# LOCALIZATION AND PATTERN FORMATION IN MODELS OF FUSION/ENERGY CONFINEMENT IN PLASMA PHYSICS:
# I. Math Framework for Non-Equilibrium Hierarchies
# II. BBGKY hierarchy and Reductions
# TOWARDS WAVELETONS IN (PLASMA) PHYSICS

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1. **The only important result we found** (after Martin Kruskal)

   A MAGNETICALLY CONFINED PLASMA CANNOT BE IN THERMODYNAMICAL EQUILIBRIUM...

   **Folklor**

2. **Current state of the Problem**

   And I walk out of space
   Into an overgrown garden of values,
   And tear up seeming stability
   And self-comprehension of causes.
   And your, infinity, textbook
   I read by myself, without people -
   Leafless, savage mathematical book,
   A problem book of gigantic radicals..

   **Osip Mandelstam**



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3. **Just for beauty**

   I learnt to distrust all physical concepts as the basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting interesting mathematics.

   G–d used beautiful mathematics in creating the world.

   I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it.

   **PAM Dirac**

4. **Just for drive**

   Wir müssen wissen. Wir werden wissen.
   We must know. We will know.

   **David Hilbert**



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## Key words:

Localization,
Localized modes,
Patterns,
Pattern formation,
Controllable patterns,
Waveletons,
(Non-linear) Pseudo-differential dynamics ($\Psi DOD$),
Multiscales,
Multiresolution,
Local/non-linear harmonic analysis (wavelet, Weyl-Heisenberg,...),
SYMMETRY (Hidden, etc),
Functional spaces,
Topology of configuration space (tokamaks vs.stellarators vs.N-kamaks)
Orbits,
Variational methods,
Minimal complexity/Effectiveness of numerics
Non-equilibrium ensembles, hierarchy of kinetics equations (BBGKY) and reasonable reductions/truncations

$\{\textbf{waveleton}\}:=\{\textbf{soliton}\} \bigsqcup \{\textbf{wavelet}\}$

**waveleton $\approx$ (meta) stable localized (controllable) pattern**

**Fusion state = (meta) stable state (long-living (meta) stable fluctuation)**

**in which most of energy of the system is concentrated in the relative small area of the whole phase space during time which is enough to take it ouside for possible usage (zero measure, min entropy).**



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# KINDERGARTEN

Base things: States, Observables, Measures, Measurement, ...

**What we are looking for?**

Pre-fusion, Fusion, Post-fusion state (KMS, SRB, ... ?)

| | |
|---|---|
| States: | Functional spaces |
| Observables: | Operator algebras |
| Key point: | **Symmetry** |
| **Filtrations**: | Tower of Scales |

It is result of Underlying Hidden Symmetry

$$\ldots V_c \subset \ldots V_n \subset \ldots V_{n+1} \subset \ldots \qquad \cap V_n = \emptyset \qquad \cup V_n = H$$

After that: operators (FWT), measures, ...

$$U(V) - W^*\{(\psi(f), \psi^*(f), \qquad \text{supp} \quad f \subset V\}$$
$$U = \overline{\cup U(V)}, \qquad u - C^*(\text{closure in norm topology})$$
$$\exists N_V = \int_V \psi^*(x)\psi(x)\mathrm{d}^3x, \qquad [\psi(x), \psi^*(y)]_\pm = \delta^3(x-y)$$
$$\psi(f) = \int \psi(x)f^*(x)\mathrm{d}^3x, \qquad [\psi(g), \psi^*(f)]_\pm = (g, f)$$
$$|1>= \psi^*(f)\Omega_0, \qquad \psi(f)\Omega_0 = 0, \qquad \Omega_0 - \text{cyclic},$$

$R(U)$ - representation, $\omega$ -state, $\omega(A)$ - expectation value of $A \in U$ in state $\omega$
GNS: representation $R_\omega(U)$ in $H_\omega$ and $\Omega \in H_\omega$ such that:

$$\omega(A) = (\Omega, R(A)\Omega), \forall A$$



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Gibbs states, equlibrium states

$\beta, \mu, V, \qquad H' = H - \mu N \qquad U_v(t) = e^{iH'_v(t)}$

Gibbs: $\omega_v(A) = Tr_v(\rho_v A) \qquad \rho_v(Tr_v e^{-\beta H'_v})^{-1} e^{-\beta H'_v}$

Well defined expectation functional, $\forall A \in U(V)$

$\exists \lim_{n\to\infty} \omega_{v_n}(A) = \omega(A), \quad V_{n+1} \supset V_n, \quad \cup V_n = \mathbb{R}^N$

KMS, Tomita-Takesaki,...



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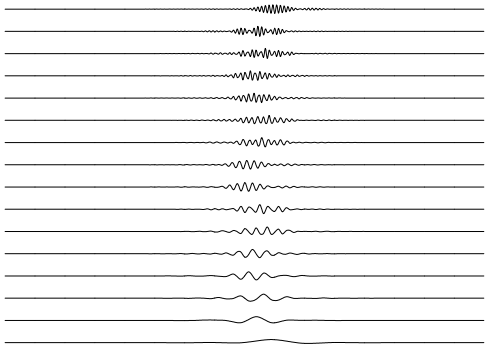

Figure 1: Localized modes.



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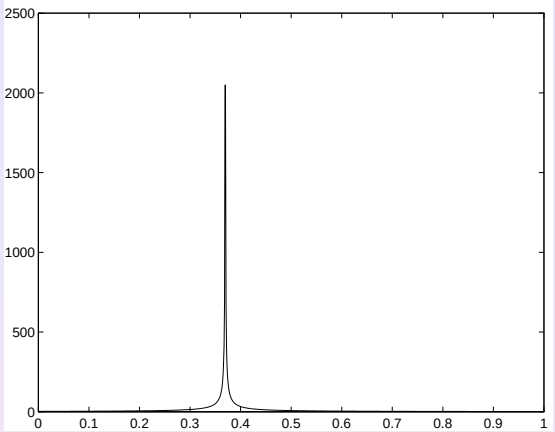


Figure 2: Kick

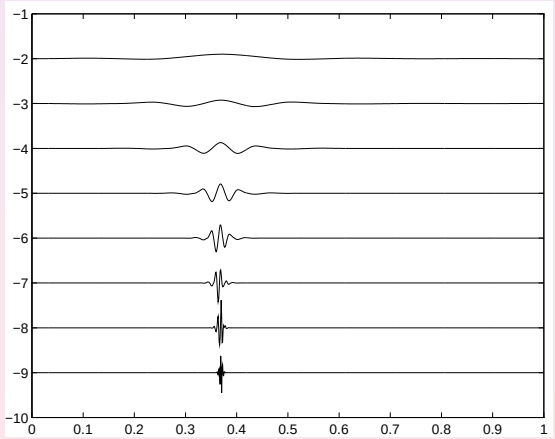


Figure 3: MRA for Kick

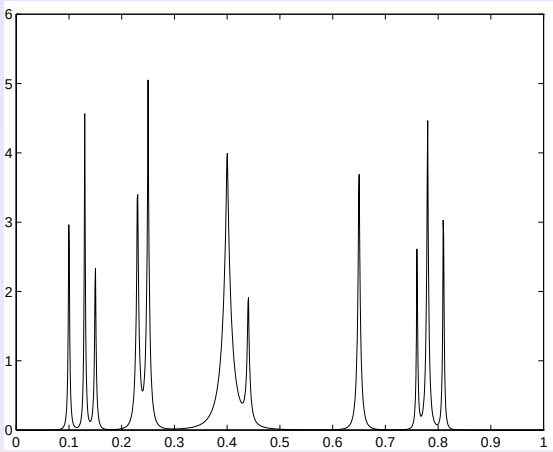


Figure 4: Multi-Kicks

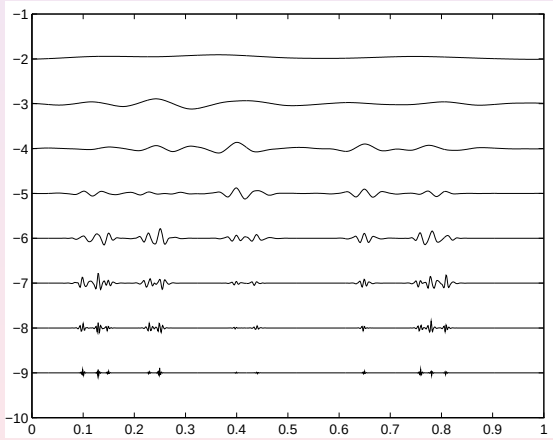


Figure 5: MRA for Multi-Kicks



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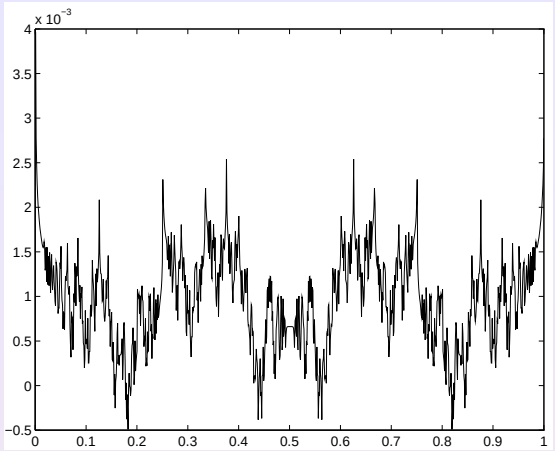


Figure 6: RW-fractal

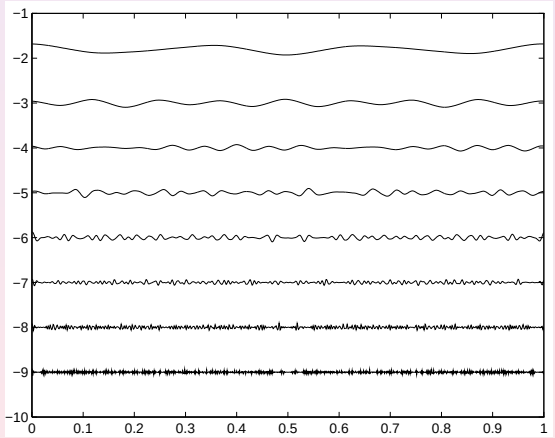


Figure 7: MRA for RW-fractal



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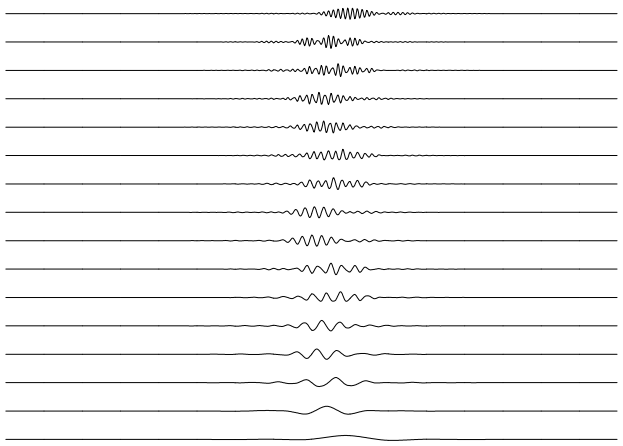

Figure 8: Localized modes.

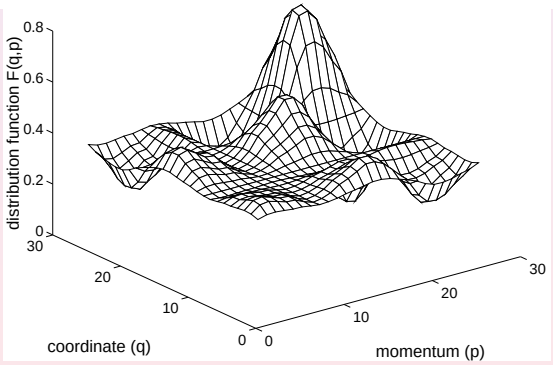


Figure 9: Localized mode contribution to distribution function.

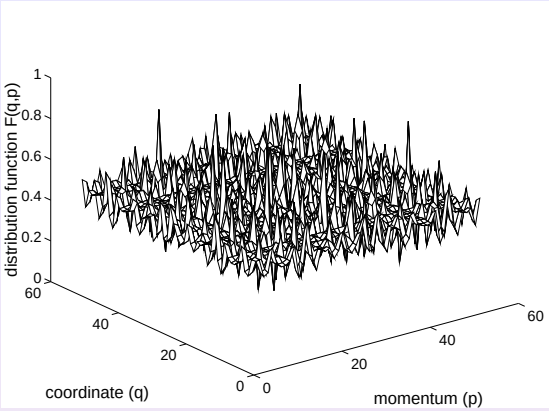


Figure 10: Chaotic-like pattern.

