed set of inputs $\|\mathbf{u}\| \leq \gamma$). Consequently, the excitability indices define the ability of the system to operate at forced or controllable oscillations caused by an input and bounded by a value of γ. The maximum for all admissible inputs with a prescribed amplitude is used in Definition 5 in view of the fact that in the general case, it is always possible to find for a given system a certain input with a preset maximum amplitude, which disrupts its oscillatory properties (some signals at the input of the system may excite it and other signals may stabilize it).

It makes sense to point out the expediency of calculating the estimates of the functions $\chi_\psi^-(\gamma)$ and $\chi_\psi^+(\gamma)$ for all values $0 \le \gamma < +\infty$. Indeed, it is possible that for a given system, the maximum amplitude of oscillations is reached for some level γ^* of an input signal, and for all $\gamma \ge \gamma^*$ the amplitude of oscillations decreases but the indices $\chi_\psi^-(\gamma)$ and $\chi_\psi^+(\gamma)$ retain their values. Consequently, to determine the critical level γ^* of the input action, it is necessary to construct the complete plots of the functions $\chi_\psi^-(\gamma)$ and $\chi_\psi^+(\gamma)$.

On the other hand, the excitability indices specify the robustness of the oscillation property introduced in Definition 4 in the following sense. If for some $0 < \gamma^* \le \gamma < +\infty$ the equality $\chi_\psi^-(\gamma) = \chi_\psi^+(\gamma)$ is valid, then this implies that the system loses the oscillation property for input signals with the amplitude that is higher than γ^*. However, it does not follow from this that any input (with the amplitude that is higher than γ^*) breaks down oscillations in the system in view of the fact that the maximum over all inputs is sought in Definition 5. A similar conclusion is valid for the case of $\chi_\psi^+(\gamma) = +\infty$. The conditions of the existence of oscillations in system (1) are stated in the following theorem.

Theorem 1. *Let for system* (1) *there exist two continuously differentiable Lyapunov functions* $V_1 : R^n \to R_{\ge 0}$ *and* $V_2 : R^n \to R_{\ge 0}$ *that satisfy the inequalities*

$$\upsilon_1(|\mathbf{x}|) \le V_1(\mathbf{x}) \le \upsilon_2(|\mathbf{x}|), \quad \upsilon_3(|\mathbf{x}|) \le V_2(\mathbf{x}) \le \upsilon_r(|\mathbf{x}|), \quad \upsilon_1\upsilon_2, \upsilon_3, \upsilon_4 \in \mathcal{K}_\infty,$$
$$l_1(\mathbf{x}, \mathbf{u}) \le L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V_1(\mathbf{x}), \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V_2(\mathbf{x}) \le l_2(\mathbf{x}, \mathbf{u}).$$

Let solutions of the system with static feedback $\mathbf{u} = \mathbf{k}(\mathbf{x})$ *be defined locally as a minimum and*

$$l_1(\mathbf{x}, \mathbf{k}(\mathbf{x})) > 0 \quad \textit{for} \quad 0 < |\mathbf{x}| < X_1 \quad \textit{and} \quad \mathbf{x} \notin \Xi;$$
$$l_2(\mathbf{x}, \mathbf{k}(\mathbf{x})) < 0 \quad \textit{for} \quad |\mathbf{x}| > X_2 \quad \textit{and} \quad \mathbf{x} \notin \Xi, \quad X_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(X_2),$$

where $\Xi \subset R^n$ *is a certain set of the zero measure. If the set*

$$\Omega = \left\{ \mathbf{x} : \upsilon_2^{-1} \circ \upsilon_1(X_1) < |\mathbf{x}| < \upsilon_2^{-1} \circ \upsilon_4(X_2) \right\}$$

does not contain equilibrium positions of the closed system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$, *then the system is oscillatory.*

Proof. In the analysis of properties of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$, we will restrict the consideration to the set of initial values of the state vector, which does not contain the equilibrium positions of the system (which form the set Ξ). Then, according to the conditions of the theorem, solutions of the system are defined locally as a minimum, and from the condition

$$\dot{V}_2 < 0 \quad \text{for} \quad |\mathbf{x}| > X_2$$

the global boundedness of solutions of the system follows (determined in this case for all $t \ge 0$). In view of the boundedness of the trajectory $\mathbf{x}(t)$, $t \ge 0$, for this trajectory there exists a nonempty compact and closed ω-limited set contained in the set Ω. Indeed, the function $V_2(t)$ asymptotically satisfies the inequality $V_2(t) < \upsilon_4(X_2)$ at $|\mathbf{x}(t)| < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. Using a similar reasoning, it is possible to show the property of the boundedness above of the function $V_1(t)$, the values of which in asymptotics satisfy the estimate $V_1(t) > \upsilon_1(X_1)$, from which $|\mathbf{x}(t)| > \upsilon_2^{-1} \circ \upsilon_1(X_1)$. Based on the assumption, the set Ω does not contain equilibrium positions of the closed system and hence the ω-limiting set of the trajectory $\mathbf{x}(t)$ does not also include such invariant subsets. Then, there exists an index i, $1 \le i \le n$, such that the solution is the $\lfloor \pi^-,\ \pi^+ \rfloor$-oscillation with respect to the output $\psi = |x_i|$ for $\upsilon_2^{-1} \circ \upsilon_1(X_1) < \pi^- < \pi^+ < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. Let us assume that this output does not exist, which means that at all $1 \le i \le n$ the equality $\psi = |x_i|$ is fulfilled for $\pi^- = \pi^+$. However, a similar equality can be fulfilled only for equilibrium positions that, according to the assumption, are excluded from the set Ω, which is the contradiction. Consequently, for almost all initial conditions there exists an oscillatory output, which, by Definition 4, implies the property of oscillation of system (1) with feedback $\mathbf{u} = \mathbf{k}(\mathbf{x})$.

