Localization, Multiscales and Complex Quantum Patterns

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We present a family of methods, analytical and numerical, which can describe behaviour in (non) equilibrium ensembles, both classical and quantum, especially in the complex systems, where the standard approaches cannot be applied. We demonstrate the creation of nontrivial (meta) stable states (patterns), localized, chaotic, entangled or decoherent, from basic localized modes in various collective models arising from the quantum hierarchy of Wigner-von Neumann-Moyal-Lindblad equations, which are the result of "wignerization" procedure of classical BBGKY hierarchy.

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It is widely known that the currently available experimental techniques (and, apparently, those which will become avaiable in the nearest future) in the area of quantum physics as a whole and in that of quantum computations in particular, as well as the present level of understanding of phenomenological models, outstripped the actual level of mathematical/theoretical description. Considering, for example, the problem of describing the realizable states, one should not expect that planar waves and (squeezed) gaussian coherent states would be enough to characterize such complex systems as qCPU-like devices. Complexity of the set of relevant states, including entangled (chaotic) states is still far from being clearly understood and moreover from being realizable.

As a starting point for our approach let us consider the following well-known example of GKP (Gottesman, Kitaev, Preskill)scheme with DV (Discrete Variables)/qubit (with finite-dimensional code space embedded in the infinite-dimensional Hilbert space) or CV (Continuous Variables) for (optical) quantum computations, containing as a part (optical) nonlinearities, described by Kerr interaction or more general polynomial Hamiltonians which are needed to realize the state preparation and provide the process of CV quantum computation.

It is an important example because:

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(a) its classical counterpart is described by polynomial Hamiltonians;
(b) the proper qudits or building states (DV or CV) are well localized (but not well-defined mathematically, as we shall explain later).
One of the questions which motivated our approach is whether it is possible to keep **(a)** and at the same time improve **(b)**. Our other motivations arise from the following general questions:
(A) How can we represent well localized and reasonable state in mathematically correct form?
(B) Is it possible to create entangled and other relevant states by means of these new building blocks?

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In GKP scheme unphysical and not clearly defined mathematically logical qubit states are represented via infinite series of δ functions:
$|0>=\sum_{s=-\infty}^{\infty}\delta(x-2s\sqrt{\pi})|x>, |1>=\sum_{s=-\infty}^{\infty}\delta(x-(2s+1)\sqrt{\pi})|x>$
and approximated by the set of gaussian envelopes:
$<x|0>=N_0\sum_{-\infty}^{+\infty}e^{-1/2(2sk\sqrt{\pi})^2}e^{-1/2(\frac{x-2s\sqrt{\pi}}{\triangle})^2}$,
$<x|1>=N_1\sum_{-\infty}^{+\infty}e^{-1/2(2s+1)k\sqrt{\pi})^2}e^{-1/2(\frac{x-(2s+1)\sqrt{\pi}}{\triangle})^2}$.
Due to numerous mathematical and computational reasons, some of which are described below, such and related choices cannot be appropriate neither as a starting point on the route to the real qCPU device nor as a satisfactory theoretical description.
One needs to sketch up the underlying ingredients of the theory (spaces of states, observables, measures, classes of smoothness, quantization set-up etc) in an attempt to provide the maximally extendable but at the same time really calculable and realizable description of the dynamics of quantum world.

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The general idea is rather simple: it is well known that the idea of "symmetry" is the key ingredient of any reasonable physical theory from classical (in)finite dimensional (integrable) Hamiltonian dynamics to different sub-planckian models based on strings, etc.
A starting point for us is a possible model for (continuous) "qudit" with subsequent description of the whole zoo of possible realizable (controllable) states/patterns which may be useful from the point of view of quantum experimentalists and engineers. The proper representation theory is well known as "local nonlinear harmonic analysis", in particular case of simple underlying symmetry–affine group–aka wavelet analysis.

