

**THE ELLIPTIC SOLUTION OF THE SINH-GORDON EQUATION**

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The elliptic solution of the two-dimensional Toda chain with the Cartan matrix of the Kac–Moody algebra  $A_1^{(1)}$ , parametrized by an arbitrary (anti) holomorphic function is obtained, as well as its particular realization – the sinh-Gordon equation – with the use of the generalized Pohlmeyer transformation and the generalization of the elliptic solution of the  $CP^1$  model.

*1. Introduction.* In ref. [1] one of the authors proposed to consider the generalization of the Pohlmeyer transformation connecting the  $O(3)\sigma$  model and the sine–Gordon equation as a method of constructing new non-linear systems with exponential interaction, which at the time have a set of solutions. The realization of the general relativity principle permits us to establish local equivalence between the general Kähler chiral models and their equivalent systems with potential exponential interaction. The set of solutions of the initial chiral model is transformed into solutions of the reduced model. The most simple example of this construction is the generation of the Liouville equation and the corresponding formulae of its solution by the  $O(3)\sigma$  model and its instanton sector (in the euclidean case) [1].

Let the action of the  $CP^1$  model be  $A$ :

$$A = \int h_{CP^1} (u_z \bar{u}_{\bar{z}} + u_{\bar{z}} \bar{u}_z) d^2x, \tag{1}$$

where  $u = u(z, \bar{z})$  is the initial chiral field,  $h = h_{CP^1} = (1 + u\bar{u})^{-2}$  the metric of the  $CP^1$ ,  $z = x + iy$ ,  $\bar{z} = x - iy$  in the euclidean metric,  $z = x + y$ ,  $\bar{z} = x - y$  in the Minkowski metric,  $u_z = \partial u / \partial z$ .

It is demonstrated in ref. [1] that if  $u$  is the solution of the Euler – Lagrange equation corresponding to the action (1):

$$h u_{z\bar{z}} + (\partial h / \partial u) u_z u_{\bar{z}} = 0, \tag{2}$$

that is out of the instanton sector and

$$\exp B^1 = h_{CP^1} u_z \bar{u}_{\bar{z}}, \quad \exp B^2 = h_{CP^1} u_{\bar{z}} \bar{u}_z, \tag{3}$$

then the new dynamic variables  $B^1$  and  $B^2$  satisfy the equations

$$\begin{aligned} B^1_{z\bar{z}} + 2 \exp B^1 - 2 \exp B^2 &= 0, \\ B^2_{z\bar{z}} - 2 \exp B^1 + 2 \exp B^2 &= 0. \end{aligned} \tag{4}$$

This is the two-dimensional generalized Toda chain corresponding to the two-dimensional Cartan matrix of the Kac–Moody algebra  $A_1^{(1)}$  [3].

All solutions with finite action (in the euclidean case) are exhausted by the instanton sector, that is transformed into the Liouville equation [1]. Outside the instanton sector lie the meron solutions as well as recently found elliptic solutions interpolating between the instantons and merons [4,5].

The aim of this letter is the construction of the solution of the system (4) and its particular realization – the sinh-Gordon equation, using the generalization [2] of the elliptic solutions found in refs. [4,5] and the transformations formulae (3).

*2. The generalized elliptic solution of the  $O(3)\sigma$  model.* As was shown in ref. [2] the Ansatz [4,5] that is not constrained by the boundary condition at infinity gives the most general solution (in the limits of the given Ansatz [6]), parametrized by an arbitrary holo-

morphic function. Thus the solution of the equation of motion (2) takes the form [2]

$$u = \tan(A)(f/\bar{f})^a \exp(iac), \tag{5}$$

where  $A = A(x)$  satisfies the equation

$$A_{xx} = a^2 \sin 4A, \tag{6}$$

where

$$x = \frac{1}{2} \ln f\bar{f},$$

and  $f = f(z), f_{\bar{z}} = 0$  an arbitrary holomorphic function,  $a, c$  arbitrary real constants.