Note 1. As in [9], it is possible to use the Lyapunov function for the system linearized in the neighborhood of the origin of coordinates as the function V_1 for defining the property of the local instability of the system at zero. Further, the requirement for the existence of the Lyapunov function V_2 can be reduced to the property of the boundedness of solutions of the system $\mathbf{x}(t)$ with the known upper bound, which can be estimated with the use of other approaches that do not concern the analysis of properties of the time derivative of some Lyapunov functions. In this case, the assertion of Theorem 1 can be brought closer to the result of Theorem 3.2 in [20].

Example 1. Let us consider the Van der Pol model

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2,$$

where the parameter $\varepsilon > 0$. According to Theorem 1, to verify this system for the availability of the oscillation property, it is necessary to implement two Lyapunov functions that enable us to establish the local instability and the global boundedness of solutions of the system. Let us note that the given system has only one equilibrium position at the origin of coordinates and hence the set Ω will not contain this equilibrium position. We will analyze the properties of the following Lyapunov functions:

$$V_1(\mathbf{x}) = 0.5\left(x_1^2 + x_2^2\right);$$
$$V_2(\mathbf{x}) = 0.5\left(\varepsilon^{-1}x_2 - 2x_1 + 1/3\,x_1^3\right)^2 + 1/12\,x_1^4,$$

the complete time derivatives of which, taken in view of equations of the system, have the form

$$\dot{V}_1 = \varepsilon x_2^2 - \varepsilon x_2^2 x_1^2;$$
$$\dot{V}_2 = -\left[\frac{\sqrt{\varepsilon}}{2}\left(2 - \frac{1}{\varepsilon^2}\right)x_1 - \frac{x_2}{\sqrt{\varepsilon}}\right]^2 - \frac{x_1^4}{3\varepsilon} + \left[\frac{\varepsilon}{4}\left(2 - \frac{1}{\varepsilon^2}\right)^2 + \frac{2}{\varepsilon}\right]x_1^2.$$

The function $\dot{V}_1$ is strictly positive for all $0 < |x_1| < 1$ and $x_2 \ne 0$, but the submanifold $x_2 = 0$ does not contain invariant solutions of the system beyond the equilibrium position at the origin of coordinates and thus $\dot{V}_1(t) > 0$ for almost all $t \ge 0$ such that $0 < |x_1(t)| < 1$ and

$$|\mathbf{x}| < X_1 \Rightarrow \dot{V}_1 \ge 0,$$

where $X_1 = 1$. Let us note that similar conclusions were obtained in [21] for $X_1 = \sqrt{3}$. The local instability can also be verified with the use of the properties of a linearized system, whose

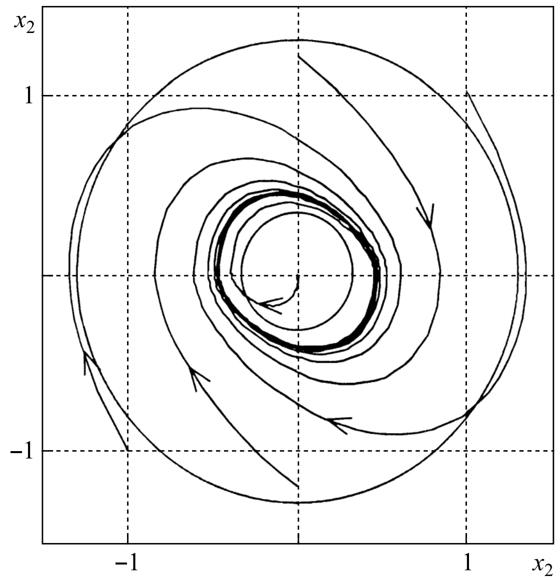

Fig. 1. Trajectories and the set Ω for the Van der Pol system.

eigenvalues always have a positive real part for $\varepsilon > 0$:

$$\lambda_{1,2} = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}.$$

Investigating the function $\dot{V}_2$, we can obtain the inequality

$$X_2 \leq \sqrt{3\left[\frac{\varepsilon^2}{4}\left(2 - \frac{1}{\varepsilon^2}\right)^2 + 2\right]}.$$

In this example, the functions $v_1(s) = v_2(s) = 0.5\,s^2$ and the functions $v_3(s)$ and $v_4(s)$ can be derived numerically for a given value ε. The results of the calculation of the set Ω and the computer modeling of trajectories of the system for $\varepsilon = 1$ are displayed in Fig. 1, where dark circles define the size and the form of the set Ω. Let us point out that in [22], the nonquadratic Lyapunov function with a discontinuous time derivative was suggested for the Van der Pol system.

