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Excitation of Oscillations in Nonlinear Systems under Static Feedback

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Abstract— **New conditions for oscillatority of a system in sense of Yakubovich are proposed. These conditions are applicable to nonlinear systems decomposed into two nonlinear parts. Upper and lower bounds for oscillation amplitude are obtained. The relation between the oscillatority bounds and excitability indices for the systems with the input are established. Example illustrating proposed results by computer simulation is given.**

I. Introduction

MOST works on analysis or synthesis of nonlinear systems are devoted to studying stability-like behavior. Their typical results show that the motions of a system are close to a certain limit motion (limit mode) that either may exist in the system or may be created by a controller. Evaluating deflection of the system trajectory from a limit mode, one may obtain quantitative information about system behavior [11], [20].

During recent years an interest in studying more complex behavior of the systems related to oscillatory and chaotic modes has grown significantly. The works belonging to this class are mostly dealing with relaxed stability properties (orbital stability, Zhukovsky stability, partial stability) of some periodic limit modes [12], [13]. However the need for studying irregular, chaotic behavior demands for development of analysis and design methods for non-periodic oscillations. One of such methods based on the concept of excitability index is aimed at evaluation of oscillation amplitude for the systems excited with a bounded control [4], [5].

An important and useful concept for studying irregular oscillations is that of "oscillatority" introduced by V.A.Yakubovich in 1973 [23]. Frequency domain conditions for oscillatority were obtained for Lurie systems, composed of linear and nonlinear parts [12], [23], [24]. However, when studying physical systems in many cases it is more natural to decompose the system description into two nonlinear parts. Among the systems of such a class are, e.g. mechanical systems where energy plays a role of Lyapunov function. Extension of analysis and design methods to oscillations in such a class of systems is still to appear.

In this paper an approach to oscillations detecting and oscillatority feedback design for a class of nonlinear systems is suggested. Section 2 contains some useful auxiliary statements and definitions (two preliminary results are placed in Appendix). Main definitions and oscillations existence conditions are presented in Section 3. Section 4 deals with task of static feedback design, which ensures oscillations appearance in closed loop system with predefined bounds on amplitude. Conclusion finishes the paper in Section 5. An examples of analytical calculations with computer simulations of proposed solutions is included.

II. Preliminaries

Let us consider general model of nonlinear dynamical system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u})\,;\ \mathbf{y} = \mathbf{h}(\mathbf{x})\,, \qquad (1)$$

where $\mathbf{x} \in R^n$ is state space vector; $\mathbf{u} \in R^m$ is input vector; $\mathbf{y} \in R^p$ is output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous functions on R^n, $\mathbf{h}(0)=0$ and $\mathbf{f}(0)=0$. Solution $\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)$ of the system (1) with initial condition $\mathbf{x}_0 \in R^n$ and input $\mathbf{u}$ is defined at the least locally for $t \le T$, $\mathbf{y}(\mathbf{x}_0,\mathbf{u},t) = \mathbf{h}\big(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)\big)$ (further we will simply write $\mathbf{x}(t)$ or $\mathbf{y}(t)$ if all other arguments are clear from the context). If $T=+\infty$ then such system is called forward complete. In this work we will consider feedback connection of system (1) with static system $\mathbf{u}=\mathbf{k}(\mathbf{y})$.

As usual, it is said, that a function $\rho: R_{\ge 0} \to R_{\ge 0}$ belongs to class $\mathcal{K}$, if it is strictly increasing and $\rho(0)=0$; $\rho \in \mathcal{K}_\infty$ if $\rho \in \mathcal{K}$ and $\rho(s) \to \infty$ for $s \to \infty$; Lebesgue measurable function $\mathbf{x}: R_{\ge 0} \to R^n$ is essentially bounded, if $\|\mathbf{x}\| = ess\sup\{|\mathbf{x}(t)|,\ t \ge 0\} < +\infty$, where $|\cdot|$ denotes usual Euclidean norm, $R_{\ge 0} = \{\tau \in R : \tau \ge 0\}$. We will use the property of dissipativity [22].

