Poissonology for Vlasovid

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Wild Wild East

1984, release 3.0

After: Kruskal, Ebin, Marsden, Holm, Weinstein, Duistermaat, Sternberg, Guillemin, Kirillov, Berezin, Arnold, Lichnerowicz, Dikii, Manin, Kostant, Gelfand, Zakharov, Morrison, Drinfeld, Sokolov, Weil, Weyl, Leray, Faddeev, Khesin, Witten, Adler, Floer, Gromov,

Lieelogy

Lie Algebras, Lie Groups

<u>def</u> G – $\boxed{\text{Lie group}}$: Topological Group $\cap$ Manifold

$G \times G \xrightarrow{\varphi} G$: $\varphi(x,y) = xy^{-1}$ – smooth

Hilbert 5 : $C^0 \to C^\omega$

<u>Th.</u> G – Lie group, H – closed subgroup $\Longrightarrow$ G/H – smooth manifold where G acts by smooth transformations. If H – invar. subgroup, G/H – Lie group.

Lie Algebra

<u>def</u> Lie algebra under field K ($\mathbb{R}, \mathbb{C}, \ldots$)

Lie algebra under K with linear space L under K with additional operation, $[x,y]$ – commutator, Lie bracket

1. bilinear
2. antisymmetric, $[x,x] = 0$, $\forall x \in L$
3. Jacobi ident.: $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$, $\forall x,y,z \in L$

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^{k} x_k, \quad c_{ij}^{k} + c_{ji}^{k} = 0$$

structure constants: $\frac{n^2(n-1)}{2}$ c_{ij}^{k}, $i<j$

Ex. If A - assoc. algebra under K

$$[x,y] = xy - yx \ , \quad x,y \in A$$

<u>def.</u> <u>Differentiating</u> of <u>algebra</u> A under K (A - non-assoc. !)

$\mathcal{D} : A \to A$ (linear)

$$\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$$

<u>Prop.</u> Space of all diff. of algebra A is Lie algebra regarding operation

$$[\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2\mathcal{D}_1$$

<u>Cons.</u> $\boxed{\text{Vect } M}$ - smooth vector fields on manifold M is Lie algebra regarding $[\ ,\ ]$.

Here $A = C^{\infty}(M)$

<u>Hom</u> {Liealg} - <u>category</u>

<u>Mor</u> - homomorphisms Lie algebras

$$\varphi : L_1 \to L_2 : \quad \varphi([x,y]) = [\varphi(x), \varphi(y)], \quad \forall x,y \in L_1$$

Hom: $L_1 \to L_2$

$\boxed{\text{Representation}}$ <u>Linear</u> <u>repr.</u>

L_2 - linear, $[A,B] = AB - BA$,

Th. Ado: <u>Every finite dim. Lie algebra have exact finite dimensional linear repr.</u>

Matrix Algebras Lie

def. L - Lie algebra, $\forall X \in L$:

$\operatorname{ad} X$ - operator in L:

$$\operatorname{ad} X(y) = [X, y]$$

prop. ad - differentiating, $X \mapsto \operatorname{ad} X$ - repr. of Lie algebra L in space L (regular, adj. repr.)

Kernel repr. = center L

def. M - ideal in L, if $[L, M] \subset M$

$M \subset L$

$L_1 = [L, L]$, $L_2 = [L, L_1]$, ... $L_{n+1} = [L, L_n]$, ...

$L^1 = [L, L]$, $L^2 = [L^1, L^1]$, ... $L^{n+1} = [L^n, L^n]$

def. Lie alg.: solvable / nilpotent, if

$$L^\infty / L_\infty = \{0\}$$

Th. Engel: Alg. Lie - nilpotent iff $\operatorname{ad} X$, $\forall X \in L$ nilpotent

$$(\operatorname{ad} X)^n = 0, \quad n = n_0$$

def. Killing - Cartan form

$$B(X, y) = \operatorname{tr}(\operatorname{ad} X \cdot \operatorname{ad} y)$$

def. Lie algebra: simple: no nontrivial ideals

Semisimple no nontrivial commutative ideals

Th. Cartan: $B(X, y)$ - non-degenerate iff L - semisimple

$B(X, y) = 0$: nilpotent

Semi-simple Lie algebras

$\mathbb{C}$: 4 infnt. series, 5 - exceptional
$\mathbb{R}$: 12 infnt. series, 23 - exceptional

A_n, B_n, C_n, D_n

Cartan subalgebra, Dynkin diagrams, ...

ad X, $X \in H$ (Cartan) - diagonalize

$[X, X_\alpha] = \lambda_\alpha(X) X_\alpha, \quad X \in H$

$[X_\alpha, X_\beta]$, Σ - roots

Lie Group - Lie algebra

G, e, $T_e G$

def. Adjoint representation G on $\mathfrak{g}$

$A(g)$: diffeo. G:

$A(g): h \mapsto g h g^{-1}, \quad h, g \in G$

$A(g)$ - Aut G (aut G), e-fixed point

$A(g)_*(e): \mathfrak{g} \to \mathfrak{g}$ (differentiating)