Example 2. Let us examine a model of the Lorentz system

$$\dot{x} = \sigma(y - x);$$
$$\dot{y} = rx - y - xz;$$
$$\dot{z} = -bz + xy,$$

where the parameters $\sigma = 10$, $r = 97$, and $b = 8/3$. It is known that the selected model displays a chaotic mode of motion at prescribed values of the parameters. We will demonstrate the application of Theorem 1, taking, as an example, this system. First of all, we will note that the linear approximation matrix of the system at the origin of coordinates

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

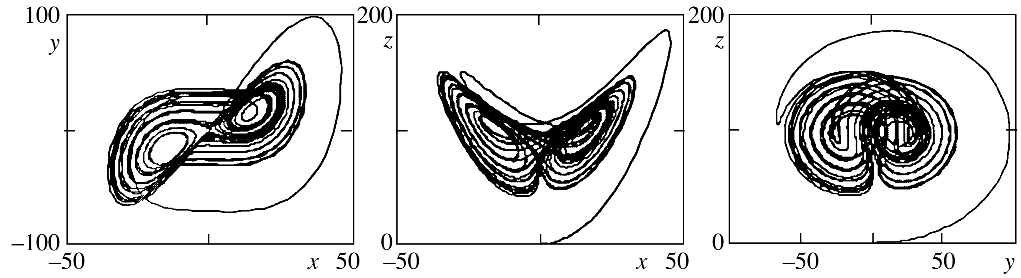

Fig. 2. Trajectories of the Lorentz system.

has one eigenvalue with the positive real part at the selected values of the parameters. Consequently, the system is locally unstable. Let us analyze the total time derivative of the function

$$V(x, y, z) = 0.5\left(\sigma^{-1}x^2 + y^2 + (z - r)^2\right).$$

In view of the system under consideration, for the derivative there is an inequality

$$\dot{V} = -x^2 + xy - y^2 - bz^2 + rbz \leq -0.5\,x^2 - 0.5\,y^2 - 0.5\,bz^2 + 0.5\,br^2,$$

from which the global boundedness of trajectories of the system follows. According to Remark 1, all requirements of Theorem 1 are satisfied and this system is oscillatory in the sense of Definition 4. An example of the phase trajectory is shown in Fig. 2.

Let us note that the properties from Definition 4 were introduced for the zero input $\mathbf{u} = 0$, but in Theorem 1, the system with the control $\mathbf{k}$ was used. There are two causes of the mention of feedback in the statement of Theorem 1. First, this enables us to point to one of the possible approaches to the application of given results in practice, which relies on the decomposition of the initial system into the nominal portion and static feedback. Second, such a statement of the result facilitates the establishment of the relation between the oscillation property and the index of excitability of the system, as this is done in the following corollary.

Corollary 1. *Let all the conditions of Theorem 1 be fulfilled for system* (1) *and the solution* $\mathbf{x}(\mathbf{x}_0, \mathbf{k}(\mathbf{x}), t)$, $\mathbf{x}_0 \in R^n$, *be the* $\lfloor\pi^-,\ \pi^+\rfloor$*-oscillation relative to some output* $\psi = \eta(\mathbf{x})$ *in the sense of Definition 4, so that*

$$\pi^+ - \pi^- \leq \upsilon_3^{-1} \circ \upsilon_4(X_2) - \upsilon_2^{-1} \circ \upsilon_1(X_1), \quad \pi^+ - \pi^- \leq \chi^+_{\psi,\mathbf{x}_0}(\gamma) - \chi^-_{\psi,\mathbf{x}_0}(\gamma),$$

where $\gamma \geq \gamma^*$, $\gamma^* = \sup\limits_{|\mathbf{x}| \leq \Gamma} |\mathbf{k}(\mathbf{x})|$, $\Gamma = \upsilon_3^{-1} \circ \upsilon_4(\max\{X_2, |\mathbf{x}_0|\})$.

Proof. According to the results of Theorem 1, the solution of system (1) with feedback $\mathbf{k}$ satisfies the constraint $|\mathbf{x}(t)| \leq \Gamma$ at all $\mathbf{x}_0 \in R^n$ and $t \geq 0$. Then, the input $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is bounded above by a value γ^* and the result follows from Definitions 4 and 5.

Consequently, to calculate estimates of the indices of excitability, it is sufficient to find a certain control $\mathbf{k}$ for system (1), which ensures the oscillation ability of the closed system.

