Quantum mechanics at different scales. Part 1: Resolution of a (quantum) point

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Abstract

We consider some generalization of the theory of quantum states, which is based on the analysis of long standing problems and unsatisfactory situation with the possible interpretations of quantum mechanics. We demonstrate that the consideration of quantum states as sheaves can provide, in principle, more deep understanding of some phenomena. The key ingredients of the proposed construction are the families of sections of sheaves with values in the category of the functional realizations of infinite-dimensional Hilbert spaces with special (multiscale) filtration.

1 Quantum states: functions vs. patterns/sheaves

During a relative long period, it is well-known that there is a great difference between (at least) the mathematical levels of the investigation of quantum phenomena in different regions. Really, in the area of the so-called Strings (Math)Physics, one needs all existing machinery [1], like deformation quantization, non-commutative geometry, etc while more applicable in real life Quantum Mechanics/Physics uses old routines mainly. Of course, Physics at planckian scales demands a new vision regarding Physics of (realizable) Quantum Devices (like future CPU) but, at the same time, we deal with the same phenomenon (at least on qualitative level). It is impossible to imagine that some hypothetical object can be more quantum or less quantum. It seems that it can be only quantum or not quantum, i.e. classical (surely, we ignore, at the moment, quasi-classics). At the same time, even advanced modern Mathematics cannot help us in the final (at least practically accepted) analysis of the long standing quantum phenomena and the final classification of a zoo of interpretations. The well-known incomplete list is as follows:

(L)
entanglement, measurement, wave function collapse, decoherence, Copenhagen interpretation, consistent histories, many-worlds interpretation/ multiverse (MWI), Bohm interpretation, ensemble interpretation, (Dirac) self-interference, "instantaneous" quantum interaction, hidden variables, etc. [2]

As a result, beyond a lot of fundamental advanced problems at planckian scales we are still even unready to create the proper theoretical background for the reliable modeling and constructing of quantum devices far away from planckian scales.

So, let us expose one more approach in an attempt to understand, in unified framework, some set of well-known phenomena.

Mathematical side

Let us presume that Quantum Dynamics can be properly described via Wigner-Weyl approach which covers other ones. It means that:

i) we use the language, machinery and ideology of the theory of pseudo-differential operators;

ii) we work with the symbols of operators instead of the operators;

iii) Quantum Evolution is described by (pseudo-differential) Wigner-like equations (Wigner-von Neumann-Moyal-Lindblad);

iv) the adequate analysis of the full set of possible phenomena described by **iii)** demands the using of Microlocal Analysis [3] (although there are some phenomena which can be described more traditionally);

v) we need to consider Quantum State not as a function but as a sheaf [3].

It is actually more proper (at least from formal, mathematical, point of view) if we actually want to take into account a lot of internal arguments from points **i)-iv)** above.

Physical side

It is very hard to believe that trivial simple solutions, like gaussians, can exhaust all variety of possible quantum states needed for the resolution of all contradictions, hidden inside the list **(L)** mentioned above. So, let us propose the following (physical) hypothesis:

(H1)
the physically reasonable really existing Quantum States cannot be described by means of functions. Quantum state is a complex pattern which demands a set/class of functions/patches instead of one function for proper description and understanding.

There is nothing unusual in **(H1)** for physicists since Dirac's description of monopole. All the more, there is nothing unusual for mathematicians who successfully used sheaves, germs, etc in different areas. Definitely, the introducing of **(H1)** causes a number of standard topics, the most important of them are motivations, formal (exact) definition and (at least) particular realizations. Actually, why need we to change our ideology after a century (since Planck) of success? The answer is trivial and related to the list **(L)** which is overcompleted with contradictions and misunderstanding after many decades of discussions.

2 On the route to right description: (quantum) patterns as sheaves

2.1 Motivations

In the following description, it is possible to find some features or reminiscences of previous models and interpretations from the list **L**, where the most important points for allusions are the hidden variables, localization, ensemble/statistical interpretation, MWI, Dirac's "self-interference". Let us sketch the main ingredients.