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Definition 1. *The system (1) is dissipative if there exists continuous function $V: R^n \to R_{\geq 0}$ such that for $\forall \mathbf{x}_0 \in R^n$ and Lebesgue measurable $\mathbf{u}: R_{\geq 0} \to R^m$, $t \geq 0$ the following inequality is satisfied*

$$V(\mathbf{x}(t)) \leq V(\mathbf{x}_0) + \int_0^t \varpi(\mathbf{x}(\tau), \mathbf{y}(\tau), \mathbf{u}(\tau))\, d\tau. \quad (2)$$

The functions ϖ and V are called supply rate and storage functions of the system (1). ■

In the case when storage function is continuously differentiable, inequality (2) can be rewritten in more simpler form:

$$\dot{V}(\mathbf{x}, \mathbf{u}) = L_{\mathbf{f}(\mathbf{x},\mathbf{u})} V(\mathbf{x}) \leq \varpi(\mathbf{x}, \mathbf{u}, \mathbf{y}).$$

Definition 2. *Dissipative system (1) is called*

– *passive if $\varpi(\mathbf{x}, \mathbf{y}, \mathbf{u}) = \mathbf{y}^T \mathbf{u} - \beta(\mathbf{x})$, where β is a continuous function reflecting the dissipation rate in the system; if $\beta(\mathbf{x}) \geq \hat{\beta}(|\mathbf{x}|)$, $\beta \in \mathcal{K}$, then system (1) is called strictly passive [10];*

– *h-dissipative, if it has continuously differentiable storage function V and*

$$\underline{\alpha}(|\mathbf{y}|) \leq V(\mathbf{x}) \leq \overline{\alpha}(|\mathbf{x}|),\ \omega(\mathbf{y}, \mathbf{u}) = -\alpha(|\mathbf{y}|) + \sigma(|\mathbf{u}|),$$

$\sigma \in \mathcal{K}$, $\alpha, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$;

– *input-output-to-state stable (IOSS), if it has continuously differentiable storage function W and*

$$\alpha_1(|\mathbf{x}|) \leq W(\mathbf{x}) \leq \alpha_2(|\mathbf{x}|),\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty,$$

$$\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}) = -\alpha_3(|\mathbf{x}|) + \sigma_1(|\mathbf{u}|) + \sigma_2(|\mathbf{y}|),$$

$\alpha_3 \in \mathcal{K}_\infty$, $\sigma_1, \sigma_2 \in \mathcal{K}$ *[19];*

– *input-to-state stable (ISS), if it has continuously differentiable storage function U and [14]*

$$\alpha_4(|\mathbf{x}|) \leq U(\mathbf{x}) \leq \alpha_5(|\mathbf{x}|),\ \alpha_4, \alpha_5 \in \mathcal{K}_\infty;$$

$$\omega(\mathbf{x}, \mathbf{y}, \mathbf{u}) = -\alpha_6(|\mathbf{x}|) + \delta(|\mathbf{u}|),\ \alpha_6 \in \mathcal{K}_\infty,\ \delta \in \mathcal{K}. \quad ■$$

If inequality (2) for case $\varpi(\mathbf{x}, \mathbf{y}, \mathbf{u}) = \mathbf{y}^T \mathbf{u} - \beta(\mathbf{x})$ can be recomposed as equality, then it is said that system possesses passivity property with known dissipation rate β.

Term h-dissipativity was introduced with some differences in [2]. Important example of such kind of systems is $\mathbf{y}$-strict passive system [10]. Also passive system (1) can be transformed to h-dissipative under suitable feedback transformation.

Storage functions for IOSS and ISS systems are called Lyapunov functions [16], [19]. In papers [14] and [19] both terms were introduced as solutions properties and existence of corresponding Lyapunov functions is the equivalent characterization of ISS and IOSS properties.

The relation of properties introduced in Definition 2 is stated in the Lemma A1 (see Appendix), which was proved in [1] with more restrictive requirement for h-dissipativity storage function:

$$\alpha_7(|\mathbf{x}|) \leq V(\mathbf{x}) \leq \alpha_8(|\mathbf{x}|),\ \alpha_7, \alpha_8 \in \mathcal{K}_\infty.$$

General result in this direction was obtained in [11], where was proved that input-to-output stability (this property is closely connected with h-dissipativity, see also [17] for more details) and IOSS equivalent to ISS property for the system (1). Having in mind Lemma A1 let us continue with oscillation existence conditions development.

III. Oscillatority Conditions

At first it is necessary to give a precise definition of the term "oscillation" placed in the title of this Section and the paper. There are several approaches to define oscillation phenomena for dynamical nonlinear systems [12]. The most general one is the concept introduced by Yakubovich [23], [24]. Here we recover definitions from [23], [24] with some mild modifications [8], [12] dealing with high dimension and general form of the system.