In the proof of Theorem 1, the norm of a component of the state vector of the system was suggested as an example of the oscillatory output, but the availability of such an output precludes the complete uncovering of all specific features of the oscillations observed in the system. This is one of the disadvantages of the suggested solution because the result does not afford the constraints on the set of possible oscillatory variables in the system. To overcome this disadvantage, it is possible to reformulate the assertion of Theorem 1 in the case of the property of oscillation with respect to the output $\mathbf{y}$ in the following way:

$$\upsilon_1(|\mathbf{y}|) \leq V_1(\mathbf{x}) \leq \upsilon_2(|\mathbf{y}|), \quad \upsilon_3(|\mathbf{y}|) \leq V_2(\mathbf{x}) \leq \upsilon_4(|\mathbf{y}|),$$
$$l_1(\mathbf{x}, \mathbf{k}(\mathbf{x})) > 0 \quad \text{for} \quad 0 < |\mathbf{y}| < Y_1;$$
$$l_2(\mathbf{x}, \mathbf{k}(\mathbf{x})) < 0 \quad \text{for} \quad |\mathbf{y}| > Y_2, \quad Y_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(Y_2),$$

and the set $\Omega = \left\{\mathbf{y} : \upsilon_2^{-1} \circ \upsilon_1(Y_1) < |\mathbf{y}| < \upsilon_3^{-1} \circ \upsilon_4(Y_2)\right\}$. In this case, the system is oscillatory if the given set does not contain equilibrium positions of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$. A more efficient result defining the composition of oscillatory variables in the system is stated below.

Lemma 1. *Let system* (1) *satisfy all conditions of Lemma A.1. We assume that static feedback* $\mathbf{u} = \mathbf{k}(\mathbf{x})$*, where* $\mathbf{k} : R^n \to R^m$ *is a certain function ensuring the local existence of solutions in the closed system, satisfies the conditions*

i) $\alpha_6(|\mathbf{x}|) > \delta(|\mathbf{k}(\mathbf{x})|)$ *at* $|\mathbf{x}| > X \geq 0$ *and* $\mathbf{x} \notin \Xi$*;*

ii) $l(\mathbf{x}, \mathbf{u}) \leq L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x})$, $l(\mathbf{x}, \mathbf{k}(\mathbf{x})) > 0$ *at* $0 < |\mathbf{h}(\mathbf{x})| \leq Y$ *and* $\mathbf{x} \notin \Xi$

for some constants X *and* Y *on condition that* $Y < \alpha^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X)$ *(here, the functions* α_4*,* α_5*,* α_6*, and* δ *are derived in the course of the proof of Lemma A.1). The set* $\Xi \subset R^n$ *displays the zero measure. If the set* $\Omega = \{V(\mathbf{x}) : \alpha(Y) \leq V(\mathbf{x}) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X)\}$ *does not contain the equilibrium positions of the closed system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$*, then this system is endowed with the oscillation property.*

Proof. In this case, the system is provided with the SISV property with respect to the input $\mathbf{u}$ and according to Condition (i), the solution of the closed system is bounded and, hence, defined for all $t \geq 0$. As before, the variables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ have nonempty closed and compact ω-limiting sets and satisfy the estimate given below (the form of the functions α_4 and α_5 is presented in the statement of Lemma A.1):

$$|\mathbf{x}| \leq \alpha_4^{-1} \circ \alpha_5(X).$$

Using Condition (ii) of the lemma, it is possible to justify the fulfillment of the inequality $\dot{V} > 0$ for small $0 < |\mathbf{y}| \leq Y$. Then, the set of ω-limiting trajectories for the function $V(t)$ resides in the set Ω. Further, repeating the final steps of the proof of Theorem 1, we can justify the assertion of the lemma.

Here, the function V depends only on a portion of the variables, which bounds the set of oscillatory variables in the system and, in addition, points to one of the possible approaches to the derivation of the functions V_1 and V_2 from Theorem 1. The results of the suggested Theorem 1 and Lemma 1 do not concern the methods of deducing the law of feedback $\mathbf{k}$ that ensures the existence in the system of oscillatory modes of motion with desired parameters. The next section deals with the given problem.

4. CONTROL OF OSCILLATORY MODES

The material of this section relies on the results of Lemma A.2 that is the consequence of Lemma A.1 (in Lemma A.2, conditions are stated under which static feedback ensures the SISV property for a SIOSV and passive system). The conditions imposed on the law of the control $\mathbf{k}$ in Lemma A.2 display a complex mathematical formulation, but their checking for the given system creates no difficulties. For example, if σ_1 and σ_2 are quadratic functions of their own arguments, then the control $\mathbf{k}$ with the linear gain relative to $\mathbf{y}$ satisfies all constraints of the lemma. The basic result of this section is presented in the following theorem.

Theorem 2. *Let system* (1) *be passive with the known dissipation rate* β *and SIOSV in the sense of Definitions 1 and 3. Then, the control law*

$$\mathbf{u} = -\mathbf{k}(\mathbf{x}) + \mathbf{d}; \quad |\mathbf{k}(\mathbf{x})| \le \lambda(|\mathbf{y}|) + K, \quad \lambda \in \mathcal{K}, \quad 0 < K < +\infty;$$
$$\mathbf{y}^{\mathrm{T}}\mathbf{k}(\mathbf{x}) = \beta(\mathbf{x}) + \mu(|\mathbf{d}|) + \mu(K) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^{\mathrm{T}}\mathbf{d}, \quad \kappa \in \mathcal{K}_\infty, \quad \mu \in \mathcal{K};$$
$$\lim_{s\to+\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty;$$
$$\mathbf{y}^{\mathrm{T}}\mathbf{k}(\mathbf{x}) > \beta(\mathbf{x}) \quad at \quad 0 < |\mathbf{y}| < Y < +\infty,$$
$$Y < \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K),$$

where $\mathbf{d} \in R^m$ *is a new vector of the input of the system (the Lebesgue measurable time function bounded almost everywhere), ensures the following:*