From our point of view the advantages of such approach are as follows:
i) natural realization of localized states in any proper functional realization of (Hilbert) space of states
ii) hidden symmetry of chosen realization of proper functional model provides the (whole) spectrum of possible states via the so-called multiresolution decomposition.
It is obvious, that consideration of symbols of operators instead of operators themselves is the starting point as for the mathematical theory of pseudodifferential operators as for quantum dynamics formulated in the language of Wigner-like equations. It should be noted that in such picture we can naturally include the effects of self-interaction ("quantum non-linearity") on the way of construction and subsequent analysis of nonlinear quantum models. So, our consideration will be in the framework of (Nonlinear) Pseudodifferential Dynamics (ΨDOD). As a result of i), ii), we'll have:
iii) most sparse, almost diagonal, representation for a wide class of operators included in the set-up of the whole problems.
It's possible by using the so-called Fast Wavelet Transform representation for algebra of observables.
Then points i)–iii) provide us by

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iv) natural (non-perturbative) multiscale decomposition for all dynamical quantities, as states as observables.
The simplest case we will have, obviously, in Wigner-Weyl representation. Existence of such internal multiscales with different dynamics at each scale and transitions, interactions, and intermittency between scales demonstrates that quantum mechanics, despite its linear structure, is really a serious part of physics from the mathematical point of view. It seems, that well-known underlying quantum complexity is a result of transition by means of (still rather unclear) procedure of quantization from complexity related to nonlinearity of classical counterpart to the rich pseudodifferential (more exactly, microlocal) structure on the quantum side.

v) localized modes (basis modes, eigenmodes) and constructed from them chaotic or entangled, decoherent (if we change Wigner equation for (master) Lindblad one) patterns.
It should be noted that these bases modes are nonlinear in contrast with usual ones because they come from (non) abelian generic group while the usual Fourier (commutative) analysis starts from $U(1)$ abelian modes (plane waves). They are really "eigenmodes" but in sense of decomposition of representation of the underlying hidden symmetry group which generates the multiresolution decomposition. The set of patterns is built from these modes by means of variational procedures more or less standard in mathematical physics. It allows to control the convergence from one side but, what is more important,
vi) to consider the problem of the control of patterns (types of behaviour) on the level of reduced (variational) algebraical equations.
We need to mention that it is possible to change the simplest generic group of hidden internal symmetry from the affine (translations and dilations) to much more general, but, in any case, this generic symmetry will produce the proper natural high localized eigenmodes, as well as the decomposition of the functional realization of space of states into the proper orbits; and all that allows to compute dynamical consequence of this procedure, i.e. pattern formation, and, as a result, to classify the whole spectrum of proper states.

For practical reasons controllable patterns (with prescribed behaviour) are the most useful. We mention the so-called waveleton-like pattern which we regard as the most important one. We use the following allusion in the space of words:
$\{\text{waveleton}\}{:=}\{\text{soliton}\} \bigsqcup \{\text{wavelet}\}$
vii) waveleton $\approx$ (meta)stable localized (controllable) pattern
To summarize, the approach described below allows one
viii) to solve wide classes of general ΨDOD problems, including generic for quantum physics Wigner-like equations, and
ix) to present the analytical/numerical realization for physically interesting patterns.
We would like to emphasize the effectiveness of numerical realization of this program (minimal complexity of calculations) as additional advantage. So, items i)-ix) point out all main features of our approach.

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Class of Models

Here we describe a class of problems which can be analysed by methods described in Introduction. We start from individual dynamics and finish by (non)-equilibrium ensembles. All models belong to the ΨDOD class and can be described by finite or infinite (named hierarchies in such cases) system of ΨDOD equations:

a). Individual classical/quantum mechanics (cM/qM): linear/nonlinear; $\{cM\} \subset \{qM\}$, $*$ - quantized for the class of polynomial Hamiltonians $H(p, q, t) = \sum_{i,j} a_{ij}(t) p^i q^j$.

b). QFT-like models in framework of the second quantization (dynamics in Fock spaces).

c.) Classical (non) equilibrium ensembles via BBGKY Hierarchy (with reductions to different forms of Vlasov-Maxwell/Poisson equations).

d.) Wignerization of a): Wigner-Moyal-Weyl-von Neumann-Lindblad.

e.) Wignerization of c): Quantum (Non) Equilibrium Ensembles.

