O(2,1) σ-MODEL GENERATES THE SOLUTION OF THE TWO-DIMENSIONAL EINSTEIN EQUATION PARAMETRIZED BY ARBITRARY FUNCTIONS

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The solutions of the two-dimensional Einstein equation or Ernst equation, parametrized by arbitrary functions and generated by the solutions of the corresponding O(2,1) σ-model and by a special choice of the determinant of the metric are presented. The metric is also given. For the Einstein–Maxwell equations analogous results are obtained. The solutions have a non-trivial curvature tensor.

1. The Einstein equations that admit two commuting Killing vectors are being actively considered now [1]. In this case the Einstein equations are reduced to two-dimensional nonlinear equations which can be analyzed by powerful methods [2]. The presence of the Geroch symmetry which has been recently identified with the Kac–Moody algebra SL(2,R) $\otimes$ R(t, t^{-1}) is of special interest [3]. This symmetry according to the Geroch hypothesis should generate all the solutions of that equation. The problem naturally arises how to construct the solution that corresponds to these symmetries. The corresponding solution should be complete, i.e. should be parametrized by the necessary number of arbitrary functions (for example, the formula for the Liouville equation). The presence of arbitrary functions in the solution may increase the class of metrics under consideration. A large number of papers deal with the construction of the solutions [1,2]. For example, a construction was obtained that permits one to parametrize the solution of the initial equation by the solution of the radial part of the Laplace equation [4]. This construction generalizes the well-known classes of Weyl and Papapetrou [5]. But the given constructions do not provide the presence of an arbitrary function in the solution. In this situation, in accordance with the idea proposed by one of the authors in the chiral and Toda models [6] it is useful to refrain from some number of arbitrary functions in the solution and to obtain a simply enough explicit formula containing an arbitrary function. Such formulae were obtained by one of us for O(3) and O(2,1) σ-models and by means of a generalized Pohlmeyer transformation [7] for the $A_1^{(1)}$ Toda chain [6]. In ref. [6] the simplest case of the Ernst equation was also considered. The consideration, as in ref. [6], is based on reducing the Ernst equation to an equation for the O(2,1) σ-model. We use a special choice of the determinant of the metric instead of using the Weyl coordinates. Such a consideration gives the possibility to obtain a non-trivial solution, parametrized by arbitrary functions by means of a solution of the corresponding O(2,1) σ-model. It should be noted that in ref. [6] another Ansatz gives an elliptic solution of the O(2,1) σ-model and that the case of the Ernst equation considered in ref. [6] corresponds to the meron sector of the O(2,1) σ-model. It should also be noted that the simplest of the solutions considered in this letter has a non-trivial curvature tensor: not all components R^i_{jkl} are equal to zero.

In section 3 we give the solution of the Ernst equation, in section 4 the main part of the metric, in section 5 the full metric and the curvature in the simples case, and in section 6 we consider the Einstein–Maxwell case.

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2. The two-dimensional Einstein equation or Ernst equation (we follow the notation of ref. [8]) corresponds to the metric

$$-\mathrm{d}s^2 = H(t,x)(\mp \mathrm{d}t^2 + \mathrm{d}x^2) + f_{AB}(t,x)\mathrm{d}x^A \mathrm{d}x^B\,, \quad A,B=2,3\,, \quad x_2=y\,, \quad x_3=z \tag{1}$$

The cases of an axially symmetric stationary vacuum and gravitational waves are considered in the same way. The upper sign corresponds to the waves, the lower one to the vacuum. In the first case the solution is parametrized by two arbitrary functions depending on light-cone variables and in the second case by an arbitrary holomorphic function. We consider the new variables u and v, $u=x+\mathrm{i}t$, $v=x-\mathrm{i}t$ and $u=\overline{v}$ in the vacuum case, and $u=\frac{1}{2}(x+t)$, $v=\frac{1}{2}(t-x)$ in the wave case. In the coordinates u, v the Ernst equations are

$$f_{uv} + (\tau_u f_v + \tau_v f_u)/2\tau - 2f_u f_v/(f+\overline{f}) = 0\,, \quad \tau_{uv}=0\,, \quad \tau=\overline{\tau}\,, \quad f\neq\overline{f}\,, \quad \tau^2 = \det f_{AB}\,. \tag{2}$$