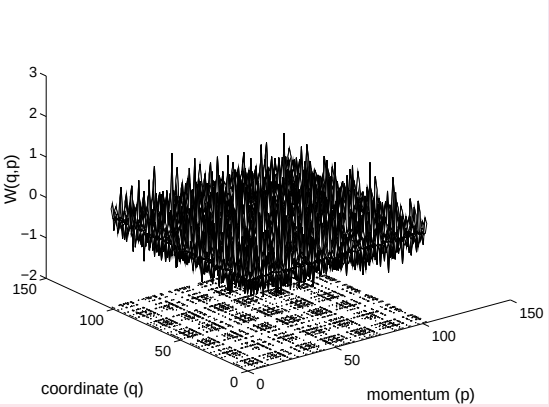


Figure 11: Entangled-like pattern.

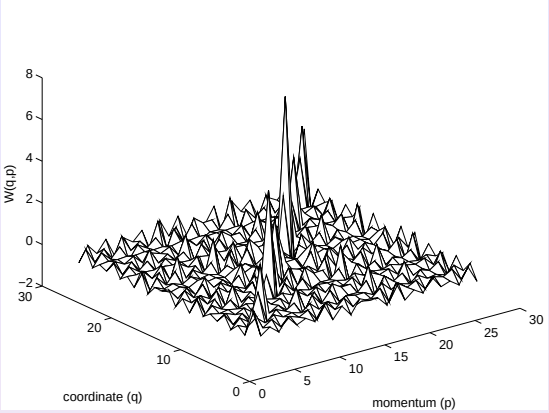


Figure 12: Localized pattern: waveleton.

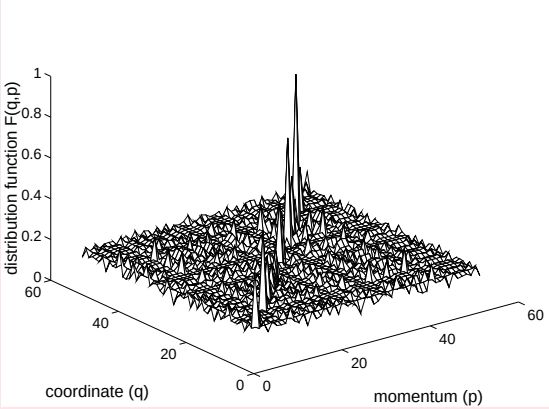


Figure 13: Localized pattern: waveleton.

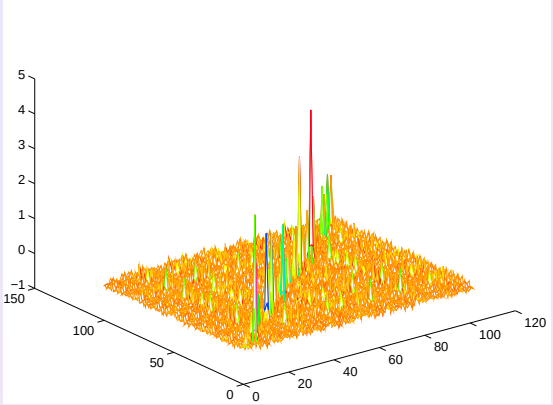


Figure 14: Waveleton Pattern

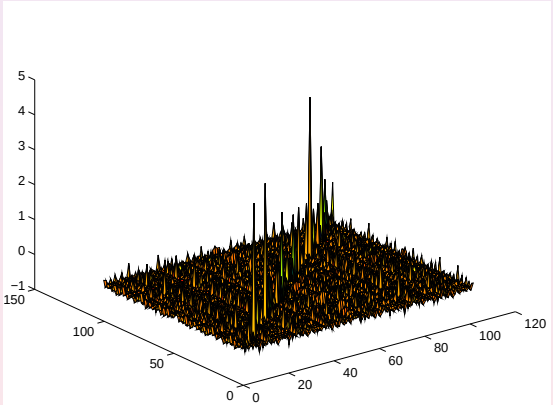


Figure 15: Waveleton Pattern



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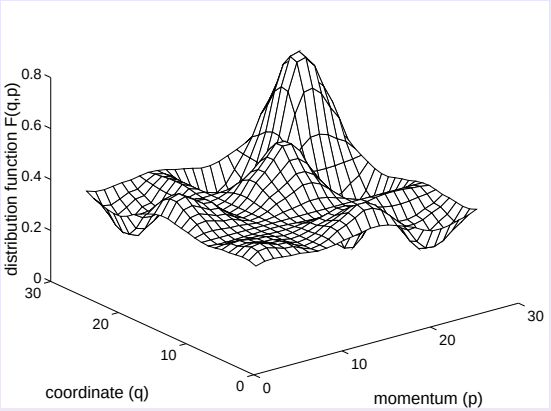


Figure 16: Localized mode contribution to distribution function.

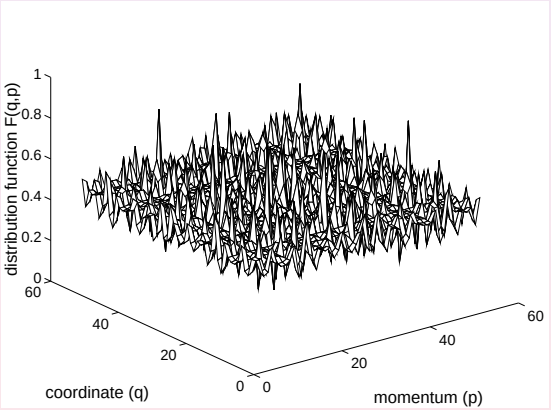


Figure 17: Chaotic-like pattern.



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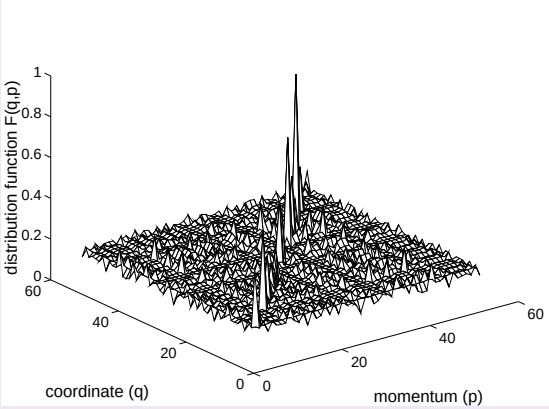


Figure 18: Localized waveleton pattern.

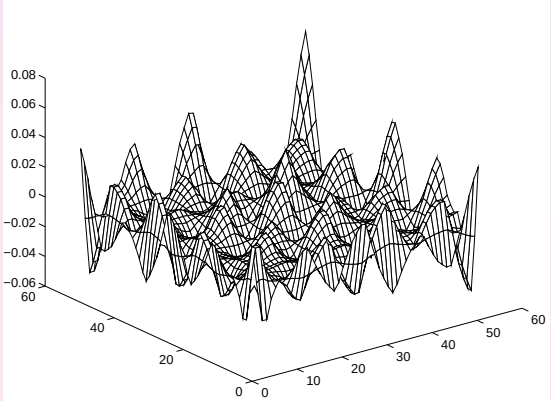


Figure 19: Eigenmode of level 1.

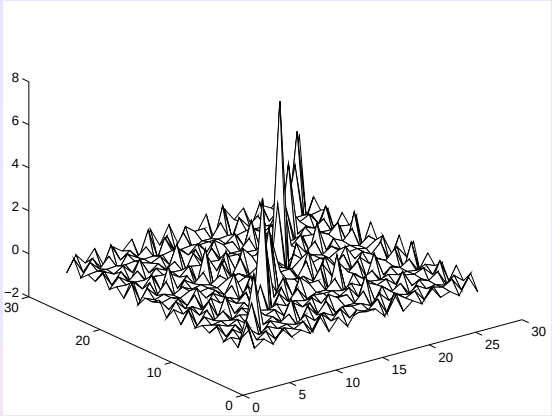


Figure 20: Stable waveleton pattern.

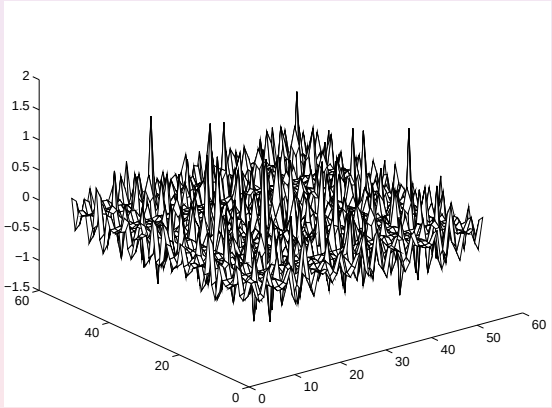


Figure 21: Chaotic-like behaviour.



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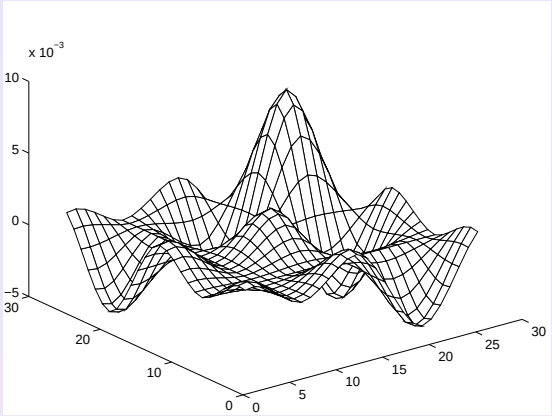


Figure 22: Waveleton-like distribution.

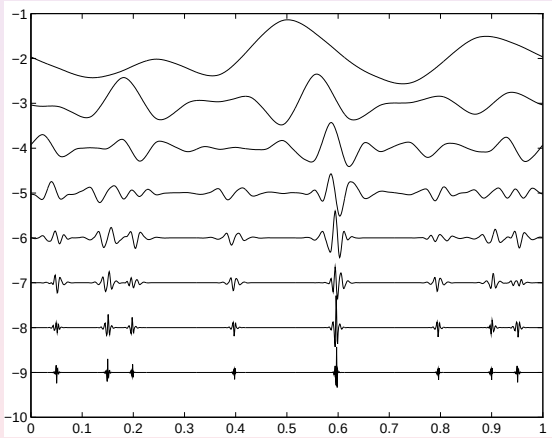


Figure 23: Multiscale decomposition.

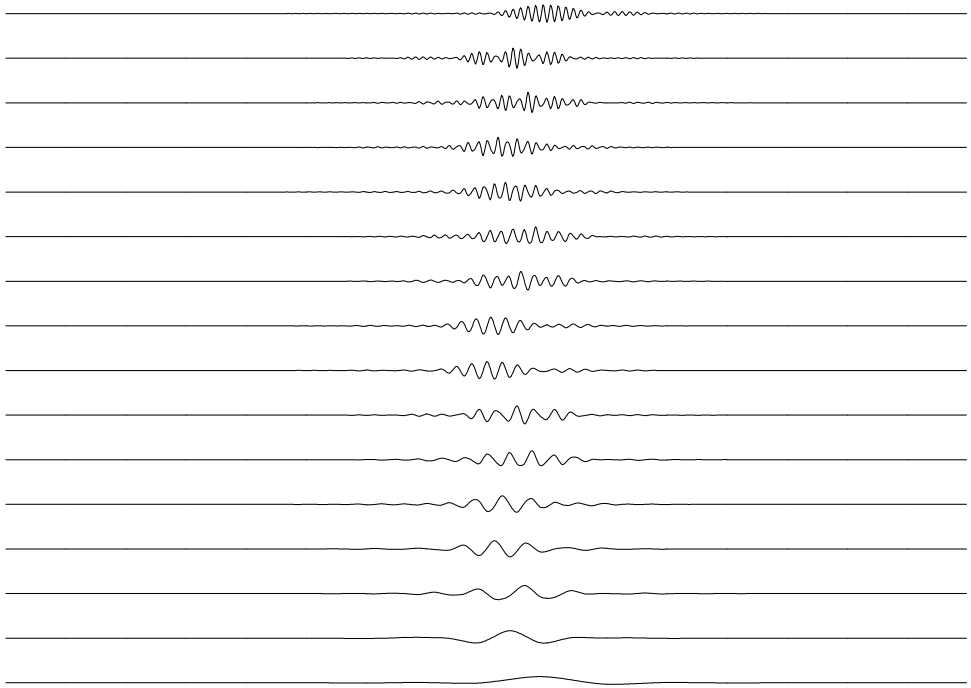

Figure 24: Localized modes.



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# MATH

How much is it needed?

| | |
|---|---|
| 0 % (subset of measure zero) | $\gg$ 100 % |
| CPU Intel Pentium 1-5<br>DVD<br>Cell Phone | Super String Physics<br>Sub-Planckian Physics,<br>Quantum Gravity, Higgs |

**? $\longleftarrow$ Plasma Fusion $\longrightarrow$ ?**

Is MATH really needful thing to describe Plasma World? —
Or it is a place for engineers only :-) :-(

**MOTIVES:**

Folklor: a magnetically confined plasma cannot be in thermodynamical equilibrium...
Instabilities...(meta) stability (long living fluctuations)
VERY complex system: lineariazation*, perturbation methods, U(1)
Fourier analysis (SIN, COS, EXP $i(kx - \omega t)$) are not proper
*) wave eq. vs. sine - Gordon (or KdV or KP etc): solitons, breathers, finite-gap solutions differ from solutions of linear equations.



