NONLINEAR EFFECTS IN ACCELERATOR PHYSICS: FROM SCALE TO SCALE VIA WAVELETS

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Abstract

This is the first part of our two papers in which we present applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In the general case we have the solution as a multiresolution expansion in the base of compactly supported wavelet basis. We give extension of our previous results to the case of periodic orbital particle motion in storage rings. Then we consider more flexible variational method which is based on biorthogonal wavelet approach.

1 INTRODUCTION

This is the first part of our two presentation in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1], [2], which is based on approach of two of us from [3], [4] to investigation of nonlinear problems – general, with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum, which are considered in the framework of local(nonlinear) Fourier analysis, or wavelet analysis. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases.

In [3], [4] we considered application of multiresolution representation to general nonlinear dynamical system with polynomial type of nonlinearities. Starting with variational formulation of initial dynamical problem we construct explicit representation for all dynamical variables in the base of compactly supported (Daubechies) wavelets. Our solutions are parametrized by solutions of a number of reduced algebraical problems one from which is nonlinear with the same degree of nonlinearity and the rest are the linear problems which correspond to particular method of calculation of scalar products of functions from wavelet bases and their derivatives. In this paper we consider further extension of our previous results. In section 2 we consider modification of our previous construction to the periodic case, in section 3 we consider generalization of our approach from [1], [2] to variational formulation in the biorthogonal bases of compactly supported wavelets.

Our main example is calculation of orbital particle motion in storage rings. Starting from Hamiltonian which described classical dynamics in storage rings [5] $\mathcal{H}(\vec{r},\vec{P},t)=c\{\pi^2+m_0^2c^2\}^{1/2}+e\phi$ and using Serret-Frenet parametrization, truncation of power series expansion of square root we arrive to the following approximated Hamiltonian for particle motion in machine coordinates:

$$
\begin{aligned}
\mathcal{H} \;=\; & \frac{1}{2}\cdot\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{[1+f(p_\sigma)]}+p_\sigma- \\
& [1+K_x\cdot x+K_z\cdot z]\cdot f(p_\sigma)+ \\
& \frac{1}{2}\cdot[K_x^2+g]\cdot x^2+\frac{1}{2}\cdot[K_z^2-g]\cdot z^2- \\
& N\cdot xz+\frac{\lambda}{6}\cdot(x^3-3xz^2)+ \\
& \frac{\mu}{24}\cdot(z^4-6x^2z^2+x^4)+ \\
& \frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]
\end{aligned}
\qquad (1)
$$

Then we use series expansion of function $f(p_\sigma)$ and the corresponding expansion of RHS of Hamiltonian equations of motions. In the following we take into account only an arbitrary polynomial (in terms of dynamical variables) expressions and neglecting all nonpolynomial types of expressions, i.e. we consider such approximations of RHS, which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variable s.

2 VARIATIONAL WAVELET APPROACH FOR PERIODIC TRAJECTORIES

We start with extension of our approach [1], [2] to the case of periodic trajectories. The equations of motion corresponding to Hamiltonian (1) may be formulated as a particular case of the general system of ordinary differential equations $dx_i/dt=f_i(x_j,t)$, $(i,j=1,...,n)$, $0\le t\le 1$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. According to our variational approach [3], [4] we have the solution in the following form

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t), \qquad (2)$$

where λ_i^k are the roots of reduced algebraical systems of equations with the same degree of nonlinearity and $\varphi_k(t)$ corresponds to useful type of wavelet bases (frames). It should be noted that coefficients of reduced algebraical system are the solutions of additional linear problem and also depend on particular type of wavelet construction and type of bases. This linear problem is our second reduced algebraical problem. Our construction is based on multiresolution approach. Because affine group of translation and dilations is inside this approach our method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace V_j corresponds to level j of resolution, or to scale j. We consider a sequence of successive approximation by subspaces V_j: $...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset ...$ satisfying the following properties: $\bigcap_{j\in\mathbf{Z}} V_j = 0$, $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$, $f(x) \in V_j <=> f(2x) \in V_{j+1}$ There is a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k)_{k\in\mathbf{Z}}\}$ forms a Riesz basis for V_0. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x) = \varphi(x-i)$. Scaling relation that defines φ, ψ are $\varphi(x) = \sum\limits_{k=0}^{N-1} a_k \varphi(2x-k) = \sum\limits_{k=0}^{N-1} a_k \varphi_k(2x)$, $\psi(x) = \sum\limits_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x+k)$.
Let be $f: \mathbf{R} \longrightarrow \mathbf{C}$ and the wavelet expansion is $f(x) = \sum\limits_{\ell\in\mathbf{Z}} c_\ell \varphi_\ell(x) + \sum\limits_{j=0}^{\infty} \sum\limits_{k\in\mathbf{Z}} c_{jk} \psi_{jk}(x)$
The indices k, ℓ and j represent translation and scaling, respectively $\varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - \ell)$, $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. The set $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ forms a Riesz basis for V_j. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}. Just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$.
All expansions which we used are based on the following properties: $\{\varphi_{jk}\}_{j\geq 0, k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$, $V_{j+1} = V_j \bigoplus W_j$, $L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}$, $\{\varphi_{0,k}, \psi_{j,k}\}_{j\geq 0, k\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{R})$, $\{\varphi_{j,k}, \psi_{\ell,k}; 0 \leq j \leq J \leq \ell, k \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$. If in formulae (4) $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x) = \sum\limits_{\ell\in\mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet $\psi(x)$, which has k vanishing moments $\int x^k \psi(x)dx = 0$, or equivalently $x^k = \sum c_\ell \varphi_\ell(x)$ for each k, $0 \leq k \leq K$. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0, N-1]$ and $N/2$ vanishing moments. There exists $\lambda > 0$ such that DN has λN continuous derivatives; for small $N, \lambda \geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi_\ell^n = d^n\varphi_\ell(x)/dx^n$. We derive the wavelet - Galerkin approximation of a differentiated $f(x)$ as $f^d(x) = \sum_\ell c_l \varphi_\ell^d(x)$ and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$ [6]: $\phi_\ell^d(x) = \sum\limits_m \lambda_m \varphi_m(x)$, $\lambda_m = \int\limits_{-\infty}^{\infty} \varphi_\ell^d(x)\varphi_m(x)dx$. The coefficients λ_m are 2-term connection coefficients. In general we need to find