Effects we are interested in

1. Hierarchy of internal/hidden scales (time, space, phase space).

2. Non-perturbative multiscales: from slow to fast contributions, from the coarser to the finer level of resolution/decomposition.

3. Coexistence of hierarchy of multiscale dynamics with transitions between scales.

4. Realization of the key features of the complex quantum world such as the existence of chaotic and/or entangled states with possible destruction in "open/dissipative" regimes due to interactions with quantum/classical environment and transition to decoherent states.

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At this level we may interpret the effect of mysterious entanglement or "quantum interaction" as a result of simple interscale interaction or intermittency (with allusion to hydrodynamics), i.e. the mixing of orbits generated by multiresolution representation of hidden underlying symmetry. Surely, the concrete realization of such a symmetry is a natural physical property of the physical model as well as the space of representation and its proper functional realization. So, instantaneous interactions (or transmission of "quantum bits" or "teleportation") materialize not in the physical space-time variety but in the space of representation of hidden symmetry along the orbits/scales constructed by proper representations. Dynamical/kinematical principles of usual space-time varieties, definitely, do not cover kinematics of internal quantum space of state or, in more weak formulation, we still have not such explicit relations.

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One additional important comment:
{QMF (Quadratic Mirror Filters)} $\longrightarrow$ Loop groups $\longrightarrow$ Cuntz operator algebra $\longrightarrow$ Quantum Group structure.
It should be noted the appearance of natural Fock structure inside this functorial sequence above with the creation operator realized as some generalization of Cuntz-Toeplitz isometries. All that can open a new vision of old problems and bring new possibilities.

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We finish this part by the following qualitative definitions of key objects (patterns). Their description and understanding in different physical models is our main goal in this direction.
By localized states (localized modes) we mean the building blocks for solutions or generating modes which are localized in maximally small region of the phase (as in c- as in q-case) space.
By an entangled/chaotic pattern we mean some solution (or asymptotics of solution) which has random-like distributed energy (or information) spectrum in a full domain of definition. In quantum case we need to consider additional entangled-like patterns, roughly speaking, which cannot be separated into pieces of sub-systems.

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By a localized pattern (waveleton) we mean (asymptotically) (meta) stable solution localized in a relatively small region of the whole phase space (or a domain of definition). In this case the energy is distributed during some time (sufficiently large) between only a few localized modes (from point 1). We believe it to be a good model for plasma in a fusion state (energy confinement) or a model for quantum continuous "qubit" or a result of the decoherence process in open quantum system when the full entangled state degenerates into localized (quasiclassical) pattern.

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Representation theory of internal/hidden/underlying symmetry, Kinematical, Dynamical, Hidden.
Arena (space of representation): proper functional realization of (Hilbert) space of states.
Harmonic analysis on (non)abelian group of internal symmetry.
Local/Nonlinear (non-abelian) Harmonic Analysis (e.g, wavelet/gabor etc. analysis) instead of linear non-localized $U(1)$ Fourier analysis.
Multiresolution (multiscale) representation. Dynamics on proper orbit/scale (inside the whole hierarchy of multiscales) in functional space. The key ingredients are the appearance of multiscales (orbits) and the existence of high-localized natural (eigen)modes.
Variational formulation (control of convergence, reductions to algebraic systems, control of type of behaviour).

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Set-up/Formulation

Let us consider the following generic ΨDOD dynamical problem

$$L^j\{Op^i\}\Psi = 0,$$

described by a finite or infinite number of equations which include general classes of operators Op^i such as differential, integral, pseudodifferential etc

Surely, all Wigner-like equations/hierarchies are inside.