1). Arena for Quantum Evolution

First of all, we need to divide the kinematical and dynamical features of a set of Quantum States (QS). From the formal point of view it means that one needs to consider some bundle (X, H, H_x) whose sections are the so-called $|\psi>$ functions or QS. Here X is (kinematical) space-time base space with the proper kinematical symmetry group (like Galilei or Poincare ones), H is a total formal Hilbert space and $H(x) = H_x$ are fibers with their own internal structures and hidden symmetries. In addition, such a bundle has the corresponding structure group which connects different fibers. Of course, in a very particular case we have the constant bundle with the trivial structure group but non-trivial fiber symmetry. Anyway, as we shall demonstrate later, it is very reasonable to provide the one-to-one correspondence between Quantum States and the proper sections

$$|QS>: \quad X \longrightarrow H, \qquad QS: \quad x \longmapsto H_x = H(x).$$

As a result, we have, at least, three different symmetries inside this construction: kinematical one on space-time, hidden one inside each fiber and the gauge-like structure group of the bundle as a whole. It is obvious that the kinematical laws (like relativity principles) depend on the proper type of symmetry and are absolutely different in the base space and in the fibers. It should be noted that the functional realizations of fibers and the total space are very important for our aims. Roughly speaking, it can be supposed that physical effects depend on the type of the particular functional realization of formal (infinite dimensional) Hilbert space. E.g., it is impossible to use infinite smooth approximations, like gaussians, for the reliable modeling of chaotic/fractal phenomena. So, the part of Physics at quantum scales is encoded in the details of the proper functional realization.

2). Localization and a Tower of Scales

It is well-known that nobody can prove that gaussians (or even standard coherent states, etc) are an adequate and proper image for Quantum States really existing in the Nature. We can suggest that at quantum scales other classes of functions or, more generally, other functional spaces (not C^∞, e.g.) with the proper bases describe the underlying physical processes. There are two key features we are interested in. First of all, we need the best possible localization properties for our trial base functions. Second, we need to take into account, in appropriate form, all contributions from all internal hidden scales, from coarse-grained to finest ones. Of course, it is a hypothesis but it looks very reasonable:

(H2)

there is a (infinite) tower of internal scales in quantum region that contributes to the really existing Quantum States and their evolution.

So, we may suppose that the fundamental generating physical "eigen-modes" correspond to a selected functional realization and are localized in the best possible way. Let us note the role of the proper hidden symmetries which are responsible for the quantum self-organization and resulting complexity.

3). An Ensemble of Scales: Self-interaction

As a result of the description above, we may have non-trivial "interaction" inside an infinite hierarchy of modes or scales. It resembles, in some sense, a sort of turbulence or intermittency. Of course, here the generating avatar is a representation theory of hidden symmetries which create the non-trivial dynamics of this ensemble of hierarchies.

4). Hidden Parameters and Hidden Symmetry

It is well-known that symmetries generate all things (at least) in fundamental physics. Here, we have a particular case where the generic symmetry corresponds to the internal hidden symmetry of the underlying functional realization. Moreover, as it is proposed above, we have even the more complicated structure because we believe that QS is not a function but a sheaf. As a result, we have interaction between two different symmetries, namely hidden symmetry in the fiber, that corresponds to the internal symmetry of the functional realization, and the structure "gauge" group of a sheaf, which provides multi-fibers transition/dynamics. Both these algebraic structures can be parametrized by the proper group parameters which can play the role of famous "hidden variables" introduced many decades ago.

5). MWI

Of course, MWI or Multiverse interpretation can be covered by the structure sketched above. Quantum States are the sections of our fundamental sheaf, so we can consider them as a collection of maps between the patches of base space and fibers. All such maps simultaneously exist and, as an equivalence class, represent the same Quantum State. We postpone the detailed description to the next Section but here let us mention that each member of the full family can be considered as an object belonged to some fixed World. Obviously, before measurement we cannot distinguish samples but after measurement we shall have the only copy at our hands.

2.2 On the way to definition

The main reason to introduce sheaves as a useful instrument for the analysis of Quantum States is related to their main property which allows to assign to every region U in space-time X some family $F(U)$ of algebraic or geometric objects such as functions or differential operators. The family can be restricted to smaller regions, and the compatible collections of families can be glued to give a family over larger regions, so it provides connection between small and large scales, local and global data[3].

Informal construction is as follows.