The metric f_{AB} from (1) is obtained from the solution of eqs. (2):

$$f_{AB} = \tau \begin{vmatrix} -2/(f+\overline{f}) & \mathrm{i}(f-\overline{f})/(f+\overline{f}) \\ \mathrm{i}(f-\overline{f})/(f+\overline{f}) & -2f\overline{f}/(f+\overline{f}) \end{vmatrix}\,, \tag{3a}$$

and H is obtained from the solution of the linear equations.

$$(\ln H)_u = (\ln\tau)_{uu}/(\ln\tau)_u + \mathrm{Sp}\,A^2/4\tau\tau_u\,, \quad (\ln H)_v = (\ln\tau)_{vv}/(\ln\tau)_v + \mathrm{Sp}\,B^2/4\tau\tau_v\,,$$

$$A = -\tau(f_{AB})_u(f_{AB})^{-1}\,, \quad B = \tau(f_{AB})_v(f_{AB})^{-1}\,. \tag{3b}$$

If τ = const eqs. (2) give the equation of motion of the O(2,1) σ-model. Its lagrangian is:

$$L = h(f_u\overline{f}_v + f_v\overline{f}_u)\,, \tag{4}$$

where $h = h_{\mathrm{O}(2,1)} = (f+\overline{f})^{-2}$ is the metric on the group O(2,1).

3. We look for a solution of the form [6]

$$f(U,V) = A(y)\exp \mathrm{i}N(y)\,, \tag{5}$$

where

$$y=y(u,v)\,, \quad y=\overline{y}\,, \quad y_{uv}=0\,, \tag{6}$$

and

$$\tau=\tau(s)\,, \quad s=s(u,v)\,, \quad s=\overline{s}\,, \quad s_{uv}=0\,,$$

where the condition

$$\tau_u f_v + \tau_v f_u = 0\,, \quad \tau\neq \text{const}\,, \tag{7a}$$

in eq. (2) gives the relation

$$s_v y_u + s_u y_v = 0\,. \tag{7b}$$

If (7b) holds eq. (2) transforms into a system of two differential equations for the functions $A=A(y)$ and $N=N(y)$:

$$(A_y/A)_y + 2(A_y/A)N_y\tan N = 0\,, \quad N_{yy} - (A_y/A)^2\tan N + N_y^2\tan N = 0\,. \tag{8}$$

From the first of them with

$$B = A_y/A\,, \tag{9}$$

we obtain

$$B = B_0 \cos^2 N(y)$$

and

$$A = A_0 \exp B_0 \int \cos^2 N(y)\, dy\,. \tag{10}$$

The second one gives

$$N_{yy} + N_y^2 \tan N - B_0^2 \cos^4 N \tan N = 0\,. \tag{11}$$

From this we obtain

$$y - y_0 = \int \frac{dN}{\cos(N)(C + B_0^2 \sin^2 N)^{1/2}} \tag{12}$$

C, B_0, y_0, A_0 are real constants. Further let us consider the three cases: (1) $B_0 = 0, C \neq 0$; (2) $C = 0, B_0 \neq 0$; (3) $C \neq 0, B_0 \neq 0$. In case (1) we obtain:

$$f = A_0 \left(\frac{2 \exp \sqrt{C}(y - y_0)}{1 + \exp 2\sqrt{C}(y - y_0)} + \mathrm{i}\, \frac{\exp 2\sqrt{C}(y - y_0) - 1}{1 + \exp 2\sqrt{C}(y - y_0)} \right). \tag{13}$$