The main objects are:

(Hilbert) space of states, $H = \{\Psi\}$, with a proper functional realization, e.g.,: L^2, Sobolev, Schwartz, C^0, C^k, ... C^∞, ...;

Definitely, $L^2(R^2)$, $L^2(S^2)$, $L^2(S^1 \times S^1)$, $L^2(S^1 \times S^1 \times Z_n)$ are different objects proper for different physics inside.

Decompositions

$$\Psi \approx \sum_i a_i e^i$$

via high-localized bases (wavelet families, generic wavelet packets etc), frames, atomic decomposition, etc. with the following main properties: (exp) control of convergence, maximal rate of convergence for any Ψ in any H. Figure 1 demonstrates the full set of localized nonlinear basis eigenmodes.

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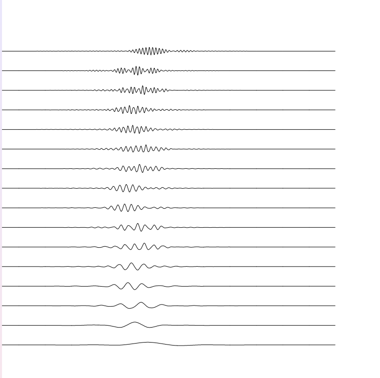

Figure: 1 Localized Basis Nonlinear Eigenmodes.

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Observables/Operators (ODO, PDO, ΨDO, SIO,..., Microlocal analysis of Kashiwara-Shapira (with change from functions to sheafs)) satisfy the main property – the matrix representation in localized bases

$$< \Psi|Op^i|\Psi >$$

is maximum sparse:

$$\begin{pmatrix} D_{11} & 0 & 0 & \dots \\ 0 & D_{22} & 0 & \dots \\ 0 & 0 & D_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This almost diagonal structure is provided by the so-called Fast Wavelet Transform.

Class of smoothness. The proper choice determines natural consideration of dynamics with/without Chaos/Fractality property.
Measures: multifractal wavelet measures $\{\mu_i\}$ together with the class of smoothness are very important for analysis of complicated analytical behaviour.
Variational/Projection methods, from Galerkin to Rabinowitz minimax, Floer (in symplectic case of Arnold-Weinstein curves with preservation of Poisson/symplectic structures). Main advantages are the reduction to algebraic systems, which provides a tool for the smart subsequent control of behaviour and control of convergence.

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Multiresolution or multiscale decomposition, *MRA* (or wavelet microscope) consists of the understanding and choosing of

1). (internal) symmetry structure, e.g., affine group $= \{$translations, dilations$\}$ or many others; construction of

2). representation/action of this symmetry on $H = \{\Psi\}$.
As a result of such hidden coherence together with using point vi) we'll have:

   a). LOCALIZED BASES   b). EXACT MULTISCALE DECOMPOSITION with the best convergence properties and real evaluation of the rate of convergence via proper "multi-norms".

Figures 2, 3 and 4, 5 demonstrate examples of MRA representations of complicated (singular) functions via basis high-localized nonlinear eigenmodes. Figures 6, 7 demonstrate MRA decomposition for "fractal-like" object: Riemann-Weierstrass Function

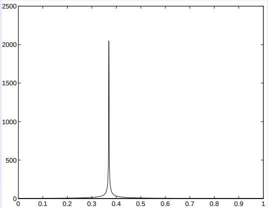

Figure: 2 Kick-like Function.

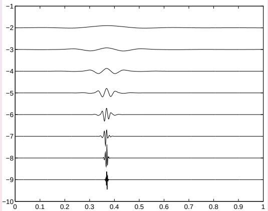

Figure: 3 MRA Decomposition for Kick.

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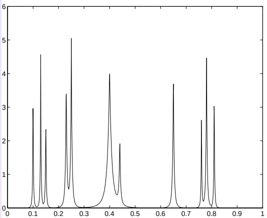

Figure: 4 Multi-Kicks.