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# HOPES, ILLUSIONS (lost), . . . , PARADIGMS, LESSONS (after Martin Kruskal)

**Tori**

**FPU** $\longrightarrow$ **Liouvillean** $\longrightarrow$ **KAM** $\longrightarrow$ **Cantori** $\longrightarrow \ldots$

**L-Tori Windings** (**Soliton Era** 1967-1984/9)

KP example

$$\frac{3}{4}u_y = w_x, \quad w_y = u_t - \frac{1}{4}(6uu_x + u_{xxx})$$

$$\frac{3}{4}u_{yy} = \frac{\partial}{\partial x}[u_t - \frac{1}{4}(6uu_x + u_{xxx})]$$

$$\frac{\partial L}{\partial t} - \frac{\partial A}{\partial y} = [A, L]$$

$$L = \partial x^2 + u, \quad A = \partial_x^3 + \frac{3}{4}(u\partial_x + \partial_x u) + w$$

**Zero - curvature representation, Lax pair**



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$$
\begin{aligned}
u(x,y,t) &= 2\partial_x^2 \log \Theta(xU + yV + tW + z_0) + C \\
w(x,y,t) &= \frac{3}{2}\partial_x \partial_y \log \Theta(xU + yV + tW + z_0) + C_1
\end{aligned}
$$

$\Theta = \Theta(z)$ - theta function of Riemann surface $\Gamma$ wrt a basis of cycles $a_1, b_1, \ldots b_g$, $U_i, V_i, W_i$ - b-periods

$$
U_i = \oint_{b_i} \Omega_1, \quad V_i = \oint_{b_i} \Omega_2, \quad W_i = \oint_{b_i} \Omega_3,
$$

$\Omega_1, \Omega_2, \Omega_3$ - normalized second kind diff.



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# KP Hierarchy

$\Gamma$ - Riemann surface, BA function

$$\Psi(x, t_2, t_3, \ldots, p) = \left(1 + \sum_{i=1}^{\infty} \frac{\zeta_i}{k^i}\right) e^{kx + k^2 t_2 + k^3 t_3 + \ldots}$$

$$KP(t_2 = y, t_3 = t, t_1 = x)$$

$$\frac{\partial \Psi}{\partial t_n} = A_n \Psi$$

$$A_1 = \partial_x, \quad A_2 = L, \quad A_3 = A, \quad A_n = \partial_x^n + \sum_{i=2}^{n} u_i^n \partial_x^{n-i}$$

$u_2^n, \ldots, u_n^n$ via $\zeta_1, \ldots, \zeta_{n-1}$ recursively and satisfy infinite system of differential equations, KP hierarchy:

$$\left[-\frac{\partial}{\partial t_n} + A_n, -\frac{\partial}{\partial t_m} + A_m\right] = \frac{\partial A_n}{\partial t_m} - \frac{\partial A_m}{\partial t_n} + [A_n, A_m] = 0,$$

$$n, m = 1, 2, \ldots$$



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## Zero-curvature representation: $F_{\mu\nu} = 0$,

Plane connection $A_\mu \sim \partial_\mu g g^{-1}$
Moduli space (Hitchin)
Casimirs, BT, Kac-Moody/Virasoro, Lie-Poisson, Drinfeld-Sokolov, Coalgebras, Hopf, Quantum Groups

**KAM, etc.**

**Hilbert 16**; Homoclinic; Heteroclinic; Stable/Unstable; Bifurcations;

**Heteroclinic** bifurcations: resonance, transverse

**Resonance**: birth and death of periodic orbit (**Magnetic-reconection**);

**Transverse**: (un)stability of heteroclinic cycle;



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**Arnold diffusion**:

$$H(I, p, q, \Phi, t), \quad I(0) < \delta, I(T) > K, \quad T > 0$$

**SRB** - New Ergodic Theory

**Hyperbolic dynamics: Anosov Flows**

**Attractor(s) - repeller(s) coexistence**

**Attractor**: Fixed point, Limit cycle, limit tori, Strange attractor (dim $\notin \mathbb{N}$)

**Sensitivity** to initial data (Chaos)

**Hidden symmetries** (Fuchsian, Kleinian, PSL(2,Z), Modular)

Thuston-Milnor-Sullivan: **Quasiconformal Dynamics**



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## J. Leray's way:

Algebraical Analysis (Sato)

**Microlocalizations**( Kashiwara-Shapira) ΨDO

**Representation** Theory of **Hidden Symmetries** of Functional Spaces

Local Nonlinear Harmonic **Analysis** on the Orbits

Frames, Atomic Decomposition, Basis Pursuit, Modulation Spaces/ Feichtinger's algebra

SLE, Dimers (Kenyon), (**Random**) Matrix models

## Classical/Quantum

**Moyal/Deformation Quantization**

**Wigner-Weyl** (ΨDO)

**Sheaves as states: quantum points** arXiv:1109.5035, 1109.5042

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# INTRODUCTION

**I. Class of Models**

a). Individual $cM/qM$ (linear/nonlinear; $\{cM\} \subset \{qM\}$), ($*$ - Quantization of) Polynomial/Rational Hamiltonians:

$$H(p,q,t) = \sum_{i.j} a_{ij}(t) p^i q^j$$

**Important example: Orbital motion (in Storage Rings).**

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x,y),$$

where $f_n$ is a homogeneous function of $x$ and $y$ of order $n$. The cases $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decupole.

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The corresponding Hamiltonian is:

$$H(x, p_x, y, p_y, s) = \frac{p_x^2 + p_y^2}{2} + \left(\frac{1}{\rho^2(s)} - k_1(s)\right) \cdot \frac{x^2}{2} + k_1(s)\frac{y^2}{2} - \mathcal{R}e\left[\sum_{n\geq 2} \frac{k_n(s) + ij_n(s)}{(n+1)!} \cdot (x+iy)^{(n+1)}\right]$$

Terms corresponding to kick type contributions of rf-cavity:

$$A_\tau = -\frac{L}{2\pi k} \cdot V_0 \cdot \cos\left(k\frac{2\pi}{L}\tau\right) \cdot \delta(s - s_0)$$

or localized cavity $V(s) = V_0 \cdot \delta_p(s - s_0)$ with $\delta_p(s - s_0) = \sum_{n=-\infty}^{n=+\infty} \delta(s - (s_0 + n \cdot L))$ at position $s_0$.

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The second example: using Serret–Frenet parametrization, we have after truncation of power series expansion of square root the following approximated (up to octupoles) Hamiltonian for orbital motion in machine coordinates:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]} \\
&+ p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \cdot f(p_\sigma) \\
&+ \frac{1}{2} \cdot [K_x^2 + g] \cdot x^2 + \frac{1}{2} \cdot [K_z^2 - g] \cdot z^2 - N \cdot xz \\
&+ \frac{\lambda}{6} \cdot (x^3 - 3xz^2) + \frac{\mu}{24} \cdot (z^4 - 6x^2 z^2 + x^4) \\
&+ \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right]
\end{aligned}$$

Then we use series expansion of function $f(p_\sigma)$:
$f(p_\sigma) = f(0) + f'(0)p_\sigma + f''(0)p_\sigma^2/2 + \ldots = p_\sigma - p_\sigma^2/(2\gamma_0^2) + \ldots$ and the corresponding expansion of RHS of equations.

b). QFT (e.g. second quantization/Fock)
c). BBGKY Hierarchy (with reductions to different forms VMP, rms/envelope dynamics (belong to a).) and all that)
d). Wignerization of a): Wigner-Moyal-Weyl-von Neumann-Lindblad
e). Wignerization of c): Quantum (Non) Equilibrium Ensembles
Important remark: Symbols of Op instead of Op: ΨDO picture.
(Non)Linear ΨDO Dynamics (all $qM \subset \Psi DOD$)



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II. Effects (what we are interested in)
i).Hierarchy of (internal) scales (time, space, phase space). Multiscales (non-perturbative!): from slow to fast, from coarses to finest level of resolution/decomposition. Coexistence of hierarchy of multiscale dynamics (with transitions between scales ...for rmns: cascades in hydrodynamics).
If that, then we need (at least)
ii). Localized (max!) modes/"harmonics" –
Nonlinear LOCALIZED Harmonics
instead of plane waves/($sin/cos$) – farewell to 'stupid' Fourier $U(1)$ analysis, coherent states (any)/gaussians)...
What is a MODE/qubit? We take it from our zoo/library! What is (simple) PATTERN/STATE? IT'S THE SOLUTION OF $\Psi DOD$ DECOMPOSED VIA THE BASES OF LOCALIZED (eigen) MODES! (first, naive def.).

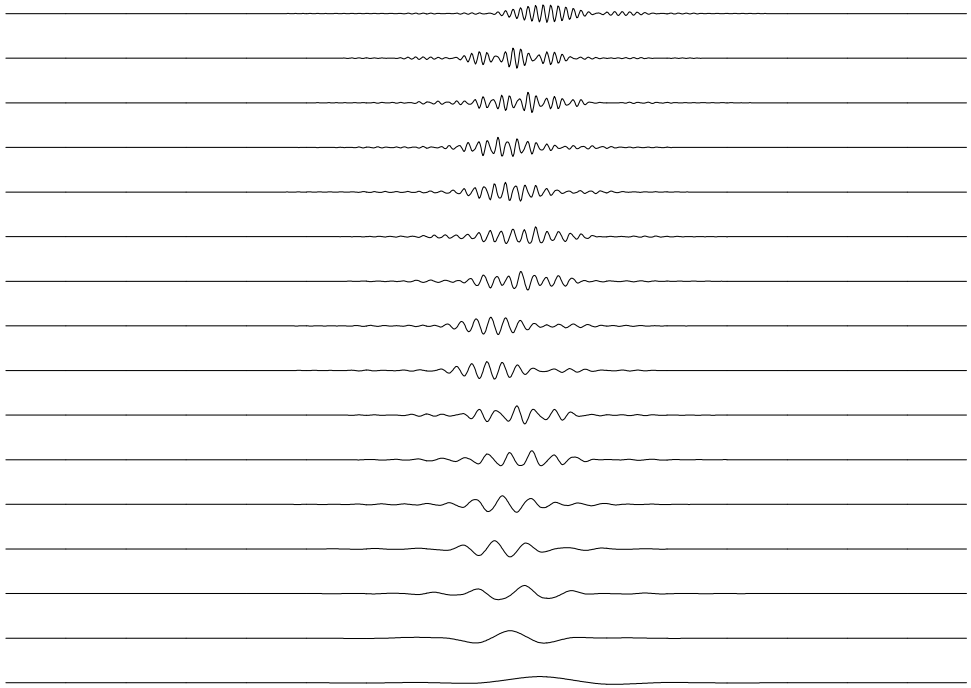

Figure 25: Localized modes.



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iii).Ensembles of states/qubits: Zoo of Patterns (controllable by construction)

ALL THAT inside the Class of Models ($\Psi$DO Dynamics) $\longrightarrow$ localized, entangled/chaotic, decoherent/quasiclassical.

**III. How? Methods? SYMMETRY!** (HIDDEN symmetry)
Representation theory of internal/hidden/underlying symmetry.
Kinematical, Dynamical, Hidden.
Arena (space of representation): (Hilbert) SPACE of STATES.
Symmetry inside Functional spaces!
METHODS:
Harmonic analysis on (non)abelian Group
(kinematical/internal symmetry). Instead of $U(1)$ Fourier analysis:
Local/Nonlinear (non-abelian) Harmonic Analysis!
Dynamics on proper orbits/scales (MULTISCALES) in Functional spaces.
Remark. $\exists$ much more powerful technique because of existence of HIDDEN Symmetry*
*({QMF} $\longrightarrow$ Loop groups $\longrightarrow$ Cuntz op. alg. $\longrightarrow$ Quantum Group structure ($\exists$ natural Fock-like structure also).


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The key objects are: MULTIRESOLUTION (Multiscale) representation (e.g, wavelet/gabor etc. analysis)
$\longrightarrow$ appearance of MULTISCALES (ORBITS) and localized (NATURAL eigen-) modes.