$$\Lambda_{\ell_1\ell_2\dots\ell_n}^{d_1 d_2 \dots d_n} = \int\limits_{-\infty}^{\infty} \prod \varphi_{\ell_i}^{d_i}(x)dx \qquad (3)$$

For quadratic (Riccati) case we need to evaluate two and three connection coefficients

$$\begin{aligned} \Lambda_\ell^{d_1 d_2} &= \int_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)dx, \quad d_i \geq 0, \\ \Lambda^{d_1 d_2 d_3} &= \int\limits_{-\infty}^{\infty} \varphi^{d_1}(x)\varphi_\ell^{d_2}(x)\varphi_m^{d_3}(x)dx \end{aligned}$$

Now we consider the same objects and procedure of their calculations but in the base of periodic wavelet functions on the interval [0,1] and corresponding expansion (2) inside our variational approach [3], [4]. Periodization procedure gives us

$$\begin{aligned} \hat\varphi_{j,k}(x) &\equiv \sum_{\ell\in Z} \varphi_{j,k}(x-\ell) \qquad (4) \\ \hat\psi_{j,k}(x) &= \sum_{\ell\in Z} \psi_{j,k}(x-\ell) \end{aligned}$$

So, $\hat\varphi, \hat\psi$ are periodic functions on the interval [0,1]. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k' mod(2^j)$, we may consider only $0 \leq k \leq 2^j$ and as consequence our multiresolution has the form $\bigcup\limits_{j\geq 0} \hat V_j = L^2[0,1]$ with $\hat V_j = span\{\hat\varphi_{j,k}\}_{k=0}^{2j-1}$ [7]. Integration by parts and periodicity gives useful relations between objects (3) ($d = d_1 + d_2$):

$$\Lambda_{k_1,k_2}^{d_1,d_2} = (-1)^{d_1}\Lambda_{k_1,k_2}^{0,d_2+d_1}, \Lambda_{k_1,k_2}^{0,d} = \Lambda_{0,k_2-k_1}^{0,d} \equiv \Lambda_{k_2-k_1}^d$$

So, any 2-tuple can be represent by Λ_k^d. Then our second additional linear problem is reduced to the eigenvalue problem for $\{\Lambda_k^d\}_{0\leq k\leq 2^j}$ by creating a system of 2^j homogeneous relations in Λ_k^d and inhomogeneous equations. So, if we have dilation equation in the form $\varphi(x) = \sqrt{2}\sum_{k\in Z} h_k\varphi(2x-k)$, then we have the following homogeneous relations $\Lambda_k^d = 2^d \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} h_m h_\ell \Lambda_{\ell+2k-m}^d$, or in such form $A\lambda^d = 2^d\lambda^d$, where $\lambda^d = \{\Lambda_k^d\}_{0\leq k\leq 2^j}$. Inhomogeneous equations are: $\sum_\ell M_\ell^d \Lambda_\ell^d = d!2^{-j/2}$,

where objects $M_\ell^d(|\ell| \leq N-2)$ can be computed by recursive procedure $M_\ell^d = 2^{-j(2d+1)/2}\tilde{M_\ell^d}$, $\tilde{M_\ell^k} =< x^k, \varphi_{0,\ell} >= \sum_{j=0}^k \binom{k}{j} n^{k-j} M_0^j$, $\tilde{M_0^\ell} = 1$. So, we reduced our last problem to standard linear algebraical problem. Then we use the same methods as in [3], [4].
As a result we obtained for closed trajectories of orbital dynamics described by Hamiltonian (1) the explicit time solution (2) in the base of periodized wavelets (4).