When  $a = \frac{1}{2}, c = 0$  from (5), (6) results:

$$u = \left( \frac{1 + k \operatorname{sn}(\frac{1}{2} \ln(f\bar{f}))}{1 - k \operatorname{sn}(\frac{1}{2} \ln(f\bar{f}))} \right)^{1/2} (f/\bar{f})^{1/2}. \tag{7}$$

The representation (7) will be used further. Here  $\operatorname{sn}$  is an elliptic Jacobi function with parameter  $k$ .

The solution may be put down also as

$$u = \left( \frac{1 + k \operatorname{sn} F}{1 - k \operatorname{sn} F} \right)^{1/2} \exp(iH), \tag{8}$$

where  $F = F(z, \bar{z}), H = H(z, \bar{z}), F = \bar{F}, H = \bar{H}$  with the properties  $F_{z\bar{z}} = H_{z\bar{z}} = 0, F_z H_{\bar{z}} + F_{\bar{z}} H_z = 0, F_z F_{\bar{z}} = H_z H_{\bar{z}}$ . The representation (8) gives also the representations (7). It should be noted that ref. [4] deals with the cases  $f(z) = z$  and  $f(z) = z^n$  and ref. [5] with the case

$$f(z) = \prod_{i=1}^n \frac{z - d_i}{z - b_i}.$$

It is noticeable that these particular cases are classified in the theory of singular harmonic mappings [7], but in section 4 we shall consider the case  $f(z) = \exp z$ . The classification of these and more complicated functions in the theory of harmonic mappings is not yet clear.

For  $k = 1$  (7), (8) give the instanton but this case is forbidden to us because it is reduced to the Liouville equation, but not to the system (4) [1]. We give formulae for  $k^2 < 1$ , for  $k^2 > 1$  they are the same [ $k = 0$  gives trivial solutions for (4)].

3. *The solution of the Toda chain.* Putting the expression (7) in the formulae (3) we get for the solution of the system (4) the formulae

$$\exp B^1 = \frac{1}{16} [k \operatorname{cn}(\frac{1}{2} \ln f\bar{f}) + \operatorname{dn}(\frac{1}{2} \ln f\bar{f})]^2 f_z \bar{f}_{\bar{z}} / f\bar{f},$$

$$\exp B^2 = \frac{1}{16} [k \operatorname{cn}(\frac{1}{2} \ln f\bar{f}) - \operatorname{dn}(\frac{1}{2} \ln f\bar{f})]^2 f_z \bar{f}_{\bar{z}} / f\bar{f}, \tag{9}$$

where  $f = f(z), f_{\bar{z}} = 0$  is an arbitrary holomorphic function, the same holds for  $g = g(\bar{z}), g_z = 0$  is an antiholomorphic function with the substitution:

$$f_z \bar{f}_{\bar{z}} \rightarrow g_{\bar{z}} \bar{g}_z, \quad f\bar{f} \rightarrow g\bar{g}.$$

Another representation results from (8):

$$\exp B^1 = \frac{1}{4} (k \operatorname{cn} F + \operatorname{dn} F)^2 F_z F_{\bar{z}},$$

$$\exp B^2 = \frac{1}{4} (k \operatorname{cn} F - \operatorname{dn} F)^2 F_z F_{\bar{z}}, \tag{10}$$

where  $F = F(z, \bar{z}), F = \bar{F}, F_{z\bar{z}} = 0$ . As the Jacobi functions are expressed through  $\theta$  functions, the answer may be represented as it is usual in  $\theta$  functions.

4. *The solutions of the sinh-Gordon equation.*

From system (4) we come to the sinh-Gordon equation through imposing the constraints  $B^1 = -B^2 = B$  in the formulae (9), (10). We get from (4) the equation

$$B_{z\bar{z}} + 4 \sinh B = 0, \tag{11}$$

and from (10) we get the solution:

$$\begin{aligned} \sinh B &= \frac{1}{2} k (d/dF) (\operatorname{sn} F) F_z F_{\bar{z}} \\ &= \frac{1}{2} k (\operatorname{cn} F \operatorname{dn} F) F_z F_{\bar{z}}, \end{aligned} \tag{12}$$

where as a consequence of the condition  $B^1 = -B^2$

$$F_z F_{\bar{z}} = \pm 4/(k^2 - 1). \tag{13}$$

To the formulae for  $k^2 < 1$  corresponds a minus, for  $k^2 > 1$  a plus. Now we can put down