(i) *the boundedness of solutions of the closed system;*

(ii) *if the set* $\Omega = \left\{V(\mathbf{x}) : \alpha(Y) \le V(\mathbf{x}) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K)\right\}$ *does not contain the equilibrium positions of the system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, -\mathbf{k}(\mathbf{x}))$, *then for* $\mathbf{d}(t) \equiv 0$, $t \ge 0$, *the closed system displays the oscillation property (here, the functions* α_4, α_5, α_6, *and* δ *are derived in the course of the proof of Lemma A.2).*

Proof. Let us consider the decomposition of the control law

$$\mathbf{u} = -\mathbf{k}(\mathbf{x}) = -\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})$$

into two additive terms satisfying the conditions

$$|\mathbf{k}_1(\mathbf{x})| \le \lambda(|\mathbf{y}|), \quad |\mathbf{k}_2(\mathbf{x})| \le K;$$
$$\mathbf{y}^{\mathrm{T}}\mathbf{k}_1(\mathbf{x}) + \beta(\mathbf{x}) + \mu(|\mathbf{d}|) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^{\mathrm{T}}\mathbf{d};$$
$$\mathbf{y}^{\mathrm{T}}\mathbf{k}_2(\mathbf{x}) > \beta(\mathbf{x}) + \mathbf{y}^{\mathrm{T}}\mathbf{k}_1(\mathbf{x}) \quad \text{for} \quad 0 < |\mathbf{y}| < Y < +\infty.$$

Such a decomposition corresponds to all conditions of the theorem. We will introduce an auxiliary input vector

$$\widetilde{\mathbf{d}} = \mathbf{d} + \mathbf{k}_2(\mathbf{x}),$$

which is bounded almost for all $t \ge 0$ according to the theorem conditions $\|\widetilde{\mathbf{d}}\| \le K + \|\mathbf{d}\|$. It can be noted that in the new designations, for system (1) and the control

$$\mathbf{u} = -\mathbf{k}_1(\mathbf{x}) + \widetilde{\mathbf{d}}$$

all the conditions of Lemma A.2 are fulfilled and the closed system is the SISV system with respect to the input $\widetilde{\mathbf{d}}$. The boundedness of solutions [14] follows from the boundedness of $\widetilde{\mathbf{d}}$ for the SISV system, and thus Assertion (i) of the theorem is proved. For the proof of Assertion (ii), we note that all the conditions of Lemma 1 are also valid.

The suggested theorem extends the results [23] in the case of a nonlinear dynamic system of the common kind. In addition, in this work, special attention is given to the lower bound of the amplitude of oscillations in the system at $\mathbf{d}(t) \equiv 0$, $t \ge 0$. It is pertinent to point out that the control suggested in the theorem is chosen on the basis of the fulfillment of some sector constraints relative to the output $\mathbf{y}$. In this case, it is possible to use the velocity gradient method [1, 24] for the synthesis of the laws of control in practical applications.

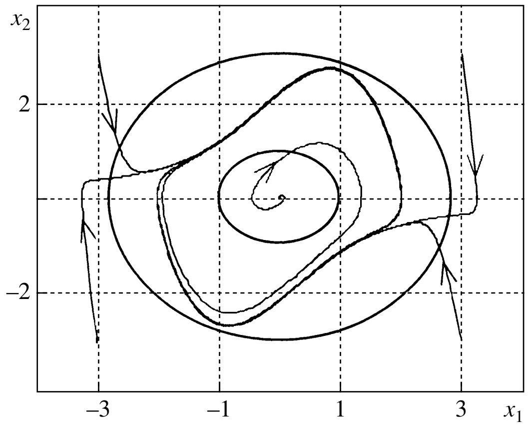

Fig. 3. Trajectories of a linear oscillator with nonlinear feedback.

Example 3. Let us consider the following model of a linear controllable system:

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = -x_1 + u,$$

provided with the passivity property relative to the reserve function

$$V(\mathbf{x}) = 0.5\left(x_1^2 + x_2^2\right), \quad \dot{V} = x_2 u,$$

and the SIOSV with the appropriate Lyapunov function

$$W(\mathbf{x}) = 0.5\left(x_1^2 + (x_1 + x_2)^2\right), \quad \dot{W} \le -0.5\left(x_1^2 + x_2^2\right) + x_2^2 + u^2$$

in the output $y = x_2$ ($\sigma_1(s) = \sigma_2(s) = s^2$). In this case, the control $u = -k_1(\mathbf{x}) + k_2(\mathbf{x})$ at $k_1(\mathbf{x}) = ax_2$, $a > 0.5$, and $k_2(\mathbf{x}) = K\,\mathrm{sgn}\,(x_2)$ conforms to all conditions of Theorem 2 for

$$\lambda(s) = as, \quad \kappa(s) = (a - 0.5)s^2, \quad \mu(s) = 0.5\,s^2.$$

All functions σ_2, $\sigma_1 \circ \lambda$, and κ are quadratic and, hence,

$$\lim_{s \to +\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty;$$

and the inequality

$$x_2 k_2(\mathbf{x}) > x_2 k_1(\mathbf{x})$$

is fulfilled for $0 < |x_2| < Y = K/a$. This system is the SISV system for the control $u = -k_1(\mathbf{x}) + d$ with the SISV Lyapunov function:

$$U(\mathbf{x}) = W(\mathbf{x}) + \frac{1 + 2a^2}{q - 0.5} V(\mathbf{x}), \quad \dot{U} \le -0.5\left(x_1^2 + x_2^2\right) + \left(2 + \frac{0.5 + a^2}{a - 0.5}\right) d^2.$$