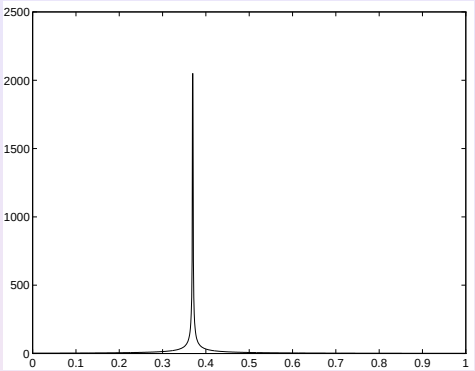


Figure 26: Kick

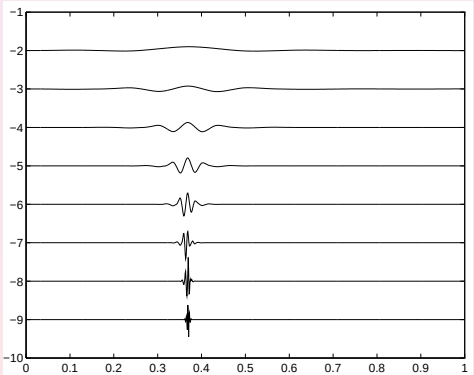


Figure 27: MRA for Kick

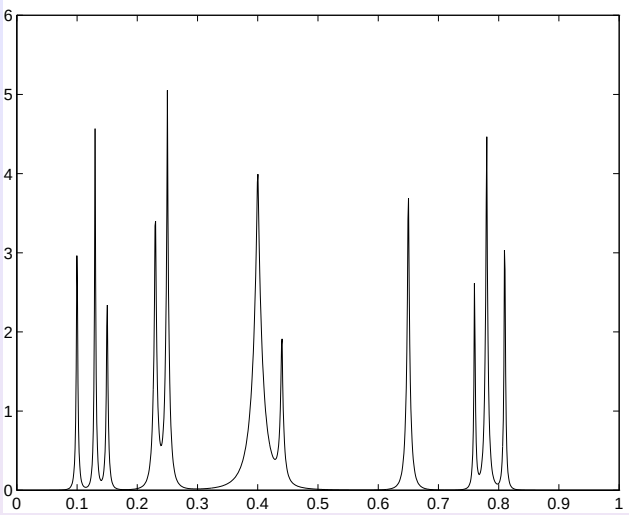


Figure 28: Multi-Kicks

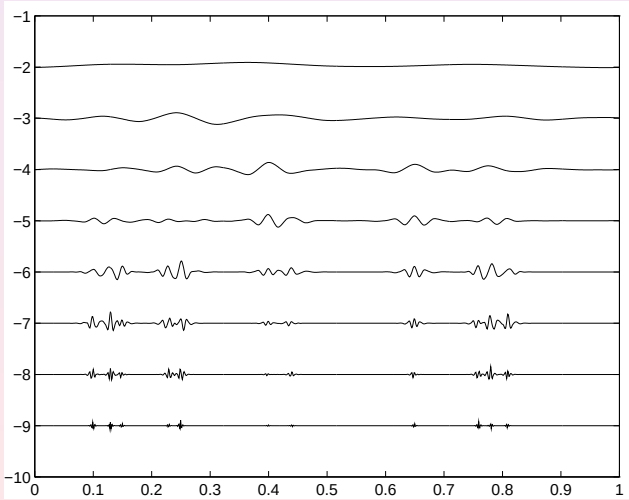


Figure 29: MRA for Multi-Kicks

$\bigoplus$ Variational formulation (CMP standard: control of convergence, reductions to algebraic systems, control of type of behaviour).

## IV. Set-up/Problems ($\Psi$DOD): $L^j\{Op^i\}\Psi = 0$

Objects inside:

i). SPACE of STATES. $H = \{\Psi\}$ (Hilbert) space of states: $L^2$, Sobolev, Schwartz, $C^0$, $C^k$, ... $C^\infty$, ...;

$L^2(R^2)$ vs. $L^2(S^2)$ vs. $L^2(S^1 \times S^1)$ vs. $L^2(S^1 \times S^1 \ltimes Z_n)$:

Tokamaks vs. Stellarators.

Class of smoothness:

Dynamics with/without Chaos/Fractality.

ii). DECOMPOSITIONS. $\Psi \approx \sum_i a_i e^i$ (bases, frames, atomic decomposition): (exp) control of convergence, max(!) rate of convergence for any $\Psi$ in any $H$.

iii). OBSERVABLES/OPERATORS (ODO, PDO, $\Psi$DO, SIO,..., Microlocal analysis of Kashiwara-Shapira):

$< \Psi | Op^i | \Psi >$ – Max sparse! (from functions to sheafs)

Almost (max!) diagonal according to FWT (BCR Th.):

$$\begin{pmatrix} D_{11} & 0 & 0 & \dots \\ 0 & D_{22} & 0 & \dots \\ 0 & 0 & D_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



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iv). MEASURES: multifractal wavelet measures $\{\mu_i\}$

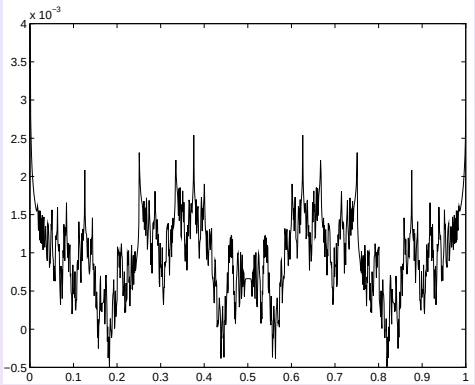


Figure 30: RW-fractal

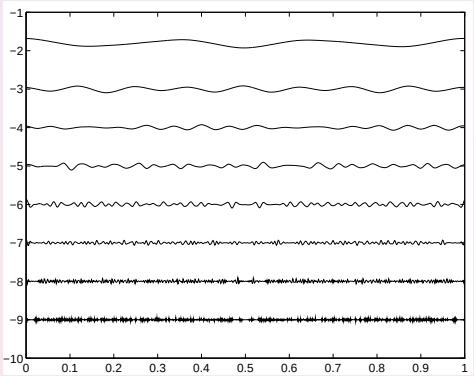


Figure 31: MRA for RW-fractal

v). VARIATIONAL/PROJECTION methods (from Galerkin to Floer/Rabinowitz, symplectic *AW* in coadjoint orbital dynamics). Preservation of Poisson/symplectic structures!

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vi). MULTIRESOLUTION or wavelet microscope

1). (internal) symmetry (kinematic):
affine group = {translations, dilations} or others...

2). representation/action of this symmetry on $H = \{\Psi\}$
$\Downarrow$ as a result (coherence!)

a). localized bases b). multiscales

vii). Effectiveness of proper numerics: CPU-time, HDD-space, minimal complexity of algorithms and (Shannon) entropy of calculations.
Finally: LOCALIZES MODES, ZOO of PATTERNS: LOCALIZED, CHAOTIC/ENTANGLED,...
LOCALIZED PATTERN: CONFINEMENT of ENERGY (FUSION STATE = WAVELETON).

**V. Quantization: $*$ Star Product**,

Deformation, Quantum Group, ...Renormalization,...

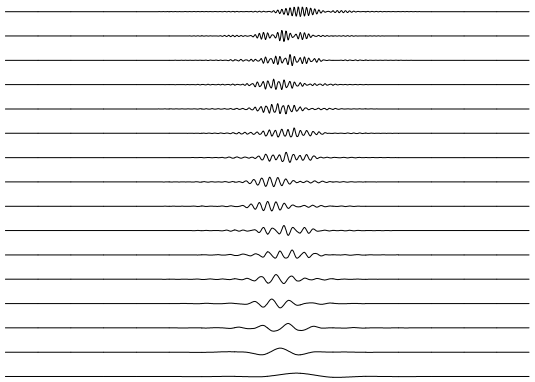

Figure 32: Localized modes.

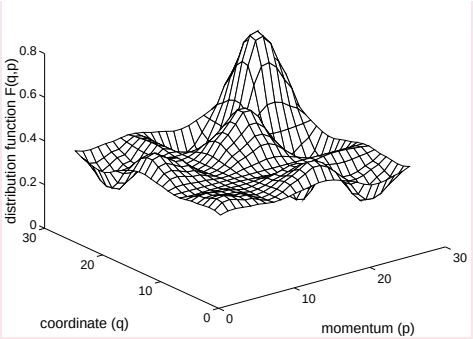


Figure 33: Localized mode contribution to distribution function.



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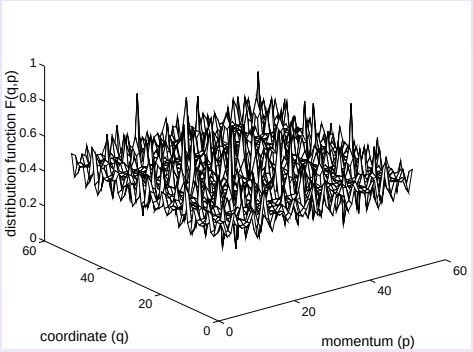


Figure 34: Chaotic-like pattern.

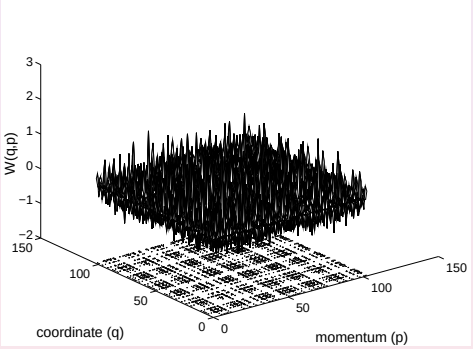


Figure 35: Entangled-like pattern.

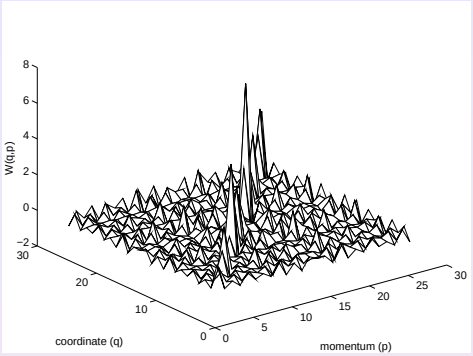


Figure 36: Localized pattern: waveleton.

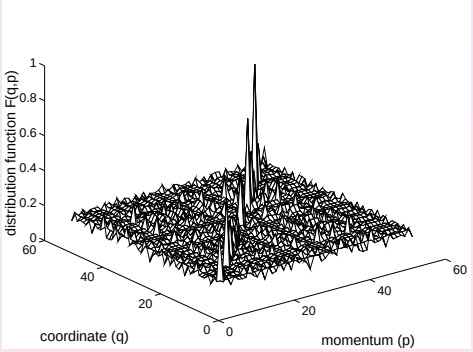


Figure 37: Localized pattern: waveleton.

# Introduction

We consider the application of a new numerical/analytical technique based on local nonlinear harmonic analysis approach for the description of complex non-equilibrium behaviour of statistical ensembles, considered in the framework of the general BBGKY hierarchy of kinetics equations and their different truncations/reductions.



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- Kinetic theory is an important part of general statistical physics related to phenomena which cannot be understood on the thermodynamic or fluid models level.
- We restrict ourselves to the rational/polynomial type of nonlinearities (with respect to the set of all dynamical variables) that allows to use our results, which are based on the so called multiresolution framework and the variational formulation of initial nonlinear (pseudodifferential) problems.
- Wavelet analysis is a set of mathematical methods which give a possibility to work with well-localized bases in functional spaces and provide the maximum sparse forms for the general type of operators (differential, integral, pseudodifferential) in such bases.
- It provides the best possible rates of convergence and minimal complexity of algorithms inside and, as a result, saves CPU time and HDD space.
- Our main goals are an attempt of classification and construction of a possible zoo of nontrivial (meta) stable states: high-localized (nonlinear) eigenmodes, complex (chaotic-like or entangled) patterns, localized (stable) patterns (waveletons).

We start from the corresponding qualitative definitions:

- By a localized state (localized mode) we mean the corresponding (particular) solution (or generating mode) which is localized in maximally small region of the phase space.
- By a chaotic pattern we mean some solution (or asymptotics of solution) which has random-like distributed energy spectrum in a full domain of definition. In quantum case we need to consider additional entangled-like patterns, roughly speaking, which cannot be separated into pieces of sub-systems.
- By a localized pattern (waveleton) we mean (asymptotically) (meta) stable solution localized in relatively small region of the whole phase space (or a domain of definition). In this case an energy is distributed during some time (sufficiently large) between only a few localized modes (from point 1). We believe, it is a good image for plasma in a fusion state (energy confinement).
- In all cases above, by the system under consideration in the classical case we mean the full BBGKY hierarchy or some cut-off of it. Our construction of cut-off of the infinite system of equations is based on some criterion of convergence of the full solution.



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- This criterion is based on a natural norm in the proper functional space, which takes into account (non-perturbatively) the underlying multiscale structure of complex statistical dynamics. According to our approach the choice of the underlying functional space is important to understand the corresponding complex dynamics.
- It is obvious that we need to fix accurately the space, where we construct the solutions, evaluate convergence etc. and where the very complicated infinite set of operators, appeared in the BBGKY formulation, acts.
- We underline that many concrete features of the complex dynamics are related not only to the concrete form/class of operators/equations but depend also on the proper choice of function spaces, where operators act. It should be noted that the class of smoothness (related at least to the appearance of chaotic/fractal-like type of behaviour) of the proper functional space under consideration plays a key role in the following.
- Our main goals are an attempt of classification, construction and control of a possible zoo of nontrivial states/patterns.
- Localized (meta)stable pattern (waveleton) is a good image for fusion state in plasma (energy confinement).