$$\sinh B = \pm [2k/(k^2 - 1)] \operatorname{cn} F \operatorname{dn} F,$$

or from (9):

$$\sinh B = \pm [2k/(k^2 - 1)] \operatorname{cn}(\frac{1}{2} \ln f\bar{f}) \operatorname{dn}(\frac{1}{2} \ln f\bar{f}), \tag{14}$$

where  $f(z)$  in correspondence to (13) satisfies the equation:

$$f_z \bar{f}_{\bar{z}} / f\bar{f} = \pm 16/(k^2 - 1). \tag{15}$$

Thus the choice of arbitrary functions  $f = f(z)$  gives according to the formulae (15) the equations of some curves in the  $z$  plane, along which eq. (11) has the solutions (14).

The same is the case with the Dodd-Bullough equation [1], but it has the formulae of solution of ration-

al type, but the number of curves for the given function is infinite and only one of them being rational, all the others are transcendental. Eq. (15) can be considered not only as setting a curve in the  $z$  plane according to the given function  $f(z)$ , but also as an equation on the function  $f(z)$ . Solving eq. (15) we get according to (14) the solution of eq. (11) on the whole  $z$  plane.

Eq. (15) has only one solution:  $f(z) = \exp(az + d)$ , where  $a\bar{a} = \pm 16/(k^2 - 1)$  and  $d$  is an arbitrary complex number. Then the formula (14) takes the form:

$$\sinh B = \pm [2k/(k^2 - 1)] \operatorname{cn} F \operatorname{dn} F, \quad (16)$$

where  $F = \frac{1}{2}(az + \bar{a}\bar{z} + d + \bar{d})$ .

*5. One dimensional case.* In case the topological charge is different from zero instead of the initial equation of motion (2) the "equation of the conservation law" can be solved [1]:

$$hu_z \bar{u}_z = e^{ib}, \quad (17)$$

where  $b$  is a real constant. According to ref. [1], if  $u$  is the solution of this equation the expression

$$\exp B = hu_z \bar{u}_z,$$

satisfies the sinh-Gordon equation (11). Here we shall consider only the one-dimensional case. Let

$$u = C \exp(iD)$$

and let us put  $D_y = 1, C_y = D_x = 0$ , then eq. (17) takes the form:

$$C_x^2 = 4 + 9C^2 + 4C^4.$$

Its solution in terms of the Weierstrass elliptic functions is [8]:

$$C(x) = \frac{1}{2} \gamma'(2x)/[\gamma(2x) + a], \quad (18)$$

where the elliptic function is formed on the roots:

$$-a, \frac{1}{2}(a + e), \frac{1}{2}(a - e), \text{ where } 6a = 9/4, e^2 = 1.$$

The solution of the one-dimensional sinh-Gordon equations:

$$B_{xx} + 16 \sinh B = 0,$$

takes the form

$$\exp B = \frac{1}{4} h (C + C_x)^2, \quad (19)$$

where  $h = (1 + C^2)^{-2}$  and  $C(x)$  is given in formula

(18). The two-dimensional generalization will be considered in a separate publication.

*6. Conclusion.* We use the elliptic solution of the  $O(3)\sigma$  model, parametrized by an arbitrary holomorphic function (7), (8) [2] and reduction to the Toda chain (4) [1] in order to obtain in accordance with the formulae (3) the solution of the Toda chain parametrized by an arbitrary (anti) holomorphic function (9) or a real harmonic function (10). The elliptic solution of the sinh-Gordon equation is obtained (16) and the concept of "solution along the curve" is considered (14), (15). We also demonstrate the possibility to consider instead of the equation of motion "the equation of the conservation law" (17) in the one-dimensional case (19), (18).

As it was shown in ref. [2] the  $O(2, 1)\sigma$  model permits also the elliptic solution, parametrized by an arbitrary holomorphic function, while its reduction results in the Toda chain analogous to (4), but with the opposite sign in the exponents. The formulae obtained for it are analogous to (9), but do not coincide with them.

Thus we see that there are at least two uncorrected sets of solutions, parametrized by an arbitrary holomorphic function. It should also be noted that the analogous results are obtained for the Ernst equation that is close to the  $O(2, 1)\sigma$  model [2].

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