From (9), (10) it can be seen that this corresponds to the case of constant modulus $A = A_0$. If $C = 0$ we have from (12)

$$\sin N = \exp[B_0(y - y_0)]\,\{1 + \exp[2B_0(y - y_0)]\}^{-1/2}\,, \quad \cos N = \{1 + \exp[2B_0(y - y_0)]\}^{-1/2}$$

and by means of (9), (10) we finally have:

$$f = A_0 \frac{\exp(B_0 y)}{1 + \exp[2B_0(y - y_0)]} \{1 + \mathrm{i} \exp[B_0(y - y_0)]\}\,. \tag{14}$$

In this case $|f| \neq$ const. In case (3) with $C \neq 0$ we have

$$(y - y_0) 2(B_0^2 + C)^{1/2} = \ln \left| \frac{(C + B_0^2)^{1/2} \sin N + (B_0^2 \sin^2 N + C)^{1/2}}{-(C + B_0^2)^{1/2} \sin N + (B_0^2 \sin^2 N + C)^{1/2}} \right|, \tag{15}$$

this gives ($C > 0$

$$\sin N = \frac{\sqrt{C}[\exp(2\mathscr{A}) - 1]}{\{C[\exp(2\mathscr{A}) + 1]^2 + 4B_0^2 \exp(2\mathscr{A})\}^{1/2}}\,,$$

$$\cos N = \frac{2(C + B_0^2)^{1/2} \exp \mathscr{A}}{\{C[\exp(2\mathscr{A}) + 1]^2 + 4B_0^2 \exp(2\mathscr{A})\}^{1/2}}\,, \tag{16}$$

with $\mathscr{A} = (B_0^2 + C)^{1/2}\,(y - y_0)$. Further by using (16) and (9), (10) we have for $A(y)$ with $y_0 \neq 0$ ($B_0^2 + C > 0$)

$$A = A_0 \left(\frac{2C[\exp(2\mathscr{A}) + 1] + 4B_0^2 - 4B_0(B_0^2 + C)^{1/2}}{2C[\exp(2\mathscr{A}) + 1] + 4B_0^2 - 4B_0(B_0^2 + C)^{1/2}} \right)^{1/2}. \tag{17}$$

Now we must have the conditions (6), (7) satisfied, that is, we must give the expressions for y and s. Analogously

[6] we have in the vacuum case ($u = z$, $v = \bar{z}$):

$$y \quad \text{or} \quad s = \tfrac{1}{2}\ln[g(z)\bar{g}(\bar{z})] \quad \text{or} \quad (1/2\mathrm{i})\ln[g(z)/\bar{g}(\bar{z})]\ , \tag{18a}$$

(one of the two), $g = g(z)$, $g_{\bar{z}} = 0$; in the wave case (u, v are light-cone variables).

$$y \quad \text{or} \quad s = \tfrac{1}{2}\ln[g_1(u)g_2(v)] \quad \text{or} \quad \tfrac{1}{2}\ln[g_1(u)/g_2(v)] \tag{18b}$$

(one of the two), $g_1 = g_1(u)$, $g_2 = g_2(v)$, and

$$\tau = k_1 s + k_2\ , \tag{18c}$$

where k_1, k_2 are real constants. So, formulae (16)–(18) give the solution of eqs. (2) if $C \neq 0$ and (14), (18) if $C = 0$. These solutions are parametrized by an arbitrary holomorphic function $g = g(z)$ in the vacuum case (18a) and by two arbitrary functions $g_1 = g_1(u)$, $g_2 = g_2(v)$ in the wave case (18b), $\tau^2 = \det f_{AB}$ in formula (18c).