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Our constructions can be applied to the following general Hamiltonians:

$$H_N = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + U_i(q) \right) + \sum_{1 \le i \le j \le N} U_{ij}(q_i, q_j),$$

where the potentials $U_i(q) = U_i(q_1, \ldots, q_N)$ and $U_{ij}(q_i, q_j)$ are restricted to rational functions on the coordinates. Let $L_s$ and $L_{ij}$ be the Liouvillean operators and

$$F_N(x_1, \ldots, x_N; t)$$

be the hierarchy of $N$-particle distribution function, satisfying the standard BBGKY hierarchy ($V$ is the volume):

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{v} \int d\mu_{s+1} \sum_{i=1}^{s} L_{i,s+1} F_{s+1}$$

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Our key point is the proper nonperturbative generalization of the previous perturbative multiscale approaches. The infinite hierarchy of distribution functions is:

$$F = \{F_0, F_1(x_1; t), \dots, F_N(x_1, \dots, x_N; t), \dots\},$$

where

$$F_p(x_1, \dots, x_p; t) \in H^p,$$
$$H^0 = R, \quad H^p = L^2(R^{6p}),$$
$$F \in H^\infty = H^0 \oplus H^1 \oplus \dots \oplus H^p \oplus \dots$$

with the natural Fock space like norm (guaranteeing the positivity of the full measure):

$$(F, F) = F_0^2 + \sum_i \int F_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^{i} \mu_\ell.$$

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- Multiresolution decomposition naturally and efficiently introduces the infinite sequence of the underlying hidden scales, which is a sequence of increasing closed subspaces $V_j \in L^2(R)$:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ...$$

- Our variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second.
- As a result, the solutions of the (truncated) hierarchies have the following multiscale decomposition via high-localized eigenmodes ($\omega_l \sim 2^l$, $k_m^s \sim 2^m$)

$$\begin{aligned} F(t, x_1, x_2, \dots) &= \sum_{(i,j)\in Z^2} a_{ij} U^i \otimes V^j(t, x_1, \dots), \\ V^j(t) &= V_N^{j,slow}(t) + \sum_{l \geq N} V_l^j(\omega_l t), \\ U^i(x_s) &= U_M^{i,slow}(x_s) + \sum_{m \geq M} U_m^i(k_m^s x_s), \end{aligned}$$

which corresponds to the full multiresolution expansion in all underlying time/space scales.

- In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode.
- It should be noted that such representations give the best possible localization properties in the corresponding (phase) space/time coordinates.
- Numerical calculations are based on compactly supported wavelets and related wavelet families and on evaluation of the accuracy on the level $N$ of the corresponding cut-off of the full system regarding norm above.
- Numerical modeling shows the creation of different internal structures from localized modes, which are related to (meta)stable or unstable type of behaviour and the corresponding patterns (waveletons) formation. Reduced algebraical structure provides the pure algebraical control of stability/unstability scenario.
- CONTROLLABLE (META) STABLE WAVELETON CONFIGURATION REPRESENTS A REASONABLE APRROXIMATION FOR THE REALIZABLE CONFINEMENT STATE.



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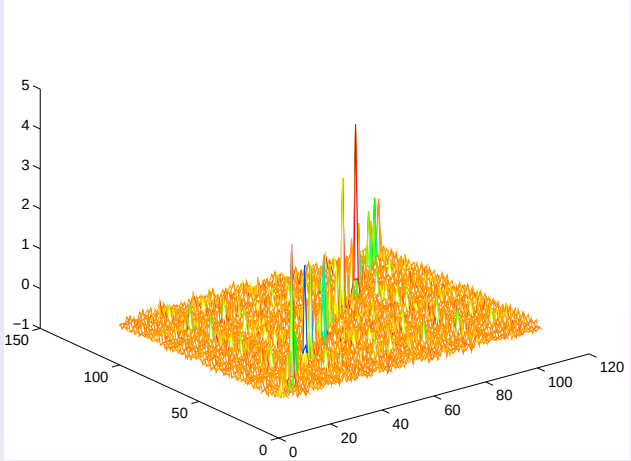


Figure 38: Waveleton Pattern

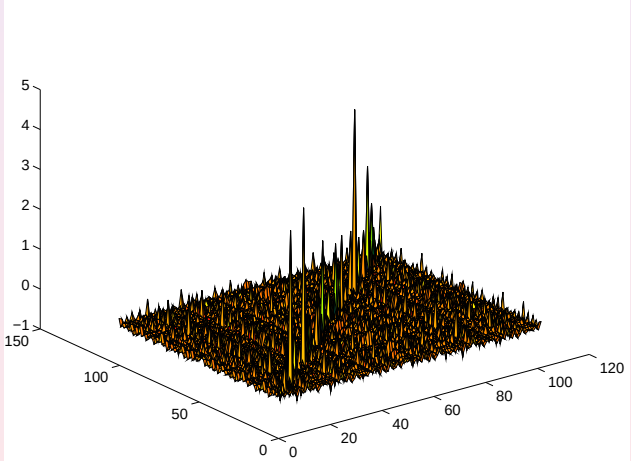


Figure 39: Waveleton Pattern



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We start from set-up for kinetic BBGKY hierarchy and present explicit analytical construction for solutions of hierarchy of equations, which is based on tensor algebra extensions of multiresolution representation and variational formulation. We give explicit representation for hierarchy of n-particle reduced distribution functions in the base of high-localized generalized coherent (regarding underlying affine group) states given by polynomial tensor algebra of wavelets, which takes into account contributions from all underlying hidden multiscales from the coarsest scale of resolution to the finest one to provide full information about stochastic dynamical process.

So, our approach resembles Bogolubov and related approaches but we don't use any perturbation technique (like virial expansion) or linearization procedures.

Numerical modeling shows the creation of different internal (coherent) structures from localized modes, which are related to stable (equilibrium) or un stable type of behaviour and corresponding pattern (waveletons) formation.

Let M be the phase space of ensemble of N particles ($\dim M = 6N$) with coordinates

$$
\begin{aligned}
x_i &= (q_i, p_i), \quad i = 1, ..., N, \\
q_i &= (q_i^1, q_i^2, q_i^3) \in R^3, \\
p_i &= (p_i^1, p_i^2, p_i^3) \in R^3, \\
q &= (q_1, \ldots, q_N) \in R^{3N}.
\end{aligned}
$$

Individual and collective measures are:

$$
\mu_i = \mathrm{d}x_i = \mathrm{d}q_i p_i, \quad \mu = \prod_{i=1}^{N} \mu_i
$$

Distribution function

$$D_N(x_1, \ldots, x_N; t)$$

satisfies Liouville equation of motion for ensemble with Hamiltonian $H_N$:

$$\frac{\partial D_N}{\partial t} = \{H_N, D_N\}$$

and normalization constraint

$$\int D_N(x_1, \ldots, x_N; t) \mathrm{d}\mu = 1$$



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where Poisson brackets are:

$$\{H_N, D_N\} = \sum_{i=1}^{N} \left( \frac{\partial H_N}{\partial q_i} \frac{\partial D_N}{\partial p_i} - \frac{\partial H_N}{\partial p_i} \frac{\partial D_N}{\partial q_i} \right)$$

Our constructions can be applied to the following general Hamiltonians:

$$H_N = \sum_{i=1}^{N} \left( \frac{p_i^2}{2m} + U_i(q) \right) + \sum_{1 \le i \le j \le N} U_{ij}(q_i, q_j)$$

where potentials

$$U_i(q) = U_i(q_1, \ldots, q_N)$$

and

$$U_{ij}(q_i, q_j)$$

are not more than rational functions on coordinates.



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Let $L_s$ and $L_{ij}$ be the Liouvillean operators (vector fields)

$$L_s = \sum_{j=1}^{s} \left( \frac{p_j}{m} \frac{\partial}{\partial q_j} - \frac{\partial u_j}{\partial q} \frac{\partial}{\partial p_j} \right) - \sum_{1 \leq i \ leq j \leq s} L_{ij}$$

$$L_{ij} = \frac{\partial U_{ij}}{\partial q_i} \frac{\partial}{\partial p_i} + \frac{\partial U_{ij}}{\partial q_j} \frac{\partial}{\partial p_j}$$

For s=N we have the following representation for Liouvillean vector field

$$L_N = \{H_N, \cdot\}$$



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and the corresponding ensemble equation of motion:

$$\frac{\partial D_N}{\partial t} + L_N D_N = 0$$

$L_N$ is self-adjoint operator regarding standard pairing on the set of phase space funct ions. Let

$$F_N(x_1, \ldots, x_N; t) = \sum_{S_N} D_N(x_1, \ldots, x_N; t)$$

be the N-particle distribution function ($S_N$ is permutation group of N element s). Then we have the hierarchy of reduced distribution functions ($V^s$ is the corresponding normalized volume factor)



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$$F_s(x_1, \ldots, x_s; t) = V^s \int D_N(x_1, \ldots, x_N; t) \prod_{s+1 \le i \le N} \mu_i$$

After standard manipulations we arrived to BBGKY hierarchy:

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{v} \int \mathrm{d}\mu_{s+1} \sum_{i=1}^{s} L_{i,s+1} F_{s+1}$$

It should be noted that we may apply our approach even to more general formulation.



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For s=1,2 we have:

$$\frac{\partial F_1(x_1;t)}{\partial t} + \frac{p_1}{m}\frac{\partial}{\partial q_1}F_1(x_1;t) = \frac{1}{\upsilon}\int \mathrm{d}x_2 L_{12} F_2(x_1,x_2;t)$$

$$\frac{\partial F_2(x_1,x_2;t)}{\partial t} + \left(\frac{p_1}{m}\frac{\partial}{\partial q_1} + \frac{p_2}{m}\frac{\partial}{\partial q_2} - L_{12}\right)$$
$$\cdot F_2(x_1,x_2;t) = \frac{1}{\upsilon}\int \mathrm{d}x_3 (L_{13}+L_{23}) F_3(x_1,x_2;t)$$

As in the general as in particular situations (cut-off, e.g.) we are interested in the cases when

$$F_k(x_1,\ldots,x_k;t) = \prod_{i=1}^{k} F_1(x_i;t) + G_k(x_1,\ldots,x_k;t),$$

where $G_k$ are correlators, really have additional reductions as in the simplest case of one-particle Vlasov/Boltzmann-like systems.



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Then by using such physical motivated reductions or/and during the corresponding cut-off procedure we obtain instead of linear and pseudodifferential (in general case) equations their finite-dimensional but nonlinear approximations with the polynomial type of nonlinearities (more exactly, multilinearities). Our key point in the following consideration is the proper generalization of naive perturbative multiscale Bogolubov's structure.



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# Multiscale analysis

The infinite hierarchy of distribution functions satisfying BBGKY system in the thermodynamical limit is:

$$F = \{F_0, F_1(x_1; t), F_2(x_1, x_2; t), \dots, F_N(x_1, \dots, x_N; t), \dots\}$$

where

$$F_p(x_1, \dots, x_p; t) \in H^p, \quad H^0 = R, \quad H^p = L^2(R^{6p})$$

(or any different proper functional space),



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$$F \in H^\infty = H^0 \oplus H^1 \oplus \cdots \oplus H^p \oplus \ldots$$

with the natural Fock-space like norm (of course, we keep in mind the positivity of the full measure):

$$(F, F) = F_0^2 + \sum_i \int F_i^2(x_1, \ldots, x_i; t) \prod_{\ell=1}^{i} \mu_\ell$$

$$F_k(x_1, \ldots, x_k; t) = \prod_{i=1}^{k} F_1(x_i; t)$$

First of all we consider $F = F(t)$ as function on time variable only, $F \in L^2(R)$, via multiresolution decomposition which naturally and efficiently introduces the infinite sequence of underlying hidden scales into the game.



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Because affine group of translations and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces.

Let the closed subspace $V_j (j \in \mathbf{Z})$ correspond to level j of resolution, or to scale j. We consider a multiresolution analysis of $L^2(R)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces $V_j$:

$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots$$

satisfying the following properties: let $W_j$ be the orthonormal complement of $V_j$ with respect to $V_{j+1}$:

$$V_{j+1} = V_j \bigoplus W_j$$



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then we have the following decomposition:

$$\{F(t)\} = \bigoplus_{-\infty<j<\infty} W_j$$

or in case when $V_0$ is the coarsest scale of resolution:

$$\{F(t)\} = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j},$$

Subgroup of translations generates basis for fixed scale number:

$$\mathrm{span}_{k\in Z}\{2^{j/2}\Psi(2^j t - k)\} = W_j.$$



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The whole basis is generated by action of full affine group:

$$\boxed{\mathrm{span}_{k\in Z, j\in Z}\{2^{j/2}\Psi(2^j t - k)\} = \mathrm{span}_{k,j\in Z}\{\Psi_{j,k}\} = \{F(t)\}}$$

Let sequence

$$\{V_j^t\}, V_j^t \subset L^2(R)$$

correspond to multiresolution analysis on time axis,

$$\{V_j^{x_i}\}$$

correspond to multiresolution analysis for coordinate $x_i$, then

$$\boxed{V_j^{n+1} = V_j^{x_1} \otimes \cdots \otimes V_j^{x_n} \otimes V_j^t}$$



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corresponds to multiresolution analysis for n-particle distribution fuction $F_n(x_1, \ldots, x_n; t)$. E.g., for $n = 2$:

$$\boxed{V_0^2 = \{f : f(x_1, x_2) = \sum_{k_1, k_2} a_{k_1, k_2} \phi^2(x_1 - k_1, x_2 - k_2), \quad a_{k_1, k_2} \in \ell^2(Z^2)\},}$$

where

$$\phi^2(x_1, x_2) = \phi^1(x_1)\phi^2(x_2) = \phi^1 \otimes \phi^2(x_1, x_2),$$

and $\phi^i(x_i) \equiv \phi(x_i)$ form a multiresolution basis corresponding to $\{V_j^{x_i}\}$. If

$$\{\phi^1(x_1 - \ell)\}, \ \ell \in Z$$

form an orthonormal set, then

$$\phi^2(x_1 - k_1, x_2 - k_2)$$

form an orthonormal basis for $V_0^2$. Action of affine group provides us by multiresolution representation of $L^2(R^2)$. After introducing detail spaces $W_j^2$, we have, e.g.

$$V_1^2 = V_0^2 \oplus W_0^2.$$



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Then 3-component basis for $W_0^2$ is generated by translations of three functions

$$\Psi_1^2 = \phi^1(x_1) \otimes \Psi^2(x_2),$$
$$\Psi_2^2 = \Psi^1(x_1) \otimes \phi^2(x_2),$$
$$\Psi_3^2 = \Psi^1(x_1) \otimes \Psi^2(x_2)$$

Also, we may use the rectangle lattice of scales and one-dimentional wavelet decomposition :

$$f(x_1, x_2) = \sum_{i,\ell;j,k} < f, \Psi_{i,\ell} \otimes \Psi_{j,k} > \Psi_{j,\ell} \otimes \Psi_{j,k}(x_1, x_2)$$

where bases functions

$$\Psi_{i,\ell} \otimes \Psi_{j,k}$$

depend on two scales $2^{-i}$ and $2^{-j}$.