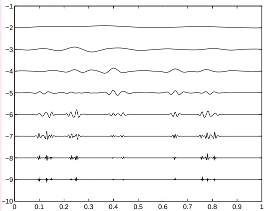

Figure: 5 MRA Decomposition for Multi-Kicks.

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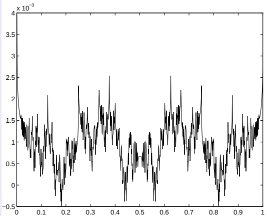

Figure: 6 RW-fractal.

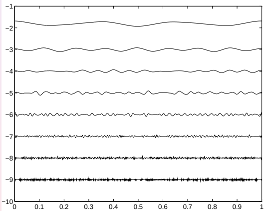

Figure: 7 MRA for RW-fractal.

Effectiveness of proper numerics: CPU-time, HDD-space, minimal complexity of algorithms, and (Shannon) entropy of calculations are provided by points i)-vii) above.

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Quantization via $*$ (Star Product) or Deformation Quantization

Omitted in the short version. Details can be found in the full version on the Web.

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Only one example: Wigner–Moyal-Lindblad (Master) Equation (Wignerization)

$$\dot{W} = \{H, W\}_{PB} + \sum_{n\geq 1} \frac{\hbar^{2n}(-1)^n}{2^{2n}(2n+1)!}\partial_q^{2n+1}U(q)\partial_p^{2n+1}W(q,p) + 2\gamma\partial_p pW + D\partial_p^2 W, \tag{1}$$

and it is more preferable for us.

Multiscale Decomposition for Space of States: Functional Realization and Metric Structure

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Some details omitted in the short version. All details can be found in the full version on the Web.

We obtain our multiscale/multiresolution representations for solutions of Wigner-like equations via a variational-wavelet approach. We represent the solutions as decomposition into localized eigenmodes (regarding action of affine group, i.e. hidden symmetry of the underlying functional space of states) related to the hidden underlying set of scales:

$$W_n(t,q,p)=\bigoplus_{i=i_c}^{\infty}W_n^i(t,q,p), \qquad (2)$$

where value i_c corresponds to the coarsest level of resolution c or to the internal scale with the number c in the full multiresolution decomposition (MRA) of the underlying functional space (L^2, e.g.) corresponding to the problem under consideration:

$$V_c\subset V_{c+1}\subset V_{c+2}\subset\dots \qquad (3)$$

and $p=(p_1,p_2,\dots),\quad q=(q_1,q_2,\dots),\quad x_i=(p_1,q_1,\dots,p_i,q_i)$ are coordinates in phase space. In the following we may consider as fixed as variable numbers of particles.

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We introduce the Fock-like space structure (in addition to the standard one, if we consider second-quantized case) on the whole space of internal hidden scales

$$H = \bigoplus_i \bigotimes_n H_i^n \tag{4}$$

for the set of n-partial Wigner functions (states):

$$W^i = \{W_0^i, W_1^i(x_1; t), \dots, W_N^i(x_1, \dots, x_N; t), \dots\}, \tag{5}$$

where $W_p(x_1, \dots, x_p; t) \in H^p$, $H^0 = C$, $\quad H^p = L^2(R^{6p})$ (or any different proper functional space), with the natural Fock space like norm:

$$(W, W) = W_0^2 + \sum_i \int W_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^{i} \mu_\ell. \tag{6}$$

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Then we have the following decomposition:

$$\{W(t)\} = \bigoplus_{-\infty<j<\infty} D_j = \overline{V_c \bigoplus_{j=0}^{\infty} D_j}, \tag{7}$$

in case when V_c is the coarsest scale of resolution. The subgroup of translations generates a basis for the fixed scale number: $\mathrm{span}_{k\in Z}\{2^{j/2}\Psi(2^j t-k)\} = D_j$. The whole basis is generated by action of the full affine group:

$$\mathrm{span}_{k\in Z, j\in Z}\{2^{j/2}\Psi(2^j t-k)\} = \mathrm{span}_{k,j\in Z}\{\Psi_{j,k}\} = \{W(t)\}. \tag{8}$$

Tensor Product Structure

Omitted in the short version. Details can be found in the full version on the Web.