Definition 3. *Solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in R^n$ of system (1) is called $[\pi^-, \pi^+]$-oscillation with respect to output $\psi = \eta(\mathbf{x})$ (where $\eta: R^n \to R$ a continuous function) if system (1) is forward complete and*

$$\underline{\lim}_{t \to +\infty} \psi(t) = \pi^-;\ \overline{\lim}_{t \to +\infty} \psi(t) = \pi^+;\ -\infty < \pi^- < \pi^+ < +\infty.$$

Solution $\mathbf{x}(\mathbf{x}_0, 0, t)$ with $\mathbf{x}_0 \in R^n$ of system (1) is called an oscillating one, if there exist some output ψ and constants π^-, π^+ such, that this solution is $[\pi^-, \pi^+]$-oscillation with respect to the output ψ. System (1) with $\mathbf{u}(t) \equiv 0$, $t \geq 0$ is called oscillatory, if for almost all $\mathbf{x}_0 \in R^n$ solutions of the system $\mathbf{x}(\mathbf{x}_0, 0, t)$ are oscillating. ■

Note that term "almost all solutions" is used to emphasize that generally system (1) for $\mathbf{u}(t) \equiv 0$, $t \geq 0$ has nonempty set of equilibrium points, thus, there exists a set of initial conditions with zero measure such, that corresponding solutions are not oscillation. Introduced in Definition 3 oscillation property defined for zero input and any initial conditions of system (1), the following property is a closely related characterization of the system behavior, which develops proposed above property for case of non zero input but for specified initial conditions [5].

Definition 4. *Let $\mathbf{u}: R_{\geq 0} \to R^m$ be Lebesgue measurable and essentially bounded function of time $t \geq 0$ and $\mathbf{x}_0 \in R^n$ be given such that $\mathbf{x}(\mathbf{x}_0, \mathbf{u}, t)$ are defined for all $t \geq 0$. The functions $\chi_\psi^-(\gamma)$, $\chi_\psi^+(\gamma)$ defined for $0 \leq \gamma < +\infty$ are called a lower and upper excitation indi-*

ces of system (1) in point $\mathbf{x}_0$ *with respect to output* $\psi = \eta(\mathbf{x})$ *(where* $\eta : R^n \to R$ *is a continuous function), if*

$$\left(\chi^-_{\psi,\mathbf{x}_0}(\gamma), \chi^+_{\psi,\mathbf{x}_0}(\gamma)\right) = \arg\sup_{(a,b)\in\mathcal{E}(\gamma)} \{b-a\}$$

$$\mathcal{E}(\gamma) = \left\{ (a,b) : \begin{pmatrix} a = \underline{\lim}_{t\to+\infty} \eta(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)), \\ b = \overline{\lim}_{t\to+\infty} \eta(\mathbf{x}(\mathbf{x}_0,\mathbf{u},t)) \end{pmatrix} \right\}_{\|\mathbf{u}\|\le\gamma}.$$

Lower and upper excitation indices of forward complete system (1) with respect to output ψ *are*

$$\chi^-_\psi(\gamma) = \inf_{\mathbf{x}_0\in R^n} \chi^-_{\psi,\mathbf{x}_0}(\gamma),\ \chi^+_\psi(\gamma) = \sup_{\mathbf{x}_0\in R^n} \chi^+_{\psi,\mathbf{x}_0}(\gamma). \quad ■$$

In the same way it is possible to introduce indices for a vector output $\boldsymbol{\psi} = \boldsymbol{\eta}(\mathbf{x})$, in which case indices would be vectors of the same dimension as that output $\boldsymbol{\psi}$.

Excitation indices characterize abilities of system (1) to forced or controllable oscillations caused by bounded inputs. It is clear that properties $\pi^- = \chi^-_\psi(0)$ and $\pi^+ = \chi^+_\psi(0)$ are satisfied. For non zero inputs indices characterize maximum (over specified set of inputs $\|\mathbf{u}\| \le \gamma$) asymptotic amplitudes $\chi^+_\psi(\gamma) - \chi^-_\psi(\gamma)$ of ψ.