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We obtain our multiscale/multiresolution representations below) via the variational wavelet approach for the following formal representation of the BBGKY system (or its finite-dimensional nonlinear approximation for the $n$-particle distribution functions) with the corresponding obvious constraints on the distribution functions.



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# Variational approach



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Let $L$ be an arbitrary (non)linear differential/integral operator with matrix dimension $d$ (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes^n})$:
$\Psi \equiv \Psi(t, x_1, x_2, \dots) = \Big(\Psi^1(t, x_1, x_2, \dots), \dots, \Psi^d(t, x_1, x_2, \dots)\Big)$,
$x_i \in \Omega \subset \mathbf{R}^6$, $n$ is the number of particles:

$$L\Psi \equiv L(Q, t, x_i)\Psi(t, x_i) = 0,$$

$$Q \equiv Q_{d_0, d_1, d_2, \dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) =$$

$$\sum_{i_0, i_1, i_2, \dots = 1}^{d_0, d_1, d_2, \dots} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots)\Big(\frac{\partial}{\partial t}\Big)^{i_0}\Big(\frac{\partial}{\partial x_1}\Big)^{i_1}\Big(\frac{\partial}{\partial x_2}\Big)^{i_2} \dots \int \mu_k$$

Let us consider now the $N$ mode approximation for the solution as the following ansatz:

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$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0, i_1, i_2, \dots = 1}^{N} a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots (t, x_1, x_2, \dots)$$

We shall determine the expansion coefficients from the following conditions (different related variational approaches are considered also):

$$\ell^N_{k_0, k_1, k_2, \dots} \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) \mathrm{d}t \mathrm{d}x_1 \mathrm{d}x_2 \cdots = 0$$

Thus, we have exactly $dN^n$ algebraical equations for $dN^n$ unknowns $a_{i_0, i_1, \dots}$. This variational approach reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second. We consider the multiresolution expansion as the second main part of our construction. The solution is parametrized by the solutions of two sets of reduced algebraical problems, one is linear or nonlinear (depending on the structure of the operator $L$) and the rest are linear problems related to the computation of the coefficients of the algebraic equations. These coefficients can be found by some methods by using the compactly supported wavelet basis functions.

As a result the solution has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes

$$F(t, x_1, x_2, \dots) = \sum_{(i,j)\in Z^2} a_{ij} U^i \otimes V^j(t, x_1, x_2, \dots)$$

$$V^j(t) = V_N^{j,slow}(t) + \sum_{l \geq N} V_l^j(\omega_l t), \quad \omega_l \sim 2^l$$

$$U^i(x_s) = U_M^{i,slow}(x_s) + \sum_{m \geq M} U_m^i(k_m^s x_s), \quad k_m^s \sim 2^m,$$

which corresponds to the full multiresolution expansion in all underlying time/space scales.
These formulae give the expansion into a slow and fast oscillating parts. So, we may move from the coarse scales of resolution to the finest ones for obtaining more detailed information about the dynamical process.



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In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each non-linear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates.

In contrast with different approaches our formulae do not use perturbation technique or linearization procedures. Numerical calculations are based on compactly supported wavelets and related wavelet families and on evaluation of the accuracy regarding norm:

$$\|F^{N+1} - F^{N}\| \leq \varepsilon$$

# Modeling of patterns



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To summarize, the key points are:
1. The ansatz-oriented choice of the (multidimensional) bases related to some polynomial tensor algebra.
2. The choice of proper variational principle. A few projection/ Galerkin-like principles for constructing (weak) solutions are considered. The advantages of formulations related to biorthogonal (wavelet) decomposition should be noted.
3. The choice of bases functions in the scale spaces $W_j$ from wavelet zoo. They correspond to high-localized (nonlinear) oscillations/excitations, nontrivial local (stable) distributions/fluctuations, etc. Besides fast convergence properties it should be noted minimal complexity of all underlying calculations, especially in case of choice of wavelet packets which minimize Shannon entropy.
4. Operator representations providing maximum sparse representations for arbitrary (pseudo) differential/ integral operators $\mathrm{d}f/\mathrm{d}x$, $\mathrm{d}^n f/\mathrm{d}x^n$, $\int T(x,y)f(y)\mathrm{d}y)$, etc.
5. (Multi)linearization. Besides the variation approach we can consider also a different method to deal with (polynomial) nonlinearities: para-products-like decompositions.

To classify the qualitative behaviour we apply standard methods from general control theory or really use the control. We will start from a priori unknown coefficients, the exact values of which will subsequently be recovered. Roughly speaking, we will fix only class of nonlinearity (polynomial in our case) which covers a broad variety of examples of possible truncation of the systems. As a simple model we choose band-triangular non-sparse matrices $(a_{ij})$ in particular case $d = 2$. These matrices provide tensor structure of bases in (extended) phase space and are generated by the roots of the reduced variational (Galerkin-like) systems. As a second step we need to restore the coefficients from these matrices by which we may classify the types of behaviour. We start with the localized mode, which is a base mode/eigenfunction, Fig. 14, corresponding to def. 1, which was constructed as a tensor product of the two Daubechies functions. Fig. 15, corresponding to def. 2, presents the result of summation of series up to value of the dilation/scale parameter equal to six on the bases of symmlets with the corresponding matrix elements equal to one. The size of matrix is 512x512 and as a result we provide modeling for one-particle distribution function corresponding to standard Vlasov-like cut-off with $F_2 = F_1^2$. So, different possible distributions of the root values of the generical algebraical system provide qualitatively different types of behaviour. The above choice provides us by a distribution with chaotic-like equidistribution.



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But, if we consider a band-like structure of matrix $(a_{ij})$ with the band along the main diagonal with finite size ($\ll 512$) and values, e.g. five, while the other values are equal to one, we obtain localization in a fixed finite area of the full phase space, i.e. almost all energy of the system is concentrated in this small volume. This corresponds to definition 3 and is shown in Fig. 16, constructed by means of Daubechies-based wavelet packets. Depending on the type of solution, such localization may be present during the whole time evolution (asymptotically-stable) or up to the needed value from time scale (e.g. enough for plasma fusion/confinement).



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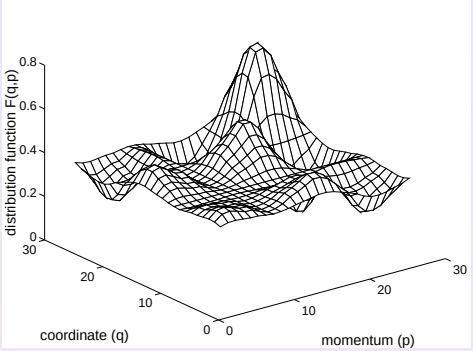


Figure 40: Localized mode contribution to distribution function.

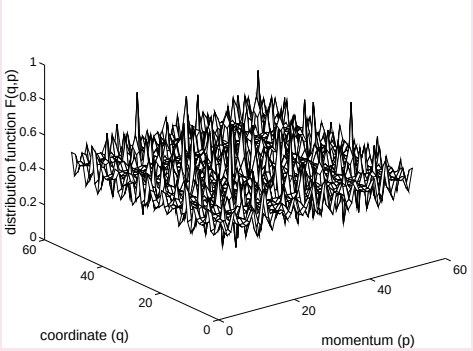


Figure 41: Chaotic-like pattern.

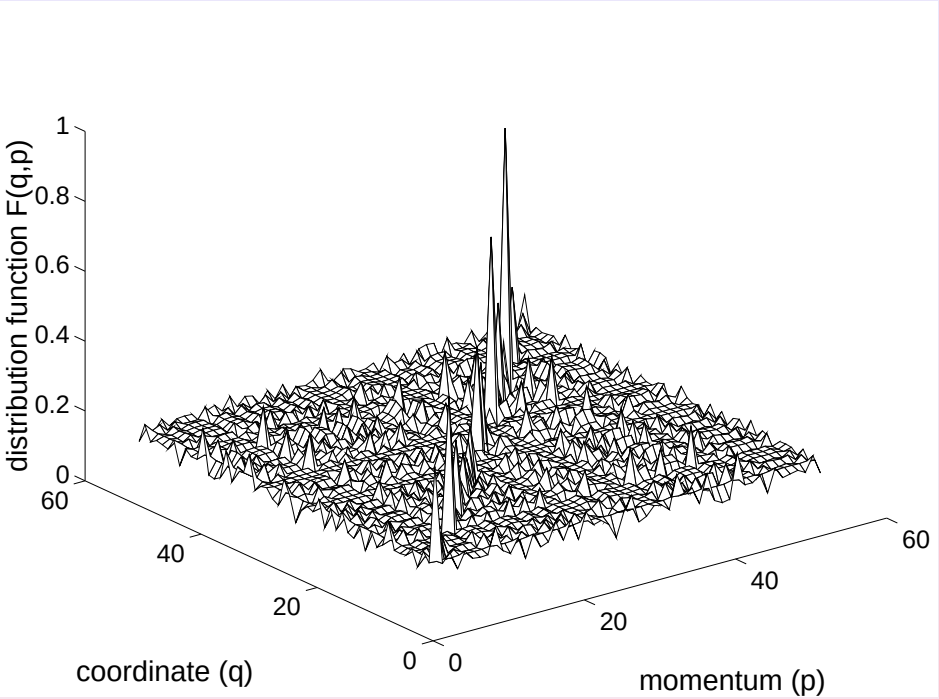


Figure 42: Localized waveleton pattern.



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# Vlasov-Poisson system

In this paper we consider the applications of numerical-analytical approach based on multiscale variational wavelet technique to the systems with collective type behaviour described by some forms of Vlasov-Poisson/Maxwell equations. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by the systems of nonlinear ordinary/partial differential/integral equations with or without some (functional) constraints. In periodic accelerators and transport systems at the high beam currents and charge densities the effects of the intense self-fields, which are produced by the beam space charge and currents, determinine (possible) equilibrium states, stability and transport properties according to underlying nonlinear dynamics. The dynamics of such space-charge dominated high brightness beam systems can provide the understanding of the instability phenomena such as emittance growth, mismatch, halo formation related to the complicated behaviour of underlying hidden nonlinear modes outside of perturbative tori-like KAM regions.

Our analysis is based on the variational-wavelet approach, which allows us to consider polynomial and rational type of nonlinearities.



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In some sense our approach is direct generaliztion of traditional nonlinear $\delta F$ approach in which weighted Klimontovich representation

$$\delta f_j = a_j \sum_{i=1}^{N_j} w_{ji} \delta(x - x_{ji}) \delta(p - p_{ji})$$

or self-similar decompostion like

$$\delta n_j = b_j \sum_{i=1}^{N_j} w_{ji} s(x - x_{ji}),$$

where $s(x - x_{ji})$ is a shape function of distributing particles on the grids in configuration space, are replaced by powerful technique from local nonlinear harmonic analysis, based on underlying symmetries of functional space such as affine or more general.



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The solution has the multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying time/phase space scales.

Starting from Vlasov-Poisson equations, we consider the approach based on multiscale variational-wavelet formulation. We give the explicit representation for all dynamical variables in the base of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by solutions of a number of reduced algebraical problems one from which is nonlinear with the same degree of nonlinearity as initial problem and the others are the linear problems which correspond to the particular method of calculations inside concrete wavelet scheme. Because our approach started from variational formulation we can control evolution of instability on the pure algebraical level of reduced algebraical system of equations. This helps to control stability/unstability scenario of evolution in parameter space on pure algebraical level. In all these models numerical modeling demonstrates the appearance of coherent high-localized structures and as a result the stable patterns formation or unstable chaotic behaviour.