In this case, the set $\Omega = \left\{\mathbf{x} : K/a \le |\mathbf{x}| \le \sqrt{1 + \frac{1.5a - 0.75}{a^2 + 0.5}}\sqrt{4 + \frac{1 + 2a^2}{a - 0.5}}K\right\}$ is always nonempty and does not contain the equilibrium positions of the system. The results of the modeling and the bounds of the set Ω are shown in Fig. 3 at $a = 1$ and $K = 1/3$.

In the case when the input signal $\mathbf{d}$ is different from zero, it is possible, using the results of Theorem 2 and Corollary 1, to obtain estimates for the index of excitability of a closed system.

Corollary 2. *Let all the conditions of Theorem 2 be fulfilled. Then, for* $\|\mathbf{d}\| \leq \gamma < +\infty$ *we have*

$$0 \leq \chi_V^-(\gamma) \leq \chi_V^+(\gamma) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K+\gamma).$$

If, in addition,

$$\mathbf{y}(t)^{\mathrm{T}} \mathbf{d}(t) \geq 0 \quad \textit{for all} \quad t \geq 0, \tag{3}$$

then

$$\alpha(Y) \leq \chi_V^-(\gamma) < \chi_V^+(\gamma) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K+\gamma).$$

Proof. The upper bound for the indices the excitability follows from the SISV property with respect to the input $\widetilde{\mathbf{d}}$ (the property of the asymptotic gain in [25]). Let us analyze properties of the time derivative of the function V:

$$\dot{V} = \mathbf{y}^{\mathrm{T}}\left(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}) + \mathbf{d}\right) - \beta(\mathbf{x}) \geq \left\lfloor \mathbf{y}^{\mathrm{T}}(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x}) \right\rfloor + \mathbf{y}^{\mathrm{T}}\mathbf{d}.$$

According to the conditions of Theorem 2, the expression in the square brackets is positive for $0 < |\mathbf{y}| < Y < +\infty$, but in view of the presence of the alternating term $\mathbf{y}^{\mathrm{T}}\mathbf{d}$, only the fulfillment of the inequality $0 \leq \chi_V^-(\gamma) \leq \chi_V^+(\gamma)$ can be justified in the general case. If, in addition, $\mathbf{y}(t)^{\mathrm{T}}\mathbf{d}(t) \geq 0$ for all $t \geq 0$, then

$$\left\lfloor \mathbf{y}^{\mathrm{T}}(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x}) \right\rfloor + \mathbf{y}^{\mathrm{T}}\mathbf{d} \geq \mathbf{y}^{\mathrm{T}}(-\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x})) - \beta(\mathbf{x}),$$

and the desired result can be obtained using the arguments similar to those presented in the proof of Theorem 2. Further, we assume that the situation $\chi_V^-(\gamma) = \chi_V^+(\gamma)$ is possible for a certain γ. However, according to Definition 5, the indices of excitability must satisfy the constraints

$$\gamma_1 \leq \gamma_2 \Rightarrow \chi_V^-(\gamma_2) \leq \chi_V^-(\gamma_1) \quad \text{and} \quad \chi_V^+(\gamma_1) \leq \chi_V^+(\gamma_2).$$

Using the concepts of the proof of Corollary 1, we can obtain the inequality

$$0 < \chi_V^+(0) - \chi_V^-(0) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K) - \alpha(Y),$$

and, consequently, the inequality $\chi_V^+(\gamma) - \chi_V^-(\gamma) > 0$ is also valid for any $\gamma \geq 0$.

Thus, a value of the index $\chi_V^+(\gamma)$ is always bounded above and, moreover, the lower and the upper index of excitability are not equal to each other at any $\gamma \in R_{\geq 0}$ under condition (3). Consequently, the system is not able to lose the oscillation property for any arbitrarily large disturbing input $\mathbf{d}$ satisfying the "consistency" condition (3). In addition, similar input actions do not lead to the formation of new equilibrium positions in the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}) + \mathbf{d})$ on the set $\Omega = \left\{V(\mathbf{x}) : \underline{\alpha}(Y) \leq V(\mathbf{x}) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(K+\gamma)\right\}$. It is worthwhile to note that the requirement for the fulfillment of condition (3) at all $t \geq 0$ can be reduced and stated in the form $t \geq T$, where $0 \leq T < +\infty$.