3 VARIATIONAL APPROACH IN BIORTHOGONAL WAVELET BASES

Now we consider further generalization of our variational wavelet approach. In [3], [4] we consider different types of variational principles which give us weak solutions of our nonlinear problems. But because integrand of variational functionals is represented by bilinear form (scalar product) it seems more reasonable to consider wavelet constructions which take into account all advantages of this structure.

As an example let us consider action functional for loops in the phase space $F(\gamma) = \displaystyle\int_\gamma pdq - \int_0^1 H(t,\gamma(t))dt$. The critical points of F are such loops γ, which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. So, (M,ω) is symplectic manifolds, $H: M \to R$, H is Hamiltonian, X_H is unique Hamiltonian vector field defined by $\omega(X_H(x), \upsilon) = -dH(x)(\upsilon)$, $\upsilon \in T_xM$, $\quad x \in M$, where ω is the symplectic structure. A T-periodic solution $x(t)$ of the Hamiltonian equations $\dot{x} = X_H(x) \quad$ on M is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1, R^{2n})$, where $S^1 = R/\mathbf{Z}$, of smooth loops in R^{2n}. Let us define a function $\Phi: \Omega \to R$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} < -J\dot{x}, x > dt - \int_0^1 H(x(t))dt, \quad x \in \Omega$$

The critical points of Φ are the periodic solutions of $\dot{x} = X_H(x)$. Computing the derivative at $x \in \Omega$ in the direction of $y \in \Omega$, we find

$$\begin{aligned}\Phi'(x)(y) &= \frac{d}{d\epsilon}\Phi(x+\epsilon y)|_{\epsilon=0} = \\ &\int_0^1 < -J\dot{x} - \bigtriangledown H(x), y > dt\end{aligned}$$

Consequently, $\Phi'(x)(y) = 0$ for all $y \in \Omega$ iff the loop x satisfies the equation $-J\dot{x}(t) - \bigtriangledown H(x(t)) = 0$, i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0) = x(1)$, i.e. periodic of period 1.

We started with two hierarchical sequences of approximations spaces [8]: $\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \dots$, $\dots \widetilde{V}_{-2} \subset \widetilde{V}_{-1} \subset \widetilde{V}_0 \subset \widetilde{V}_1 \subset \widetilde{V}_2 \dots$, and as usually, W_0 is complement to V_0 in V_1, but now not necessarily orthogonal complement. New orthogonality conditions have now the following form: $\widetilde{W}_0 \perp V_0$, $W_0 \perp \widetilde{V}_0$, $V_j \perp \widetilde{W}_j$, $\widetilde{V}_j \perp W_j$, translates of ψ span W_0, translates of $\tilde{\psi}$ span $\widetilde{W}_0$. Biorthogonality conditions are $< \psi_{jk}, \tilde{\psi}_{j'k'} >= \int_{-\infty}^{\infty} \psi_{jk}(x)\tilde{\psi}_{j'k'}(x)dx = \delta_{kk'}\delta_{jj'}$, where $\psi_{jk}(x) = 2^{j/2}\psi(2^jx-k)$. Functions $\varphi(x), \tilde{\varphi}(x-k)$ form dual pair: $< \varphi(x-k), \tilde{\varphi}(x-\ell) >= \delta_{kl}$, $< \varphi(x-k), \tilde{\psi}(x-\ell) >= 0$ for $\forall k, \forall \ell$. Functions $\varphi, \tilde{\varphi}$ generate a multiresolution analysis. $\varphi(x-k), \psi(x-k)$ are synthesis functions, $\tilde{\varphi}(x-\ell), \tilde{\psi}(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining $V_j + W_j = V_{j+1}$, $\widetilde{V}_j + \widetilde{W}_j = \widetilde{V}_{j+1}$. These are direct sums but not orthogonal sums.

So, our representation for solution has now the form $f(t) = \sum_{j,k} \tilde{b}_{jk}\psi_{jk}(t)$, where synthesis wavelets are used to synthesize the function. But $\tilde{b}_{jk}$ come from inner products with analysis wavelets. Biorthogonality yields $\tilde{b}_{\ell m} = \int f(t)\tilde{\psi}_{\ell m}(t)dt$. So, now we can introduce this more complicated construction into our variational approach. We have modification only on the level of computing coefficients of reduced nonlinear algebraical system. This new construction is more flexible. Biorthogonal point of view is more stable under the action of large class of operators while orthogonal (one scale for multiresolution) is fragile, all computations are much more simpler and we accelerate the rate of convergence. In all type of Hamiltonian calculation, which is based on some bilinear structures (symplectic or Poissonian structures, bilinear form of integrand in variational integral) this framework leads to much success.

4 REFERENCES

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