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Analysis based on the non-linear Vlasov equations leads to more clear understanding of collective effects and nonlinear beam dynamics of high intensity beam propagation in periodic-focusing and uniform-focusing transport systems. We consider the following form of equations

$$\left\{\frac{\partial}{\partial s}+p_x\frac{\partial}{\partial x}+p_y\frac{\partial}{\partial y}-\left[k_x(s)x+\frac{\partial\psi}{\partial x}\right]\frac{\partial}{\partial p_x}-\right.$$
$$\left.\left[k_y(s)y+\frac{\partial\psi}{\partial y}\right]\frac{\partial}{\partial p_y}\right\}f_b(x,y,p_x,p_y,s)=0,$$
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi=-\frac{2\pi K_b}{N_b}\int \mathrm{d}p_x\mathrm{d}p_y f_b,$$
$$\int \mathrm{d}x\mathrm{d}y\mathrm{d}p_x\mathrm{d}p_y f_b=N_b$$

The corresponding Hamiltonian for transverse single-particle motion is given by

$$H(x, y, p_x, p_y, s) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}[k_x(s)x^2 + k_y(s)y^2] + H_1(x, y, p_x, p_y, s) + \psi(x, y, s),$$

where $H_1$ is nonlinear (polynomial/rational) part of the full Hamiltonian and corresponding characteristic equations are:

$$\begin{aligned} \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + k_x(s)x + \frac{\partial}{\partial x}\psi(x, y, s) &= 0 \\ \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + k_y(s)y + \frac{\partial}{\partial y}\psi(x, y, s) &= 0 \end{aligned}$$



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# Multiscale representations

We obtain our multiscale/multiresolution representations for solutions of these equations via variational-wavelet approach. We decompose the solutions as

$$f_b(s,x,y,p_x,p_y) = \sum_{i=i_c}^{\infty} \oplus \delta^i f(s,x,y,p_x,p_y)$$

$$\psi(s,x,y) = \sum_{j=j_c}^{\infty} \oplus \delta^j \psi(s,x,y)$$

$$x(s) = \sum_{k=k_c}^{\infty} \oplus \delta^k x(s),$$

$$y(s) = \sum_{\ell=\ell_c}^{\infty} \oplus \delta^\ell y(s)$$



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where set

$$(i_c, j_c, k_c, \ell_c)$$

corresponds to the coarsest level of resolution $c$ in the full multiresolution decomposition

$$\boxed{V_c \subset V_{c+1} \subset V_{c+2} \subset \dots}$$

Introducing detail space $W_j$ as the orthonormal complement of $V_j$ with respect to

$$\boxed{V_{j+1} : V_{j+1} = V_j \bigoplus W_j,}$$

we have for

$$f, \psi, x, y \subset L^2(\mathbf{R})$$



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$$L^2(\mathbf{R}) = \overline{V_c \bigoplus_{j=c}^{\infty} W_j},$$

In some sense it is some generalization of the old $\delta F$ approach. Let $L$ be an arbitrary (non) linear differential/integral operator with matrix dimension $d$, which acts on some set of functions

$$\Psi \equiv \Psi(s,x) = \Big(\Psi^1(s,x), \ldots, \Psi^d(s,x)\Big),$$

$$s, x \in \Omega \subset \mathbf{R}^{n+1}$$

from $L^2(\Omega)$:

$$L\Psi \equiv L(R(s,x), s, x)\Psi(s,x) = 0,$$



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($x$ are the generalized space coordinates or phase space coordinates, $s$ is "time" coordinate). After some anzatzes the main reduced problem may be formulated as the system of ordinary differential equations

$$Q_i(f)\frac{\mathrm{d}f_i}{\mathrm{d}s} = P_i(f,s), \quad f = (f_1, ..., f_n),$$
$$i = 1, \ldots, n, \quad \max_i deg\ P_i = p, \quad \max_i deg\ Q_i = q$$

or a set of such systems corresponding to each independent coordinate in phase space. They have the fixed initial (or boundary) conditions $f_i(0)$, where $P_i, Q_i$ are not more than polynomial functions of dynamical variables $f_j$ and have arbitrary dependence on time. As result we have the following reduced algebraical system of equations on the set of unknown coefficients $\lambda_i^k$ of localized eigenmode expansion:

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$$L(Q_{ij}, \lambda, \alpha_I) = M(P_{ij}, \lambda, \beta_J),$$

where operators L and M are algebraization of RHS and LHS of initial problem and $\lambda$ are unknowns of reduced system of algebraical equations (RSAE). After solution of RSAE we determine the coefficients of wavelet expansion and therefore obtain the solution of our initial problem. It should be noted that if we consider only truncated expansion with N terms then we have the system of $N \times n$ algebraical equations with degree

$$\ell = max\{p, q\}$$

and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (rational) problem in the form

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$$f_i(s) = f_i(0) + \sum_{k=1}^{N} \lambda_i^k f_k(s),$$

where coefficients $\lambda_i^k$ are the roots of the corresponding reduced algebraical (polynomial) problem RSAE. Consequently, we have a parametrization of solution of initial problem by the solution of reduced algebraical problem. The obtained solutions are given in this form, where $f_k(t)$ are basis functions obtained via multiresolution expansions and represented by some compactly supported wavelets. As a result the solution of equations has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying scales starting from coarsest one For

$$\mathbf{x} = (x, y, p_x, p_y)$$

$$\Psi(s,\mathbf{x}) = \sum_{(i,j)\in Z^2} a_{ij}\mathbf{U}^i \otimes V^j(s,\mathbf{x}),$$

$$V^j(s) = V_N^{j,slow}(s) + \sum_{l\geq N} V_l^j(\omega_l s), \quad \omega_l \sim 2^l$$

$$\mathbf{U}^i(\mathbf{x}) = \mathbf{U}_M^{i,slow}(\mathbf{x}) + \sum_{m\geq M} \mathbf{U}_m^i(k_m\mathbf{x}), \quad k_m \sim 2^m,$$

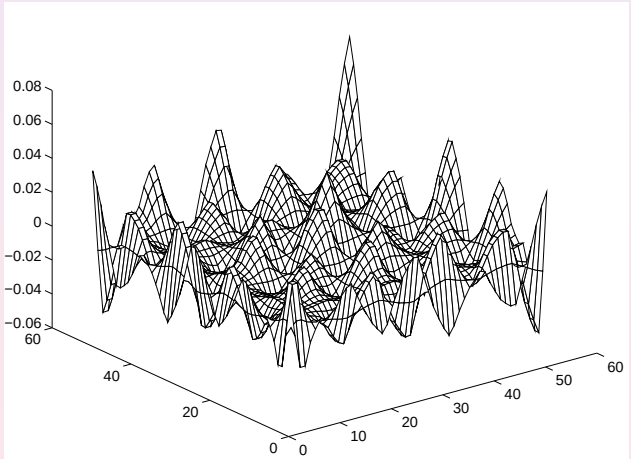


Figure 43: Eigenmode of level 1.

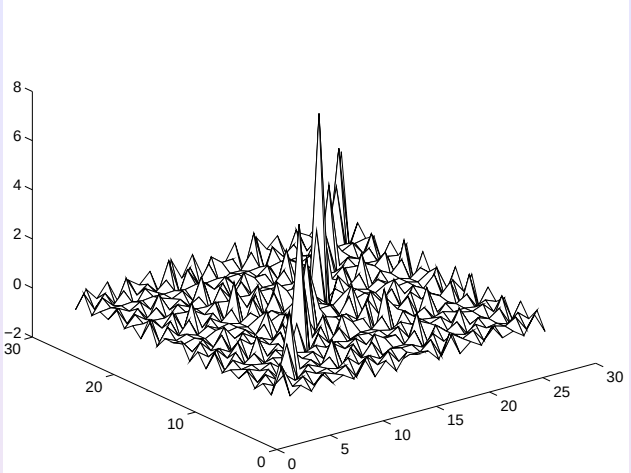


Figure 44: Stable waveleton pattern.

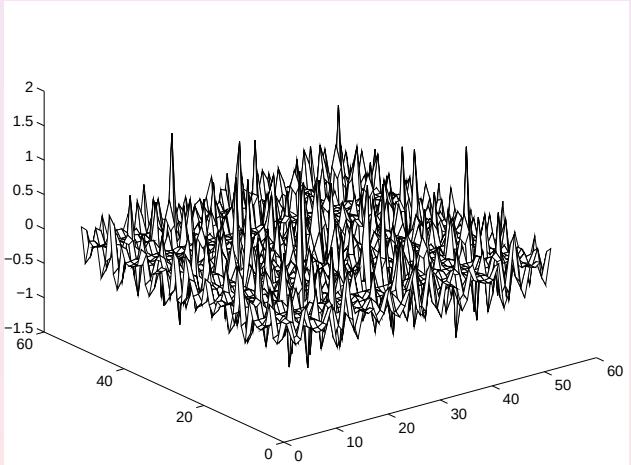


Figure 45: Chaotic-like behaviour.

This formula gives us expansion into the slow part $\Psi^{slow}_{N,M}$ and fast oscillating parts for arbitrary N, M. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first terms in the RHS correspond on the global level of function space decomposition to resolution space and the second ones to detail space. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with different approaches this formulae do not use perturbation technique or linearization procedures. So, by using wavelet bases with their good (phase) space/time localization properties we can describe high-localized (coherent) structures in spatially-extended stochastic systems with collective behaviour.



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Modelling demonstrates the appearance of stable patterns formation from high-localized coherent structures or chaotic behaviour. On Fig. 17 we present contribution to the full expansion from coarsest level (waveleton) of decomposition. Fig. 18, 19 show the representations for full solutions, constructed from the first 6 eigenmodes (6 levels in our formula), and demonstrate stable localized pattern formation and chaotic-like behaviour outside of KAM region.

We can control the type of behaviour on the level of reduced algebraical system.

# RATE/RMS models



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In this paper we consider the applications of a new numerical--analytical technique based on the methods of local nonlinear harmonic analysis or wavelet analysis to nonlinear rms/rate equations for averaged quantities related to some particular case of nonlinear Vlasov-Maxwell equations.

Our starting point is a model and approach proposed by R. C. Davidson e.a.. We consider electrostatic approximation for a thin beam. This approximation is a particular important case of the general reduction from statistical collective description based on Vlasov-Maxwell equations to a finite number of ordinary differential equations for the second moments related quantities (beam radius and emittance). In our case these reduced rms/rate equations also contain some disribution averaged quantities besides the second moments, e.g. self-field energy of the beam particles. Such model is very efficient for analysis of many problems related to periodic focusing accelerators, e.g. heavy ion fusion and tritium production. So, we are interested in the understanding of collective properties, nonlinear dynamics and transport processes of intense non-neutral beams propagating through a periodic focusing field.

Our approach is based on the variational-wavelet approach that allows to consider rational type of nonlinearities in rms/rate dynamical equations containing statistically averaged quantities also.

The solution has the multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes (waveletons), which corresponds to the full multiresolution expansion in all underlying internal hidden scales. We may move from coarse scales of resolution to the finest one to obtain more detailed information about our dynamical process. In this way we give contribution to our full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode.

Starting from some electrostatic approximation of Vlasov-Maxwell system and rms/rate dynamical models we consider the approach based on variational-wavelet formulation. We give explicit representation for all dynamical variables in the bases of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by the solutions of a number of reduced standard algebraical problems. We present also numerical modelling based on our analytical approach.



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# RATE equations

In thin-beam approximation with negligibly small spread in axial momentum for beam particles we have in Larmor frame the following electrostatic approximation for Vlasov-Maxwell equations:

$$\frac{\partial F}{\partial s} + x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} - \left(k(s)x + \frac{\partial \psi}{\partial x}\right)\frac{\partial F}{\partial x'} - \left(k(s)y + \frac{\partial \psi}{\partial y}\right)\frac{\partial F}{\partial y'} = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\frac{2\pi K}{N}\int \mathrm{d}x'\mathrm{d}y' F$$

where $\psi(x, y, s)$ is normalized electrostatic potential and

$$F(x, y, x', y', s)$$

is distribution function in transverse phase space



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$$(x, y, x', y', s)$$

with normalization

$$N = \int \mathrm{d}x\mathrm{d}y n, \qquad n(x, y, s) = \int \mathrm{d}x'\mathrm{d}y' F$$

where $K$ is self-field perveance which measures self-field intensity. Introducing self-field energy

$$E(s) = \frac{1}{4\pi K} \int \mathrm{d}x\mathrm{d}y |\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2|$$

we have obvious equations for root-mean-square beam radius $R(s)$



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$$R(s) = < x^2 + y^2 >^{1/2}$$

and unnormalized beam emittance

$$\varepsilon^2(s) = 4(< x'^2 + y'^2 >< x^2 + y^2 > - < xx' - yy' >),$$

which appear after averaging second-moments quantities regarding distribution function $F$:

$$\frac{\mathrm{d}^2 R(s)}{\mathrm{d}s^2} + \Big(k(s)R(s) - \frac{K(1+\Delta)}{2R^2(s)}\Big)R(s) = \frac{\varepsilon^2(s)}{4R^3(s)}$$

$$\frac{\mathrm{d}\varepsilon^2(s)}{\mathrm{d}s} + 8R^2(s)\Big(\frac{\mathrm{d}R}{\mathrm{d}s}\frac{K(1+\Delta)}{2R(s)} - \frac{\mathrm{d}E(s)}{\mathrm{d}s}\Big) = 0,$$



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where the term

$$K(1+\Delta)/2$$

may be fixed in some interesting cases, but generally we have it only as average

$$\boxed{K(1+\Delta)/2 = -< x\partial\psi/\partial x + y\partial\psi/\partial y >}$$

regarding distribution $F$. Anyway, the rate equations represent reasoanable reductions for the second-moments related quantities from the full nonlinear Vlasov-Poisson system. For trivial distributions Davidson e.a. found additional reductions. For KV distribution (step-function density) the second rate equation is trivial,

$$\varepsilon(s) = \mathrm{const}$$

and we have only one nontrivial rate equation for rms beam radius. The fixed-shape density profile ansatz for axisymmetric distributions also leads to similar situation: emittance conservation and the same envelope equation with two shifted constants only.