5. CONCLUSIONS

In this work, the conditions of oscillation in the sense of Yakubovich are suggested, which are used for nonlinear systems, each decomposed into two nonlinear portions. Estimates of the amplitude of oscillations in a system are found and the relation between the notions of the oscillation property and the indices of excitability are analyzed. The class of control laws in the form of state feedback is described, which ensures the oscillation property of limiting modes. The applicability of the obtained results is illustrated by the example of analysis of the oscillation property of the Van der Pol and Lorentz systems. As an auxiliary result, we will point out the suggested nonquadratic Lyapunov function, which ensures the boundedness of solutions of the Van der Pol system.

APPENDIX

Lemma A.1. *Let for the system* (1) *the SISV Lyapunov function W and the reserve function V for the h-dissipativity property be known, as in Definitions 2 and 3. If*

$$\lim_{s\to+\infty}\frac{\sigma_2(s)}{\alpha(s)}<+\infty,$$

then system (1) *is provided with the SISV property.*

Proof. According to the conditions of the lemma and Definitions 2 and 3, the following inequalities are fulfilled at all $\mathbf{x}\in R^n$ and $\mathbf{u}\in R^m$:

$$\alpha_1(|\mathbf{x}|)\le W(\mathbf{x})\le\alpha_2(|\mathbf{x}|);\quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})}W(\mathbf{x})\le-\alpha_3(|\mathbf{x}|)+\sigma_1(|\mathbf{u}|)+\sigma_2(|\mathbf{y}|);$$
$$\underline{\alpha}(|\mathbf{y}|)\le V(\mathbf{x})\le\overline{\alpha}(|\mathbf{x}|);\quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})}V(\mathbf{x})\le-\alpha(|\mathbf{y}|)+\sigma(|\mathbf{u}|),$$

where α, α_1, α_2, α_3, $\underline{\alpha}$, $\overline{\alpha}\in\mathcal{K}_\infty$ and σ, σ_1, $\sigma_2\in\mathcal{K}$. We will set up a new SIOSV Lyapunov function in the following way:

$$\widetilde{W}(\mathbf{x})=\rho(W(\mathbf{x})),\quad \rho(r)=\int\limits_0^r q(s)ds,$$

where q is a certain function from the class $\mathcal{K}$, which will be defined later on. By the construction, the function $\widetilde{W}$ is a continuously differentiable, positively defined, and radially unbounded one in view of $\rho\in\mathcal{K}_\infty$. Its time derivative satisfies the inequality

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\widetilde{W}(\mathbf{x})\le q(W(\mathbf{x}))\big[-\alpha_3(|\mathbf{x}|)+\sigma_1(|\mathbf{u}|)+\sigma_2(|\mathbf{y}|)\big].$$

To open the left side of the inequality, we will sequentially consider three situations:

(1) $0.5\,\alpha_3(|\mathbf{x}|)\ge\sigma_1(|\mathbf{u}|)+\sigma_2(|\mathbf{y}|)$, then $L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\widetilde{W}(\mathbf{x})\le-0.5\,q(W(\mathbf{x}))\alpha_3(|\mathbf{x}|)$;

(2) $0.5\,\alpha_3(|\mathbf{x}|)<\sigma_1(|\mathbf{u}|)+\sigma_2(|\mathbf{y}|)$ and $\sigma_1(|\mathbf{u}|)\le\sigma_2(|\mathbf{y}|)$, then $L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\widetilde{W}(\mathbf{x})\le 2q(W(\mathbf{x}))\sigma_2(|\mathbf{y}|)\le 2\chi(2\sigma_2(|\mathbf{y}|))\sigma_2(|\mathbf{y}|)$, where $\chi(s)=q\circ\alpha_2\circ\alpha_3^{-1}(2s)$;

(3) $0.5\,\alpha_3(|\mathbf{x}|)<\sigma_1(|\mathbf{u}|)+\sigma_2(|\mathbf{y}|)$ and $\sigma_1(|\mathbf{u}|)>\sigma_2(|\mathbf{y}|)$, then $L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\widetilde{W}(\mathbf{x})\le 2q(W(\mathbf{x}))\sigma_1(|\mathbf{u}|)\le 2\chi(2\sigma_1(|\mathbf{u}|))\sigma_1(|\mathbf{u}|)$,

Consequently, the inequality for the time derivative of the function $\widetilde{W}$, which is taken in view of the equations of system (1), can be reduced to the form

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}\widetilde{W}(\mathbf{x})\le-0.5\,q(W(\mathbf{x}))\alpha_3(|\mathbf{x}|)+2\chi(2\sigma_2(|\mathbf{y}|))\sigma_2(|\mathbf{y}|)+2\chi(2\sigma_1(|\mathbf{u}|))\sigma_1(|\mathbf{u}|).$$

Let the function χ be chosen in accordance with the equation

$$\chi(2\sigma_2(s)) = \frac{\alpha(s)}{1 + 2\sigma_2(s)}.$$

This choice of the function χ is possible in view of

$$\lim_{s\to+\infty} \frac{\sigma_2(s)}{\alpha(s)} < +\infty$$

for $q(s) = \dfrac{\alpha \circ \sigma_2^{-1}(0.25\,\alpha_3 \circ \alpha_2^{-1}(s))}{1 + 0.5\,\alpha_3 \circ \alpha_2^{-1}(s)}$ from the class $\mathcal{K}$. Then, system (1) is the SISV one with the SISV Lyapunov function $U(\mathbf{x}) = V(\mathbf{x}) + \widetilde{W}(\mathbf{x})$ ($\alpha_4(s) = \rho \circ \alpha_1(s), \alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s)$). Indeed,