4. According to formula (3a) let us find the main part of the metric f_{AB}. The solution (13), (18) gives

$$f_{AB} = -\tau \begin{vmatrix} \cosh[\sqrt{C}(y - y_0)]/A_0 & \sinh[\sqrt{C}(y - y_0)] \\ \sinh[\sqrt{C}(y - y_0)] & A_0 \cosh[\sqrt{C}(y - y_0)] \end{vmatrix} . \tag{19}$$

The solution (14), (18) gives

$$f_{AB} = -\tau \begin{vmatrix} 1 + \exp[2B_0(y - y_0)]/A_0 \exp(B_0 y) & \exp[B_0(y - y_0)] \\ \exp[B_0(y - y_0)] & A_0 \exp(B_0 y) \end{vmatrix} . \tag{20}$$

In the wave case

$$f_{AB} = -\tfrac{1}{2}\{k_1 \ln[g_1(u)/g_2(v)] + k_2\} \begin{vmatrix} [1 + \exp(-2B_0 y_0)(g_1 g_2)^{B_0}]/A_0(g_1 g_2)^{B_0/2} & \exp(-B_0 y_0)(g_1 g_2)^{B_0/2} \\ \exp(-B_0 y_0)(g_1 g_2)^{B_0/2} & A_0(g_1 g_2)^{B_0/2} \end{vmatrix}$$

The solutions (16)–(18) give

$$f_{AB} = -\tau f_0 \begin{vmatrix} \hat{f}_{11} & \hat{f}_{12} \\ \hat{f}_{21} & \hat{f}_{22} \end{vmatrix} ,$$

where

$$\hat{f}_{11} = \hat{f}_{22}^{-1} = A_0^{-1} \left(\frac{2C[\exp(2\mathscr{A}) + 1] + 4B_0^2 + 4B_0(B_0^2 + C)^{1/2}}{2C[\exp(2\mathscr{A}) + 1] + 4B_0^2 - 4B_0(B_0^2 + C)^{1/2}} \right)^{1/2} ,$$

$$\hat{f}_{12} = \hat{f}_{21} = \frac{\sqrt{C}[\exp(2\mathscr{A}) - 1]}{\{C[\exp(2\mathscr{A}) + 1] + 4B_0^2 \exp(2\mathscr{A})\}^{1/2}} ,$$

$$f_0 = \frac{\{C[\exp(2\mathscr{A}) + 1]^2 + 4B_0^2 \exp(2\mathscr{A})\}^{1/2}}{2(B_0^2 + C)^{1/2} \exp\mathscr{A}} . \tag{21}$$

5. The function H in the metric (1) is given by the solution of the linear equations (3b). From (3b) we have in the simplest case (13), (19) (we shall consider the other cases separately)

$\mathrm{Sp}\,A^2 = 2(\tau_u^2 + \tau^2 y_u^2 C)$, $\mathrm{Sp}\,B^2 = 2(\tau_v^2 + \tau^2 y_v^2 C)$,

and for H we obtain the expression

$$H = \exp[(C/4k_1^2)\tau^2]|\tau_u \tau_v|/\sqrt{|\tau|}, \tag{22}$$

where for τ we have the expressions (18c), (18a) or (18b). It should be noted that in (22) it is possible to omit the sign of the modulus as the arbitrariness permits us to choose H real and positive. Thus the formulae (19), (22) give the metric (1) parametrized by two arbitrary real functions depending on the light-cone variables $g_1 = g_1(u)$, $g_2 = g_2(v)$ in the wave case or by one holomorphic function $g = g(z)$, $g_{\bar{z}} = 0$ in the vacuum case. In the particular wave case when $C = 1$, $k_1 = -2$, $y_0 = k_2 = 0$, $g_1(u) = \exp u$, $g_2(v) = \exp(-v)$ we have from (18c), (18b) $|\tau| = u + v = t$ and from (19), (22)

$$f_{\mathrm{AB}} = t \begin{vmatrix} A_0^{-1}\cosh\frac{1}{2}x & \sinh\frac{1}{2}x \\ \sinh\frac{1}{2}x & A_0\cosh\frac{1}{2}x \end{vmatrix}, \quad H = t^{-1/2}\exp\tfrac{1}{16}t^2 .$$

It should be noted that even this simplest case has a nontrivial curvature tensor, for example

$$R^2_{223} = -\frac{\sinh\frac{1}{2}x\,\exp -\frac{1}{16}t^2}{4\sqrt{t}}(1 + \tfrac{1}{4}t^2) .$$

We shall consider the curvature invariants separately. It should be noted that the obtained solution (13) looks like the Papapetrou solution [5] but in the last case the solution is parametrized by the solution of the radial part of the Laplace equation and therefore does not provide the presence of an arbitrary function in the solution.