It is worth to stress that it would be useful to calculate or estimate values of $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$ for all $0 \le \gamma < +\infty$ due to the following reason, for example. Let maximum oscillation amplitude be reached for some γ^* and for all $\gamma \ge \gamma^*$ the amplitude is decreased, but indices $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$ would preserve their values, hence, to catch a critical value γ^* of input amplitude, which provides maximum output amplitude for ψ, it is necessary to build full graphics of functions $\chi^-_\psi(\gamma)$ and $\chi^+_\psi(\gamma)$. Obtained characteristic will be closely related with Cauchy gain recently investigated in [15] for task of non-oscillation providing (in fact $\pi^+ - \pi^-$ or $\chi^+_{\psi,\mathbf{x}_0}(\mathbf{u}) - \chi^-_{\psi,\mathbf{x}_0}(\mathbf{u})$ are asymptotic amplitudes of $\psi(t)$ in the sense of [15] for zero or non zero input $\mathbf{u}$, while $\chi^+_\psi(\gamma)$ reflects Cauchy gain of the system (1)).

Conditions of oscillation existence in the system are summarized in the following Theorem.

Theorem 1. *Let system (1) have two continuously differentiable Lyapunov functions* V_1 *and* V_2 *satisfying inequalities:*

$$\upsilon_1(|\mathbf{x}|) \le V_1(\mathbf{x}) \le \upsilon_2(|\mathbf{x}|),\ \upsilon_3(|\mathbf{x}|) \le V_2(\mathbf{x}) \le \upsilon_4(|\mathbf{x}|),$$

for $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4 \in \mathcal{K}_\infty$ *and there exists state feedback* $\mathbf{u} = \mathbf{k}(\mathbf{x})$ *ensuring existence of closed loop system solution at the least locally and*

$$L_{\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}))} V_1(\mathbf{x}) > 0 \text{ for } 0 < |\mathbf{x}| < X_1 \text{ and } \mathbf{x} \notin \Xi;$$

$$L_{\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}))} V_2(\mathbf{x}) < 0 \text{ for } |\mathbf{x}| > X_2 \text{ and } \mathbf{x} \notin \Xi,$$

$$X_1 < \upsilon_1^{-1} \circ \upsilon_2 \circ \upsilon_3^{-1} \circ \upsilon_4(X_2),$$

where $\Xi \subset R^n$ *is a set with Lebesgue zero measure. If set*

$$\Omega = \left\{\mathbf{x} : \upsilon_2^{-1} \circ \upsilon_1(X_1) < |\mathbf{x}| < \upsilon_3^{-1} \circ \upsilon_4(X_2)\right\}$$

does not contain equilibrium points of the closed loop system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$, *then the system is oscillatory.*

Proof. First of all note that further we will consider set of initial conditions not containing equilibrium points (which belong to set Ξ) of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$, then such solution of the closed loop system is well defined at least locally and due to $\dot{V}_2 < 0$ for $|\mathbf{x}| > X_2$, also global boundedness of the solution (defined from now for all $t \ge 0$) follows. Since trajectory $\mathbf{x}(t)$, $t \ge 0$ is bounded, it has a non empty closed and compact ω-limit set, which belongs to set Ω. Indeed, $V_2(t)$ asymptotically enters into set $V_2(t) < \upsilon_4(X_2)$, then $|\mathbf{x}(t)| < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. In the same way function $V_1(t)$ is upper bounded and its asymptotic values fall into set $V_1(t) > \upsilon_1(X_1)$, then again $|\mathbf{x}(t)| > \upsilon_2^{-1} \circ \upsilon_1(X_1)$. As it was supposed, Ω does not contain equilibrium points of closed loop system. Hence, ω-limit set also does not include such invariant solutions. Then there exists an index i, $1 \le i \le n$ such, that the solution is $[\pi^-, \pi^+]$-oscillation with respect to output $\psi = |x_i|$ with $\upsilon_2^{-1} \circ \upsilon_1(X_1) < \pi^- < \pi^+ < \upsilon_3^{-1} \circ \upsilon_4(X_2)$. Suppose that there is no such output. It means that for all $1 \le i \le n$ for $\psi = |x_i|$ equality $\pi^- = \pi^+$ holds. However, the latter could be true only in equilibrium points, which are excluded from set Ω by the theorem conditions. Therefore, for almost all initial conditions solutions have such oscillating output and system (1) with the feedback $\mathbf{u} = \mathbf{k}(\mathbf{x})$ is oscillatory by Definition 3. ■

Remark 1. Note, that here and further set Ω determines lower bound for value of π^- and upper bound for values of π^+. ■