Let X be the space-time base space (some topological space) with a system of open subsets $U \subset X$, then for every U and map F the image $F(U)$ is some object with internal structure (more generally, $F(U)$ takes values in some category $\mathbf{H}$) such that for every two open subsets, U and V, $V \subset U$ there is the so-called restriction map (more generally, morphism in the category $\mathbf{H}$), $r_{V,U} : F(U) \to F(V)$ (restriction morphism). A map F will be a **presheaf** if restriction morphism satisfies the following properties:

(a) for every open subset $U \subset X$, the restriction morphism $r_{U,U} : F(U) \to F(U)$ is the identity morphism,

(b) if there are three open subsets $W \subset V \subset U$, then $r_{W,V} r_{V,U} = r_{W,U}$. This property provides the connection or ordering of the underlying scales. In other words, let $O(X)$ be the category of open sets on X, whose objects are the open sets of X and whose morphisms are inclusions. Then a presheaf $\mathbf{F}$ on X with values in category $\mathbf{H}$ is the contravariant functor from $O(X)$ to $\mathbf{H}$. $F(U)$ is called the section of $\mathbf{F}$ over U and we consider it as some pre-image for adequate Quantum State $|QS>$. But our goal, in this direction, is a **sheaf**, so we need to add two additional properties.

Let $\{U_i\}_{i\in I}$ be some family of open subsets of X, $U = \cup_{i\in I} U_i$.

(c) If Ψ_1 and Ψ_2 are two elements of $F(U)$ and $r_{U_i,U}(\Psi_1) = r_{U_i,U}(\Psi_2)$ for every U_i, then $\Psi_1 = \Psi_2$.

(d) for every i let a section $\Psi_i \in F(U_i)$. $\{\Psi_i\}_{i\in I}$ are compatible if, for all i and j, $r_{U_i\cap U_j,U_i}(\Psi_i) = r_{U_i\cap U_j,U_j}(\Psi_j)$. For every set $\{\Psi_i\}_{i\in I}$ of compatible sections on $\{U_i\}_{i\in I}$, there exists the unique section $\Psi \in F(U)$ such that $r_{U_i,U}(\Psi) = \Psi_i$ for every $i \in I$.

The section $\mathbf{\Psi}$ is called the gluing of the sections Ψ_i. Definitely, we can consider this property as allusion to the hypothesis of wave function collapse. Really, $\mathbf{\Psi}$ looks as Multiverse Quantum State Ensemble $\{\Psi_i\}$ while Ψ_i is the result of measurement in the patch U_i. And it is unique!

The next step is to specify the Quantum Category **H**. According to our Hypothesis **H2**, we consider the category of the functional realization of (infinite-dimensional) Hilbert spaces provided with proper filtration, which allows to take into account multiscale decomposition for all dynamical quantities needed for the description of Quantum Evolution. The well-known type of such filtration is the so-called multiresolution decomposition [4]. It should be noted that the whole description is much more complicated because it demands the consideration of both structures together, namely, the fiber structure generated by internal hidden symmetry of the chosen functional realization and the family of gluing sections $\mathbf{\Psi}$ in the unified framework.

2.3 Realization via multiresolution: dynamics, measurement, decoherence, etc.

In the companion paper[5], we shall consider in details one important realization of this construction based on the local nonlinear harmonic analysis which has, as the key ingredient, the so-called Multiresolution Analysis (MRA)[4]. It allows us to describe internal hidden dynamics on a tower of scales. Introducing the Fock-like space structure on the whole space of internal hidden scales [6], we have the following MRA decomposition:

$$H = \bigoplus_i \bigotimes_n H_i^n$$

for the set of n-partial Wigner functions (states):

$$W^i = \{W_0^i, W_1^i(x_1;t), \dots, \qquad W_N^i(x_1,\dots,x_N;t), \dots\},$$

where $W_p(x_1,\dots,x_p;t) \in H^p$, $H^0 = C$, $\quad H^p = L^2(R^{6p})$ (or any different proper functional space), with the natural Fock space like norm:

$$(W,W) = W_0^2 + \sum_i \int W_i^2(x_1,\dots,x_i;t)\prod_{\ell=1}^{i}\mu_\ell.$$

Now we consider some phenomenological description which presents some attempt of the qualitative description of quantum dynamics as a whole and in comparison with its classical counterpart. It is possible to take, for reminiscence, the famous Dirac's phrase that "an electron can interact only itself via the process of quantum interference". Let G be a hidden/internal symmetry group on the spaces of Quantum States, which generates, via MRA, the multiscale/multiresolution representation for all dynamical quantities, unified in object $O(t)$, such as states, observables, partitions (e.g., Wigner quasi-distributions):

$$O^i(t) = \{\psi^i(t), Op^i(t), W_n^i(t)\},$$

where i is the proper scale index.