After construction the multidimensional bases we obtain our multiscale/multiresolution representations for observables (symbols), states, partitions via the variational approaches as for c-BBGKY as for its quantum counterpart and related reductions but before we need to construct reasonable multiscale decomposition for all operators included in the set-up.

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FWT Decomposition for Observables

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As a result, we have the simple linear parametrization of matrix representation of our operators in localized wavelet bases and of the action of this operator on arbitrary vector/state in proper functional space.

Variational Approach

Now, after preliminary work with (functional) spaces, states and operators, we may apply our variational approach.

Let L be an arbitrary (non)linear differential/integral operator with matrix dimension d (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes^n})$:

$\Psi \equiv \Psi(t, x_1, x_2, \dots) = \Big(\Psi^1(t, x_1, x_2, \dots), \dots, \ \Psi^d(t, x_1, x_2, \dots)\Big)$,

$x_i \in \Omega \subset \mathbf{R}^6$, n is the number of particles:

$$\begin{aligned} L\Psi &\equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \qquad (9) \\ Q &\equiv Q_{d_0, d_1, d_2, \dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) \\ &= \sum_{i_0, i_1, i_2, \dots = 1}^{d_0, d_1, d_2, \dots} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots) \Big(\frac{\partial}{\partial t}\Big)^{i_0} \Big(\frac{\partial}{\partial x_1}\Big)^{i_1} \Big(\frac{\partial}{\partial x_2}\Big)^{i_2} \dots \int \mu_k. \end{aligned}$$

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Let us consider now the N mode approximation for the solution as the following ansatz:

$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0, i_1, i_2, \dots = 1}^{N} a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots (t, x_1, x_2, \dots) \tag{10}$$

We will determine the expansion coefficients from the following conditions (related to proper choosing of variational approach):

$$\ell^N_{k_0, k_1, k_2, \dots} \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) \mathrm{d}t \mathrm{d}x_1 \mathrm{d}x_2 \dots = 0. \tag{11}$$

Thus, we have exactly dN^n algebraical equations for dN^n unknowns $a_{i_0, i_1, \dots}$. This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second one. It allows to unify the multiresolution expansion with variational construction.

So, the solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the generic operator L) and the rest are linear problems related to the computation of the coefficients of reduced algebraic equations. It is also related to the choice of exact measure of localization (including class of smoothness) which are proper for our set-up. These coefficients can be found by some functional/algebraic methods by using the compactly supported wavelet basis functions or any other wavelet families.

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As a result the solution of the equations/hierarchies from Section 4, as in c- as in q-region, has the following multiscale or multiresolution decomposition via nonlinear high-localized eigenmodes

$$W(t, x_1, x_2, \dots) = \sum_{(i,j)\in Z^2} a_{ij} U^i \otimes V^j(t, x_1, \dots),$$

$$V^j(t) = V_N^{j,slow}(t) + \sum_{l\geq N} V_l^j(\omega_l t), \ \omega_l \sim 2^l, \tag{12}$$

$$U^i(x_s) = U_M^{i,slow}(x_s) + \sum_{m\geq M} U_m^i(k_m^s x_s), \ k_m^s \sim 2^m,$$

which corresponds to the full multiresolution expansion in all underlying time/space scales.

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The formulas (12) give the expansion into a slow part and fast oscillating parts for arbitrary N, M. So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process. In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. Formulas (12) do not use perturbation techniques or linearization procedures. Numerical calculations are based on compactly supported wavelets and wavelet packets and on evaluation of the accuracy on the level N of the corresponding cut-off of the full system regarding norm (6):

$$\|W^{N+1} - W^{N}\| \le \varepsilon.$$

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Details omitted in the short version, can be found in the full version on the Web.