Remark 2. Like in [24] one can consider Lyapunov function candidate for linearized near the origin system (1) as a function V_1 to prove local instability of the system solution. Instead of existence of storage function V_2 one can require just boundedness of the system solution $\mathbf{x}(t)$ with known upper bound. It can be obtained using another approach not dealing with time derivative of Lyapunov function analysis. In this case Theorem 1 transforms into Theorem 3.4 from [8]. ■

E x a m p l e 1 . Let us consider the Van der Pol system:

$$\dot{x}_1 = x_2 ;$$

$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2) x_2 ,$$

where $\varepsilon > 0$ some parameter. To detect presence of oscillations in this system it is necessary (according to Theorem 1) to find two Lyapunov functions, which establish local instability of equilibrium $(0,0)$ and global boundedness of system solutions. Since the system has only one equilibrium point in the origin, the set Ω from the Theorem does not contain the point $(0,0)$. Let us consider the following functions:

$$V_1(\mathbf{x}) = 0.5\left(x_1^2 + x_2^2\right);$$

$$V_2(\mathbf{x}) = 0.5\left(\varepsilon^{-1} x_2 - 2x_1 + 1/3\, x_1^3\right)^2 + 1/12\, x_1^4 ,$$

which time derivatives for the system take form:

$$\dot{V}_1 = \varepsilon x_2^2 - \varepsilon x_2^2 x_1^2 ;$$

$$\dot{V}_2 = -\left[0.5\sqrt{\varepsilon}\left(2 - \varepsilon^{-2}\right) x_1 - \varepsilon^{-0.5} x_2\right]^2 - 1/3\, \varepsilon^{-1} x_1^4 + \\ + \left[0.25\varepsilon\left(2 - \varepsilon^{-2}\right)^2 + 2\varepsilon^{-1}\right] x_1^2 .$$

Function $\dot{V}_1$ is strictly positive for all $0 < |x_1| < 1$ and $x_2 \neq 0$, but the line $x_2 = 0$ does not contain invariant solutions of the system outside the origin. Thus, $\dot{V}_1(t) > 0$ for almost all $t \geq 0$ such that $0 < |x_1(t)| < 1$ and

$$|\mathbf{x}| < X_1 \Rightarrow \dot{V}_1 \geq 0 ,$$

where $X_1 = 1$ (the same conclusion was obtained in [9] for $X_1 = \sqrt{3}$). Instability of the system also can be verified for linearized version of the system, which eigenvalues $\lambda_{1,2} = 0.5\left(\varepsilon \pm \sqrt{\varepsilon^2 - 4}\right)$ are always positive for $\varepsilon > 0$. Analyzing function $\dot{V}_2$ it is possible to obtain $X_2 \leq \sqrt{3\left[0.25\varepsilon^2\left(2 - \varepsilon^{-2}\right)^2 + 2\right]}$. Here functions $\upsilon_1(s) = \upsilon_2(s) = 0.5 s^2$, while functions $\upsilon_3(s)$ and $\upsilon_4(s)$ can be computed numerically for given ε. □

The Definition 3 was introduced for vanishing input $\mathbf{u} = 0$, but in Theorem 1 a feedback $\mathbf{k}$ was utilized. There are two reasons for this. First consists in pointing out an approach for real system analyzing basing on decomposition to feedback and initial system. Another reason is to show a link between oscillatority and excitation indices, as it is done in the following Corollary.

C o r o l l a r y 1 . *Let the system (1) satisfy all conditions of Theorem 1 and solution* $\mathbf{x}(\mathbf{x}_0, \mathbf{k}(\mathbf{x}), t)$, $\mathbf{x}_0 \in R^n$ *is* $[\pi^-, \pi^+]$*-oscillation with respect to the output* $\psi = \eta(\mathbf{x})$ *in the sense of Definition 3. Then*

$$\pi^+ - \pi^- \leq \upsilon_3^{-1} \circ \upsilon_4(X_2) - \upsilon_2^{-1} \circ \upsilon_1(X_1),$$

$$\pi^+ - \pi^- \leq \chi^+_{\psi,\mathbf{x}_0}(\gamma) - \chi^-_{\psi,\mathbf{x}_0}(\gamma),$$

for $\gamma \geq \gamma^*$, *where* $\gamma^* = \sup_{|\mathbf{x}| \leq \Gamma} |\mathbf{k}(\mathbf{x})|$,

$$\Gamma = \upsilon_3^{-1} \circ \upsilon_4\left(\max\left\{X_2, |\mathbf{x}_0|\right\}\right). \quad ■$$

The proofs of Corollary 1, Lemma 1 and Corollary 2 are omitted due to space limitation. To compute estimates of excitation indices it is enough to find some control $\mathbf{k}$ for system (1), which ensures oscillations existence in closed loop system.