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}U(\mathbf{x}) \le -0.5\,q(W(\mathbf{x}))\alpha_3(|\mathbf{x}|) + \sigma(|\mathbf{u}|) + 2\chi(2\sigma_1(|\mathbf{u}|))\sigma_1(|\mathbf{u}|) \le -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|),$$

where $\alpha_6(s) = 0.5\,q(\alpha_1(s))\alpha_3(s)$ and $\delta(s) = \sigma(s) + 2\chi(2\sigma_1(s))\sigma_1(s)$.

The next result is the consequence of Lemma A.1, which represents one of the methods of synthesis of a control law for a passive system, such that it provides the SISV property for this system.

Lemma A.2. *Let system* (1) *be passive and SIOSV in the sense of Definitions 1 and 3 and*

$$\underline{\alpha}(|\mathbf{y}|) \le V(\mathbf{x}) \le \overline{\alpha}(|\mathbf{x}|), \quad \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty.$$

Then, the control is given as

$$\mathbf{u} = -\mathbf{k}(\mathbf{x}) + \mathbf{d}, \quad |\mathbf{k}(\mathbf{x})| \le \lambda(|\mathbf{y}|), \quad \lambda \in \mathcal{K};$$
$$\mathbf{y}^{\mathrm{T}}\mathbf{k}(\mathbf{x}) + \beta(\mathbf{x}) \ge \kappa(|\mathbf{y}|) + 0.5\,|\mathbf{y}|^2, \quad \kappa \in \mathcal{K}_\infty;$$
$$\lim_{s\to+\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty,$$

where $\mathbf{d} \in R^m$ *is a new input vector (the Lebesgue measurable time function bounded almost everywhere), which imparts the SISV property to the closed system.*

Proof. According to Definitions 1 and 3, the following inequalities are fulfilled at all $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$:

$$\alpha_1(|\mathbf{x}|) \le W(\mathbf{x}) \le \alpha_2(|\mathbf{x}|); \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})}W(\mathbf{x}) \le -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|);$$
$$\underline{\alpha}(|\mathbf{y}|) \le V(\mathbf{x}) \le \overline{\alpha}(|\mathbf{x}|); \quad L_{\mathbf{f}(\mathbf{x},\mathbf{u})}V(\mathbf{x}) \le -\beta(|\mathbf{x}|) + \mathbf{y}^{\mathrm{T}}\mathbf{u}$$

with α_1, α_2, α_3, $\underline{\alpha}$, $\overline{\alpha} \in \mathcal{K}_\infty$, σ_1, $\sigma_2 \in \mathcal{K}$, and β that is a certain nonnegative function. Substituting the control into these inequalities yields

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}W(\mathbf{x}) \le -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{d} - \mathbf{k}(\mathbf{x})|) + \sigma_2(|\mathbf{y}|)$$
$$\le -\alpha_3(|\mathbf{x}|) + \sigma_1(2|\mathbf{d}|) + \sigma_1(2\lambda(|\mathbf{y}|)) + \sigma_2(|\mathbf{y}|);$$
$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})}V(\mathbf{x}) \le -\beta(|\mathbf{x}|) + \mathbf{y}^{\mathrm{T}}(\mathbf{d} - \mathbf{k}(\mathbf{x})) \le -\kappa(|\mathbf{y}|) + 0.5\,|\mathbf{d}|^2.$$

Consequently, a similar control provides for the closed system the SIOSV property and the h-dissipativity relative to the new input $\mathbf{d}$. If

$$\lim_{s\to+\infty} \frac{\widetilde{\sigma}_2(s)}{\kappa(s)} < +\infty, \quad \widetilde{\sigma}_2(s) = \sigma_2(s) + \sigma_1 \circ \lambda(s),$$

then all conditions of Lemma A.1 are met and the system is the SISV one with the SISV Lyapunov function

$$U(\mathbf{x}) = V(\mathbf{x}) + \widetilde{W}(\mathbf{x}), \quad \widetilde{W}(\mathbf{x}) = \rho(W(\mathbf{x})), \quad \rho(r) = \int\limits_0^r q(s)ds,$$

$$q(s) = \frac{\kappa \circ \widetilde{\sigma}_2^{-1}(0.25\alpha_3 \circ \alpha_2^{-1}(s))}{1 + 0.5\alpha_3 \circ \alpha_2^{-1}(s)}, \quad \alpha_4(s) = \rho \circ \alpha_1(s),$$

$$\alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s),$$

$$\alpha_6(s) = 0.5\, q(\alpha_1(s))\alpha_3(s), \quad \delta(s) = 0.5\, s^2 + 2\chi(2\sigma_1(2s))\sigma_1(2s).$$

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This paper was recommended for publication by B.T. Polyak, a member of the Editorial Board