Then, the following commutative diagram represents the details of quantum life from the point of view of the representations of G on the chosen functional realization which leads to the decomposition of the whole quantum evolution into the orbits or scales corresponding to the proper level of resolution. Morphisms $W(t)$ describe Wigner-Weyl evolution in the algebra of symbols, while the processes of interactions with open World, such as the measurement or decoherence, correspond to morphisms (or even functors) $m(t)$ which transform the infinite set of scales, characterizing the quantum object, into finite ones, sometimes consisting of one element (demolition/destructive measurement)

$$\begin{array}{ccc} & W(t) & \\ \{O^i(t_1)\} & \longrightarrow & \{O^j(t_2)\} \\ \downarrow m(t_1) & \widetilde{W(t)} & \downarrow m(t_2) \\ \{O^{i_c}(t_1)\} & \longrightarrow & \{O^{j_c}(t_2)\}, \end{array}$$

where the reduced morphisms $\widetilde{W(t)}$ correspond to (semi)classical or quasiclassical evolution. So, qualitatively, **Quantum Objects** can be represented by an infinite or sufficiently large set of coexisting and interacting subsets while **(Quasi)Classical Objects** can be described by one or a few only levels of resolution with (almost) suppressed interscale self-interaction. It is possible to consider Wigner functions as some measure of the quantum character of the system: as soon as it becomes positive, we arrive to classical regime and so there is no need to consider the full hierarchy decomposition in the MRA representation. So, Dirac's self-interference is nothing else than the multiscale mixture/intermittency. Certainly, the degree of this self-interaction leads to different qualitative types of behaviour, such as localized quasiclassical states, separable, entangled, chaotic etc. At the same time, the instantaneous quantum interaction or transmission of (quantum) information from Alice to Bob takes place not in the physical kinematical space-time but in Hilbert spaces of Quantum States in their proper functional realization where there is a different kinematic life. To describe a set of Quantum Objects, we need to realize our Space of States (Hilbert space) not as one functional space but as the so-called and well known in mathematics, scale of spaces, e.g. $B^s_{p,q}$, $F^s_{p,q}$ [5]. The proper multiscale decomposition for the scale of spaces provides us by the method of the description of a set of quantum objects in case if the "size" of one Hilbert space of states is not enough to describe the complicated internal World. We will consider it elsewhere, while here we consider the one-scale case (to avoid possible misunderstanding, we need to mention that one-scale case is also described by an infinite scale of spaces, but it is the internal decomposition of the unique, attached to the problem, Hilbert space). Definitely, the full family of sections of non-trivial sheaf, as a model of QS, demands to take into account the double-hierarchy of the underlying internal scales generated by means of the corresponding hidden symmetries. As a result, on the proper orbits, we have nontrivial entangled dynamics, especially in contrast with its classical counterpart.

3 Conclusions

It seems very reasonable that there are no chances for the solution of long standing problems and novel ones if we constraint ourselves by old routines and the old zoo of simple solutions like gaussians, coherent states and all that. Evidently, that even the mathematical background of regular Quantum Physics demands new interpretations and approaches. Let us mention only the procedures of quantization as a generic example. In this respect, we can hope that our sheaf extension of representation for QS, which is natural from the formal point of view, may be very productive for the more deep understanding of the underlying (Quantum) Physics, especially, if we consider it together with the category of multiscale filtered functional realizations decomposed into the entangled orbits generated by actions of internal hidden symmetries. In such a way, we open a possibility for the exact description of a lot of phenomena like entanglement and measurement, wave function collapse, self-interference, instantaneous quantum interaction, Multiverse, hidden variables, etc. In the companion paper [6] we consider the machinery needed for the generation of a zoo of the complex quantum patterns during Wigner-Weyl evolution.

4 Summary and perspectives

We considered some generalization of the theory of quantum states, which is based on the analysis of long standing problems and unsatisfactory situation with possible interpretations of quantum mechanics. We demonstrate that the consideration of quantum states as sheaves can provide, in principle, more deep understanding of some phenomena. The key ingredients of the proposed construction are the families of sections of sheaves with values in the category of the functional realizations of infinite-dimensional Hilbert spaces with special (multiscale) filtration.

The questions we hope to answer are:

i) How may we enlarge the amount of (physical) information stored in one (quantum) physical point?

ii) Relation between structureless geometrical points and physical points (or point objects like (point) particles) with rich (possible hidden) structure.

iii) How we may "resolve" (physical) point/quantum state to provide such a structure.

iv) A new look in new framework for localization, entanglement, measurement, and all that.

v) How we may to explain/re-interpret a standard zoo of standard interpretations/phenomena (multiverse, wave functions collapse, hidden parameters, Dirac self-interference, ensemble interpretation, etc, etc. [2]) of Quantum Mechanics in the new framework.