In the proof of Theorem 1 the norm of a state space vector component was proposed as oscillating output. However, such an output does not discover all features of oscillation processes in the system. It is a shortage of above solution since such result does not restrict the possible set of oscillating variables of the system. To avoid this obstacle we formulate the same conclusion for output oscillation characteristics of system (1) in the following lemma.

L e m m a 1 . *Let all conditions of Lemma A1 hold. Suppose static state feedback* $\mathbf{u} = \mathbf{k}(\mathbf{x})$, *that ensures existence of closed loop system solution at the least locally, admits conditions*

i) $\alpha_6(|\mathbf{x}|) > \delta(|\mathbf{k}(\mathbf{x})|)$ *for* $|\mathbf{x}| > X \geq 0$ *and* $\mathbf{x} \notin \Xi$;

ii) $L_{\mathbf{f}(\mathbf{x},\mathbf{k}(\mathbf{x}))} V(\mathbf{x}) > 0$ *for* $0 < |\mathbf{h}(\mathbf{x})| \leq Y$ *and* $\mathbf{x} \notin \Xi$,

for some positive constants X *and* Y *with* $Y < \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X)$ *(here functions* α_4, α_5, α_6 *and* δ *were calculated in Lemma A1), set* Ξ *has zero Lebesgue measure. If set* $\Omega = \left\{V(\mathbf{x}) : \underline{\alpha}(Y) \leq V(\mathbf{x}) \leq \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5(X)\right\}$ *does not contain equilibrium points of closed loop system* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$, *then the system is oscillatory.* ■

Here function V depends only on part of variables, that helps to define set of oscillating variables in the system and, additionally, a way to find functions V_1 and V_2 is pointed out ($V_1(\mathbf{x}) = V(\mathbf{x})$ and $V_2(\mathbf{x}) = U(\mathbf{x})$ from Appendix). Results of proposed Theorem 1 and Lemma 1 do not deal with construction of feedback $\mathbf{k}$. Now let us continue with task of control design that ensures desired oscillation parameters for passive systems.

IV. STABILIZATION OF OSCILLATION REGIMES

This Section is based on result of Lemma A2, which is a consequence of Lemma A1 and presents conditions for state feedback provided close loop passive and IOSS system to be ISS. Although conditions imposed on feedback $\mathbf{k}$ in the Lemma A2 look complex and hard to verify, they are very natural and can be easily solved for a variety of systems, for example, in the case when σ_1 and σ_2 are quadratic functions of their arguments. Then control $\mathbf{k}$ with linear growth rate with respect to $\mathbf{y}$ satisfies all proposed conditions. Now we are ready to prove the main result of the Section.

T h e o r e m 2. *Let system (1) be passive with known dissipation rate* β *and IOSS in the sense of Definition 2 and*

$$\underline{\alpha}(|\mathbf{y}|) \le V(\mathbf{x}) \le \overline{\alpha}(|\mathbf{x}|),\ \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty .$$

Then control

$$\mathbf{u} = \mathbf{k}(\mathbf{x}) + \mathbf{d};\ |\mathbf{k}(\mathbf{x})| \le \lambda(|\mathbf{y}|) + K,\ 0 < K < +\infty;$$

$$\beta(\mathbf{x}) - \mathbf{y}^T\mathbf{k}(\mathbf{x}) + \mu(|\mathbf{d}|) + \mu(K) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^T\mathbf{d};$$

$$\lim_{s\to+\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty;$$

$$\mathbf{y}^T\mathbf{k}(\mathbf{x}) > \beta(\mathbf{x}) \text{ for } 0 < |\mathbf{y}| < Y < +\infty,$$

$$Y < \underline{\alpha}^{-1} \circ \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K),$$

where $\lambda \in \mathcal{K}, \kappa \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ *and* $\mathbf{d} \in R^m$ *is new input (Lebesgue measurable and essentially bounded function of time) ensures that*

i) system solution is bounded;

ii) if set

$$\Omega = \left\{V(\mathbf{x}) : \underline{\alpha}(Y) \le V(\mathbf{x}) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K)\right\}$$

does not contain equilibrium points of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{k}(\mathbf{x}))$ *then for* $\mathbf{d}(t) \equiv 0$, $t \ge 0$ *closed loop system is an oscillatory one (here functions* $\alpha_4, \alpha_5, \alpha_6$ *and* δ *were obtained in Lemma A2).*