6. The Einstein–Maxwell equations are (we follow the notation for ref. [9])

$$(\mathrm{Re}\,E + \Phi\bar{\Phi})(2E_{uv} + w^{-1}(w, E)) = 2E_u E_v + 2\bar{\Phi}(\Phi, E) ,$$

$$(\mathrm{Re}\,E + \Phi\bar{\Phi})(2\Phi_{uv} + w^{-1}(w, \Phi)) = 4\bar{\Phi}\Phi_u\Phi_v + (\Phi, E) , \quad w = \bar{w} , \quad w_{uv} = 0 , \tag{23}$$

$$(w, E) \overset{\mathrm{def}}{=} w_u E_v + w_v E_u , \tag{24}$$

where Φ is the Maxwell field, E is the gravitational field. We look for a solution in the form:

$$E = E_0 \exp \mathrm{i}N(x) , \quad \Phi = B(x)\exp[\mathrm{i}M(x)] , \quad w = w(s) , \tag{25}$$

where

$$x = x(u, v) , \quad s = s(u, v) , \quad x = \bar{x} , \quad s = \bar{s} , \quad x_{uv} = 0 , \quad s_{uv} = 0 . \tag{26}$$

In this letter we consider the simples case

$$M_x = b , \quad N_x = a , \tag{27}$$

where E_0, a, b are real constants. According to (24)–(26) we have

$$(w, E) = w_s E_x(s_u x_v + s_v x_u) , \quad (w, \Phi) = w_s \Phi_x(s_u x_v + s_v x_u)$$

We choose the functions $x(u, v)$, $s(u, v)$ so that they should give

$$(w, E) = 0 , \quad (w, \Phi) = 0 . \tag{28}$$

Eqs. (23) with the use of the expression (25)–(27) are reduced to the four equations (two of them coincide)

$$2b = a\,, \quad E_0 a \sin ax = 2BB_x\,, \quad B_{xx} + b^2 B = 0\,.$$

Thus we have the solution for (23)

$$E = E_0 \exp iax\,, \quad \Phi = B(x) \exp \tfrac{1}{2} iax\,, \quad B^2(x) = E_0(1 - \cos ax)\,.$$

Now we have to satisfy the conditions (28), that is to give the expressions for x, s, w. We have

$$x \quad \text{or} \quad s = \tfrac{1}{2} \ln[g(z)\bar{g}(\bar{z})] \quad \text{or} \quad (1/2\mathrm{i}) \ln[g(z)/\bar{g}(\bar{z})]\,, \quad g = g(z)\,, \quad g_{\bar{z}} = 0\,,$$

for $u = z$, $v = \bar{z}$ and $u = \bar{v}$;

$$x \quad \text{or} \quad s = \tfrac{1}{2} \ln[g_1(u) g_2(v)] \quad \text{or} \quad \tfrac{1}{2} \ln[g_1(u)/g_2(v)]\,,$$

for u, v, the light-cone variables and $w = k_1 s + k_2$, where k_1, k_2 are real constants. It should be noted that Re $E + \Phi\bar{\Phi} = E_0$ but this *simpest* solution is nontrivial because the "background" gravitational field $E = E_0 \exp iax$ has a nontrivial curvature tensor. We shall consider these questions in detail separately.

We extend this construction also to the case of presence of matter: perfect fluid with equation of state $\epsilon = p$ [10]. We get formulae that might be looked upon as a possible description of the gravitational waves. We think so because the Ernst equation is conformally invariant just as the wave equation $f_{tt} - f_{xx} = 0$ and consequently its "wave solution" (in contrast to the "soliton sector") also must be parametrized by two arbitrary functions of the light-cone variables $g_1(t + x)$, $g_2(t - x)$.

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