To summarize, the key points are:

1. The ansatz-oriented choice of the (multidimensional) bases related to some polynomial tensor algebra.

2. The choice of proper variational principle. A few projection/ Galerkin-like principles for constructing (weak) solutions can be considered. The advantages of formulations related to biorthogonal (wavelet) decomposition should be noted.

3. The choice of bases functions in the scale spaces D_j from wavelet zoo. They correspond to high-localized (nonlinear) excitations, nontrivial local (stable) distributions/fluctuations or "continuous qudits". Besides fast convergence properties it should be noted minimal complexity of all underlying calculations, especially in case of choice of wavelet packets which minimize Shannon entropy.

4. Operator representations providing maximum sparse representations for arbitrary (pseudo) differential/ integral operators $\mathrm{d}f/\mathrm{d}x$, $\mathrm{d}^n f/\mathrm{d}x^n$, $\int T(x,y)f(y)\mathrm{d}y)$, etc.

5. (Multi)linearization. Besides the variation approach we can consider also a different method to deal with (polynomial) nonlinearities: para-products-like decompositions.

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By using wavelet bases with their best phase space localization properties, we can describe the localized (coherent) structures in quantum systems with complicated behaviour (Figs. 8, 11). The numerical simulation demonstrates the formation of different (stable) pattern or orbits generated by internal hidden symmetry from high-localized structures. Our (nonlinear) eigenmodes are more realistic for the modeling of nonlinear classical/quantum dynamical process than the corresponding linear gaussian-like coherent states. Here we mention only the best convergence properties of the expansions based on wavelet packets, which realize the minimal Shannon entropy property and the exponential control of convergence of expansions like (12) based on the norm (6). Fig. 10 corresponds to (possible) result of superselection (einselection) after decoherence process started from entangled state (Fig. 11); Fig. 12 and Fig. 13 demonstrate the steps of multiscale resolution (or degrees of interference) during modeling (quantum interaction/evolution) of entangled states leading to the growth of degree of entanglement. It should be noted that we can control the type of behaviour on the level of the reduced algebraical variational system, Generalized Dispersion Relation (11).

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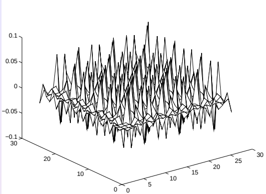

Figure: 8 Complex Pattern.

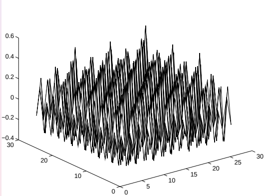

Figure: 9 Complex Pattern.

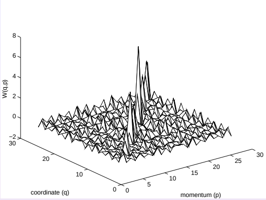

Figure: 10 Localized pattern, (waveleton) Wigner function.

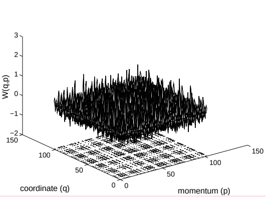

Figure: 11 Entangled-like Wigner function.

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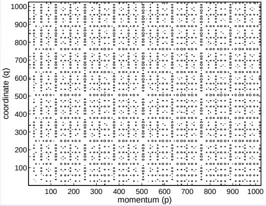

Figure: 12 Interference picture on the level 4 MRA approximation for Wigner function.

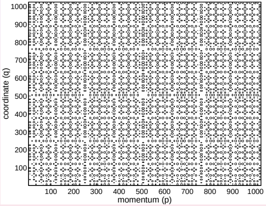

Figure: 13 Interference picture on the level 6 MRA approximation for Wigner function.