P r o o f. At first, let us introduce a partition of control input:

$$\mathbf{u} = \mathbf{k}(\mathbf{x}) = -\mathbf{k}_1(\mathbf{x}) + \mathbf{k}_2(\mathbf{x}),$$

such, that

$$|\mathbf{k}_1(\mathbf{x})| \le \lambda(|\mathbf{y}|),\ |\mathbf{k}_2(\mathbf{x})| \le K;$$

$$\mathbf{y}^T\mathbf{k}_1(\mathbf{x}) + \beta(\mathbf{x}) + \mu(|\mathbf{d}|) \ge \kappa(|\mathbf{y}|) + \mathbf{y}^T\mathbf{d};$$

$$\mathbf{y}^T\mathbf{k}_2(\mathbf{x}) > \beta(\mathbf{x}) + \mathbf{y}^T\mathbf{k}_1(\mathbf{x}) \text{ for } 0 < |\mathbf{y}| < Y < +\infty .$$

This separation is possible due to conditions of Theorem 2. Further denoting auxiliary input $\tilde{\mathbf{d}} = \mathbf{d} + \mathbf{k}_2(\mathbf{x})$, which is essentially bounded by conditions of the Theorem $\|\tilde{\mathbf{d}}\| \le K + \|\mathbf{d}\|$. It is possible to note, that for system (1) all conditions of Lemma A2 are satisfied for feedback $\mathbf{u} = -\mathbf{k}_1(\mathbf{x}) + \tilde{\mathbf{d}}$ and system is ISS with respect to input $\tilde{\mathbf{d}}$. According to ISS property [14] and boundedness of $\tilde{\mathbf{d}}$, boundedness of system solution immediately follows and statement (i) of Theorem is proven. To justify statement (ii) simply note, that conditions of Lemma 1 also hold. ■

Note that Theorem 2 extends the result from [3] and [21] to the case of general nonlinear dynamical systems. Additionally, special attention was given to the lower estimate of the oscillation amplitude for $\mathbf{d}(t) \equiv 0$, $t \ge 0$.

Exciting part $\mathbf{k}_2$ of feedback $\mathbf{k}$ defines the size of set Ω (due to constants Y and K are prescribed by $\mathbf{k}_2$) and, hence, it regulates tolerances for values of π^- and π^+

R e m a r k 3. It is necessary to stress that control in the Theorem is taken to satisfy some sector condition with respect to output $\mathbf{y}$. For design of such controls in practical application it is possible to use the speed-gradient [6], [7]. e.g. choose $\mathbf{u} = \varphi(\mathbf{y})$, where $\varphi(\mathbf{y})^T\mathbf{y} > 0$ for $0 < |\mathbf{y}| < Y_1$ and $\varphi(\mathbf{y})^T\mathbf{y} < 0$ for $|\mathbf{y}| > Y_2 > Y_1$. ■

For case of non vanishing signal $\mathbf{d}$ it is possible to obtain, basing on results of Theorem 2 and Corollary 1, estimates for excitation indices of closed loop system.

C o r o l l a r y 2. *Let all conditions of Theorem 2 hold. Then for* $\|\mathbf{d}\| \le \gamma < +\infty$

$$0 \le \chi_V^-(\gamma) \le \chi_V^+(\gamma) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K + \gamma),$$

if additionally

$$\mathbf{y}(t)^T\mathbf{d}(t) \ge 0 \text{ for all } t \ge 0, \qquad (3)$$

then

$$\underline{\alpha}(Y) \le \chi_V^-(\gamma) < \chi_V^+(\gamma) \le \overline{\alpha} \circ \alpha_4^{-1} \circ \alpha_5 \circ \alpha_6^{-1} \circ \delta(K + \gamma). \ ■$$

According to the Corollary index $\chi_V^+(\gamma)$ is always bounded, that is more, it can not be equal to $\chi_V^-(\gamma)$ for any $\gamma \in R_{\ge 0}$ with (3). Thus, system can not lose in general its oscillation ability for any large enough input disturbance possessing "coordination" condition (3).