The goal of the first part (of two) is to compare the following key objects which are basic for any type of the exposition of Quantum Mechanics (and other related areas):

Geometrical points vs. Physical Points (or Point Objects, or One-Point-Patterns);

Point functions vs. Sheaves;

Partial Differential Equations (and proper orthodox approaches)

vs.

Pseudodifferential Equations (via Microlocal analysis and all that).

We will continue heuristic Part 1 by more technical Part 2 where we compare:

Fourier/Gaussian modes vs. (pretty much) Localized (but non-gaussian) Physical Modes;

Fourier Analysis vs. Local Nonlinear Harmonic Multiscale Analysis (including wave- and other -lets and multiresolution);

Standard Quantum Images vs. Quantum Patterns generated by Multiresolution;
and as a final point,
Categorification Procedure(s) for Quantum Mechanics (QM).

It is more or less obvious that we don't have any unified framework covering a zoo of (mostly) discrepant interpretations [2] of QM, as well as satisfactory explanation/understanding of main ingredients of a phenomena like entanglement, etc.

The starting point is an idea to describe (to resolve) the key object of the area, namely point objects (physical points). Usually, for the modeling of real physical point objects, one can consider equivalence between them and standard geometrical points (at the moment, the concrete description of the set to which such points belong as one-element subsets, does not matter).

As direct consequence, our dynamical variables (like wave function or density matrix or the Wigner function) are described by means of point functions or what mathematicians mean by standard functions. But, as we can understand, a geometrical point is structureless and it seems that we need much more to enlarge the amount of data/information corresponding to this generic object. To advocate this Hypothesis in the present context it is worth noting Dirac's famous sentence "an electron can interact only itself via the process of quantum inteference". Roughly speaking, from this perspective it means that a "point particle" needs and must have a non-trivial complicated structure. It seems reasonable to have the rich structure for a model of the (quantum) physical point (usually named as "particle") in comparison with the structureless geometrical point. All above looks like Physical Hypothesis and it may be not so clear but at the same time it is more or less well known from the mathematical point of view if we accept the right (Wigner-Moyal) picture for description of Quantum World. In that framework, more exactly Strict Deformation Quantization approach, all equations are pseudodifferential and as immediate consequence we need to change point functions by sheaves what provides the clear resolution of the (Physical) Point: as a result it is no more structureless but acquires the rich (possible hidden at first glance) structure.

To summarize, our first three Hypotheses are as follows:

(Physical) Hypothesis 1

Physical Point Object (physical point, point particle) is not a structureless object and cannot be described by means of the geometrical point (in the standard math sense). Instead of, Physical Points have a rich infinite hidden structure.

(Physical version) Hypothesis 2

Quantum Dynamical Variables (wave functions, density matrix, Wigner functions) are not point functions but sheaves defined on the properly chosen space-time manifold (or topological space, or variety, or even scheme).

(Math version) Hypothesis 2

To provide the structureness of the Physical Point, allowing to enlarge a number of useful properties and increase the amount of corresponding data inside, we consider it together with a proper generalization of wave function as a section/fiber of proper sheaf (in a proper category of objects) defined on a proper model (category) of space-time.

(Physical) Hypothesis 3

Deformation Quantization Picture (roughly speaking, Wigner-Weyl-Moyal) is not in contradiction with Hypotheses 1 and 2 and allow us to consider the proper description.

(Math) Hypothesis 3

Microlocal Analysis of proper Pseudodifferential Dynamical Equations (like the Wigner one) allows to create a model of infinite hierarchy for the correct representation of Physical Point.

In such an approach **Quantum States** are (roughly speaking) sections of the so-called **coherent sheaves** or **contravariant functors** from the proper category describing space-time to other one properly describing the complex dynamics of Quantum States/Patterns. As we will explain in Part 2 [6], the objects of this category are some filtrations on the Hilbert space of States. In this picture, a result of **measurement** corresponds to **direct (inductive) limit**.

Definitely, we need a proper analytical/numerical machinery to realise such approach. It is exactly the subject of Part 2. It is not worth mentioning that all above is well known in Mathematics long ago. But we hope to convert this knowledge to a physical acceptable and computational form.

Papers/(arXiv)preprints [7]–[21] of authors can be found on web pages below.

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