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# Multiscale representations

Accordingly to our approach which allows us to find exact solutions as for Vlasov-like systems as for rms-like systems we need not to fix particular case of distribution function

$$F(x, y, x', y', s).$$

Our consideration is based on the following multiscale $N$-mode anzatz:

$$F^N(x, y, x', y', s) = \sum_{i_1, \ldots, i_5 = 1}^{N} a_{i_1, \ldots, i_5} \bigotimes_{k=1}^{5} A_{i_k}(x, y, x', y', s)$$

$$\psi^N(x, y, s) = \sum_{j_1, j_2, j_3 = 1}^{N} b_{j_1, j_2, j_3} \bigotimes_{k=1}^{3} B_{j_k}(x, y, s)$$

These formulae provide multiresolution representation for variational solutions of our system. Each high-localized mode/harmonics $A_j(s)$ corresponds to level $j$ of resolution from the whole underlying infinite scale of spaces:



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$$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \ldots,$$

where the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. The construction of such tensor algebra based multiscales bases is considered by us. We'll consider rate equations as the following operator equation.

Let $L$, $P$, $Q$ be an arbitrary nonlinear (rational in dynamical variables) first-order matrix differential operators with matrix dimension d (d=4 in our case)

corresponding to the system of equations, which act on some set of functions

$$\Psi \equiv \Psi(s) = \Big(\Psi^1(s), \ldots, \Psi^d(s)\Big), \quad s \in \Omega \subset R$$

from

$$L^2(\Omega) : Q(R,s)\Psi(s) = P(R,s)\Psi(s)$$



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or

$$L\Psi \equiv L(R, s)\Psi(s) = 0$$

where

$$R \equiv R(s, \partial/\partial s, \Psi).$$

Let us consider now the N mode approximation for solution as the following expansion in some high-localized wavelet-like basis:

$$\Psi^N(s) = \sum_{r=1}^{N} a_r^N \phi_r(s)$$



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We'll determine the coefficients of expansion from the following variational condition:

$$L_k^N \equiv \int (L\Psi^N)\phi_k(s)\mathrm{d}s = 0$$

We have exactly $dN$ algebraical equations for $dN$ unknowns $a_r$. So, variational approach reduced the initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. As a result we have the following reduced algebraical system of equations (RSAE) on the set of unknown coefficients $a_i^N$ of the expansion:

$$H(Q_{ij}, a_i^N, \alpha_I) = M(P_{ij}, a_i^N, \beta_J),$$

where operators $H$ and $M$ are algebraization of RHS and LHS of initial problem. $Q_{ij}$ ($P_{ij}$) are the coefficients of LHS (RHS) of the initial system of differential equations and as consequence are coefficients of RSAE.



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$$I = (i_1, ..., i_{q+2}), \qquad J = (j_1, ..., j_{p+1})$$

are multiindexes, by which are labelled $\alpha_I$ and $\beta_I$, the other coefficients of RSAE:

$$\boxed{\beta_J = \{\beta_{j_1 \dots j_{p+1}}\} = \int \prod_{1 \le j_k \le p+1} \phi_{j_k},}$$

where $p$ is the degree of polynomial operator $P$

$$\boxed{\alpha_I = \{\alpha_{i_1} \dots \alpha_{i_{q+2}}\} = \sum_{i_1, \dots, i_{q+2}} \int \phi_{i_1} \dots \dot{\phi}_{i_s} \dots \phi_{i_{q+2}},}$$

where $q$ is the degree of polynomial operator $Q$,

$$i_\ell = (1, ..., q+2), \quad \dot{\phi}_{i_s} = \mathrm{d}\phi_{i_s}/\mathrm{d}s.$$



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We may extend our approach to the case when we have additional constraints on the set of our dynamical variables

$$\Psi = \{R, \varepsilon\}$$

and additional averaged terms also.
In this case by using the method of Lagrangian multipliers we again may apply the same approach but for the extended set of variables. As a result we receive the expanded system of algebraical equations analogous to our system. Then, after reduction we again can extract from its solution the coefficients of the expansion. It should be noted that if we consider only truncated expansion with N terms then we have the system of $N \times d$ algebraical equations with the degree

$$\ell = max\{p, q\}$$

and the degree of this algebraical system coincides with the degree of the initial system. So, after all we have the solution of the initial nonlinear (rational) problem in the form



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$$
\begin{aligned}
R^N(s) &= R(0) + \sum_{k=1}^{N} a_k^N \phi_k(s) \\
\varepsilon^N(s) &= \varepsilon(0) + \sum_{k=1}^{N} b_k^N \phi_k(s)
\end{aligned}
$$

where coefficients

$$
a_k^N, \qquad b_k^N
$$

are the roots of the corresponding reduced algebraical (polynomial) problem RSAE. Consequently, we have a parametrization of the solution of the initial problem by solution of reduced algebraical problem. The problem of computations of coefficients

$$
\alpha_I, \quad \beta_J
$$

of reduced algebraical system may be explicitly solved in wavelet approach.

The obtained solutions are given in the form, where $\phi_k(s)$ are proper wavelet bases functions (e.g., periodic or boundary).

It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates.

In contrast with different approaches these formulae do not use perturbation technique or linearization procedures and represent dynamics via generalized nonlinear localized eigenmodes expansion.

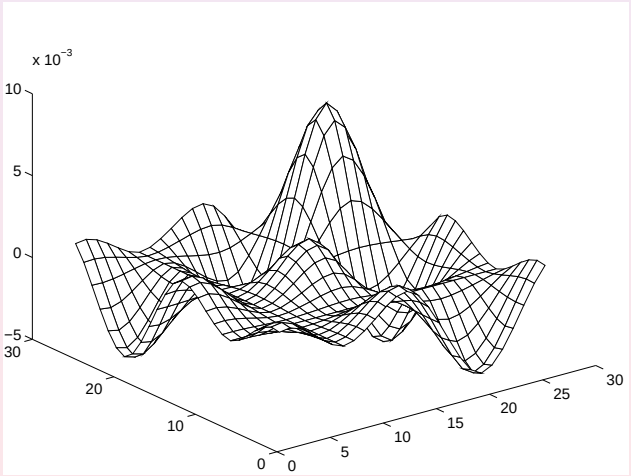


Figure 46: Waveleton-like distribution.

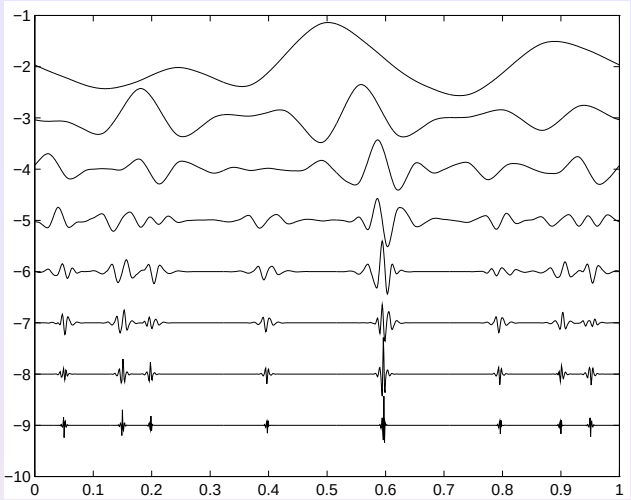


Figure 47: Multiscale decomposition.

Our N mode construction gives the following general multiscale representation:

$$
\begin{aligned}
R(s) &= R_N^{slow}(s) + \sum_{i \geq N} R^i(\omega_i s), \quad \omega_i \sim 2^i \\
\varepsilon(s) &= \varepsilon_N^{slow}(s) + \sum_{j \geq N} \varepsilon^j(\omega_j s), \quad \omega_j \sim 2^j
\end{aligned}
$$



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where $R^i(s), \varepsilon^j(s)$ are represented by some family of (nonlinear) eigenmodes and gives the full multiresolution/multiscale representation in the high-localized wavelet bases.
The corresponding decomposition is presented on Fig. 21 and two-dimensional localized mode (waveleton) contribution to distribution function is presented on Fig. 20.

As a result we can construct different (stable) patterns from high-localized (coherent) structures in spatially-extended stochastic systems with complex collective behaviour.

# Instead of conclusions (a few remarks):

- Confinement of magnetic lines
  vs. confinement of modes in (non)
  equilibrium ensemble (of particles with their own (arbitrary) individual dynamics).
- Lorentz/Flux Dynamics (dynamical variables/observables)
  vs.
  Ensemble Dynamics (dynamics of partition functions).
- Non–$U(1)$ Fourier Dispersion relations/ Pseudodispersion Relations:
  LOCALIZED SPECTRUM DOMINATES.
- As Discrete as Continuous WT/LNHA.
- Symmetries and Topology of Configuration (Flux) Space!
- (Non) linear MHD, (Alfen) Waves etc...Ansatzes.
- Knots/Braids Theory: Chern-Simons Topological Field Theory (part of QCD, Gauge Fields (YM)), Quantum Groups, CFT. World of Symmetry (dim 2, 3, 4).
- Phenomenology: Grad-Shafranov (EWBH) vs. GL/NO (rmns.: Vlasov op. problems vs. $\psi$DO).



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# Conclusions

Let us summarize our main results:

- Physical Conjectures:
- P1
  State of fusion (confinement of energy) in plasma physics may and need be considered from the point of view of non-equilibrium statistical physics. According to this BBGKY framework looks naturally as first iteration. Main dynamical variables are partitions.
- P2
  High localized nonlinear eigenmodes are real physical modes important for fusion modeling. Intermode multiscale interactions create various patterns from these fundamental building blocks, and determine the behaviour of plasma.
  High localized (meta) stable patterns (waveletons), considered as long-living fluctuations, are proper images for plasma in fusion state.



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- Mathematical framework:
- M1 Problems under consideration, like BBGKY hierarchies or their reductions are considered as $\Psi DO$ problems in the framework of proper family of methods unified by effective multiresolution approach.
- M2
  Formulae based on generalized dispersion relations (GDR) provide exact multiscale representation for all dynamical variables (partitions, first of all) in the basis of high-localized nonlinear (eigen)modes. Numerical realizations in this framework are maximally effective from the point of view of complexity of all algorithms inside. GDR provide the way for the state control on the algbraic level.
- Realizability
  According to this approach, it is possible on formal level, in principle, to control ensemble behaviour and to realize the localization of energy (confinement state) inside the waveleton configurations created from a few fundamentals modes only.

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- Open Questions
- Q1
  Definitely, all above is only very naive ensemble approach. Current level of non-equilibrium statistical physics provides us only by BBGKY generic framework. All related internal problems are well-known but we still have nothing better. At the same time possible Vlasov-like reductions or phenomenological models also look as very far from reasonable from the point of view of the fusion problem set-up.
- Q2
  Considering for allusion successful areas of physics like superconductivity, for example, we may conclude that only microscopic BCS formulation provides the full explanation although Ginzburg-Landau (GL) phenomenological approach and even Froelich's and London's ones contributed to the general picture. Whether Vlasov equations are the analogue of GL ones and whether it is possible to construct microscopic model for plasma, these two important questions remain unanswered at present time.

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- Q3
  It may be natural also that approaches proposed in this paper and related ones are wrong because the proper and adequate framework for solution of fusion problem is related to confinement of magnetic lines or loops (new physical dynamical variables instead of partitions) or fluxes instead of confinement of localized point modes (attribute of any local field theory) considered as new and really proper physical variables. Such approach demands the topological background related to proper mathematical constructions. As allusion it is possible to consider the description of (fractional) quantum Hall effect by means of Chern-Simons/anyon models which allow to describe the dynamics on (of) knots and braids analytically. Anyway, it is still possible to apply successfull methods from (M1) and (M2) here too. Remarks in Section 2.2. demonstrate interesting relations. Other open possibility is related to taking into account internal quantum properties. From this point of view our approach is very useful because we unify quantum description and its classical counterpart in the general $\Psi DO$ framework.

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