V. CONCLUSION

In this paper we propose conditions for oscillatority in sense of Yakubovich applicable to nonlinear systems decomposed into two nonlinear parts. Upper and lower bounds for oscillation amplitude are obtained. The relation between the oscillatority bounds and excitability indices for the systems with the input are established. The results are illustrated by example: evaluation of oscillations for the Van der Pol system. As a side result a nonquadratic Lyapunov function providing boundedness of the Van der Pol system solutions has been found.

APPENDIX

Proofs of the Lemmas below are omitted due to space limitation.

L e m m a A 1 . *Let system (1) have IOSS Lyapunov function* W *and* h*-dissipative storage function* V *as in Definition 2. If*

$$\lim_{s\to+\infty} \frac{\sigma_2(s)}{\alpha(s)} < +\infty ,$$

then system (1) is ISS with ISS Lyapunov function

$$U(\mathbf{x}) = V(\mathbf{x}) + \widetilde{W}(\mathbf{x}),\ \widetilde{W}(\mathbf{x}) = \rho\big(W(\mathbf{x})\big),\ \rho(r) = \int_0^r q(s)\,ds ,$$

$$q(s) = \frac{\alpha \circ \sigma_2^{-1}\left(0.25\alpha_3 \circ \alpha_2^{-1}(s)\right)}{1 + 0.5\alpha_3 \circ \alpha_2^{-1}(s)} ,$$

$$\alpha_4(s) = \rho \circ \alpha_1(s)\ , \alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s) ,$$

$$L_{\mathbf{f}(\mathbf{x},\mathbf{u})} U(\mathbf{x}) \le -\alpha_6\big(|\mathbf{x}|\big) + \delta\big(|\mathbf{u}|\big)$$

$$\alpha_6(s) = 0.5q\big(\alpha_1(s)\big)\alpha_3(s) ,$$

$$\delta(s) = \sigma(s) + 2\chi\big(2\sigma_1(s)\big)\sigma_1(s) . \qquad \blacksquare$$

The next Lemma is a corollary of Lemma A1, that present a variant of ISS stabilizing control law for a passive system.

L e m m a A 2 . *Let system (1) be passive and IOSS in the sense of Definition 2 and*

$$\underline{\alpha}\big(|\mathbf{y}|\big) \le V(\mathbf{x}) \le \overline{\alpha}\big(|\mathbf{x}|\big),\ \underline{\alpha},\overline{\alpha} \in \mathcal{K}_\infty .$$

Then control

$$\mathbf{u} = -\mathbf{k}(\mathbf{x}) + \mathbf{d} ,\ |\mathbf{k}(\mathbf{x})| \le \lambda\big(|\mathbf{y}|\big),\ \lambda \in \mathcal{K} ;$$

$$\mathbf{y}^T\mathbf{k}(\mathbf{x}) + \beta(\mathbf{x}) \ge \kappa\big(|\mathbf{y}|\big) + 0.5|\mathbf{y}|^2 ,\ \kappa \in \mathcal{K}_\infty ;$$

$$\lim_{s\to+\infty} \frac{\sigma_2(s) + \sigma_1 \circ \lambda(s)}{\kappa(s)} < +\infty ,$$

where $\mathbf{d} \in R^m$ *is new input (Lebesgue measurable and essentially bounded function of time), provides for the system ISS property with ISS Lyapunov function*

$$U(\mathbf{x}) = V(\mathbf{x}) + \widetilde{W}(\mathbf{x}),\ \widetilde{W}(\mathbf{x}) = \rho\big(W(\mathbf{x})\big),\ \rho(r) = \int_0^r q(s)\,ds ,$$

$$q(s) = \frac{\kappa \circ \widetilde{\sigma}_2^{-1}\left(0.25\alpha_3 \circ \alpha_2^{-1}(s)\right)}{1 + 0.5\alpha_3 \circ \alpha_2^{-1}(s)} ,\ \alpha_4(s) = \rho \circ \alpha_1(s) ,$$

$$\alpha_5(s) = \overline{\alpha}(s) + \rho \circ \alpha_2(s) ,\ \alpha_6(s) = 0.5q\big(\alpha_1(s)\big)\alpha_3(s) ,$$

$$\delta(s) = 0.5s^2 + 2\chi\big(2\sigma_1(2s)\big)\sigma_1(2s) . \qquad \blacksquare$$

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