$\mathrm{Ad}\, g: \mathfrak{g} \xmapsto[\text{chain rule}]{} \mathrm{Ad}\, g$: Hom. $G \to \mathrm{Aut}\, \mathfrak{g}$

$[X, Y] = \mathrm{ad}\, X\, Y$ Adjoint representation $\mathrm{Ad}_*(e) = \mathrm{ad}$

def. <u>exp mapping</u> : $\exp : \mathfrak{g} \to G$

<u>functoriality</u>

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \\ \mathfrak{g}_1 \arrow[u, "\exp"] \arrow[r, "\varphi_*(e)"'] & \mathfrak{g}_2 \arrow[u, "\exp"'] \end{tikzcd}$$

<u>Symplectic</u>, <u>Poisson</u> and <u>all that</u>

(M, ω) - Symplectic $\underset{\omega\text{-nondegen.}}{\underline{\text{Manifold}}}$ $\underset{\text{closed}}{(d\omega = 0)}$

$\frac{\omega^n}{n!}$ - symplectic / Liouville volume

de Rham: $[\omega^n] \in H^{2n}(\underset{\text{compact}}{M}, \mathbb{R}) \neq 0$

$S^{2n}, n > 1$ - never symplectic, $H^{2k}(M, \mathbb{R})$ - trivial

$H : M \to \mathbb{R}$, by nondegen., $\exists!\ X_H$ on M

$$i_{X_H}\omega = dH \quad \left(dH(Y) = \omega(X_H, Y) \right)$$

Each diffeo. $\rho_t : M \to M$ preserves ω

$$\rho_t{}^*\omega = \omega$$

$$\frac{d}{dt}\rho_t{}^*\omega = \rho_t{}^* \mathcal{L}_{X_H}\omega = \rho_t{}^*(d i_{X_H}\omega + i_{X_H} d\omega) = 0$$

Every function on (M, ω) produces family symplectomorphisms

Def. Vector field X_H s.t. $\boxed{i_{X_H}\omega = dH}$ for some $H \in C^\infty(M)$ is Hamilt. vector field with Hamilt function H

Hamilt. vector field preserve their hamilt function

$$\mathcal{L}_{X_H} H = i_{X_H} dH = i_{X_H} i_{X_H} \omega = 0$$

$\mathbb{R}^{2n}$, $\omega_0 = \sum dx_j \wedge dy_j$,

symplectic gradient

$$X_H = \sum \frac{\partial H}{\partial y_j}\frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j}\frac{\partial}{\partial y_j}$$

Eucl gradient

$$\nabla H = \sum_j \left(\frac{\partial H}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial H}{\partial y_j}\frac{\partial}{\partial y_j}\right)$$

$$\boxed{J X_H = \nabla H}$$, J - almost complex structure

Def. Vector field X on M preserving ω (s.t. $\mathcal{L}_X\omega = 0$) is a symplectic vector field

X symplectic when $i_X\omega$ is closed

X hamiltonian when $i_X\omega$ is exact!

primitive H of $i_X\omega$ - Hamiltonian function on X

$H^1_{\text{de Rham}}(M)$ - obstruction for symplectic be hamiltonian

X - diff. operator on functions:

$$X \cdot f = L_X f = df(X), \quad f \in C^\infty(M)$$

<u>Prop.</u> If X, Y are symplectic vector fields on sympl. manifold (M, ω) then $[X, Y]$ is hamiltonian vector field with hamiltonian function $\omega(Y, X)$

<u>Corr.</u> Hamilt. vector field and symplectic vector fields form Lie subalgebras for Lie bracket $[\,,\,]$.

<u>Def.</u> Poisson bracket of $f, g \in C^\infty(M)$ is function $\{f, g\} = \omega(X_f, X_g) = L_{X_g} f$

properties: $X_{\{f,g\}} = -[X_f, X_g]$

Jacobi ident.: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

Leibnitz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$

<u>Def.</u> <u>Poisson algebra</u> $(P, \{\cdot, \cdot\})$ is commutative associative algebra P with a Lie bracket $\{\cdot, \cdot\}$ satisfying Leibnitz rule

Poisson Manifolds

$\boxed{\text{Poisson structure}}$ on M : bilinear map

$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$

properties:

skew-symm. $\{f,g\} = -\{g,f\}$

Jacobi : $\{f,\{g,h\}\} + \dots = 0$

Deriv. of $C^\infty(M)$ in first argument: $\{fg,h\} = f\{g,h\} + g\{f,h\}$

PB $\{f,g\}$ depends only on differentials df, dg.

Any Poisson structure gives a <u>map</u> from cotangent bundle to tangent bundle sending df to X_f.

<u>Map</u>: bivector field η on M, Poisson bivector, $\eta \in \Lambda^2 TM$, s.t.

$$\{f,g\} = \langle df \otimes dg, \eta \rangle$$

This bracket satisfy the Jacobi identity and defines Poisson structure iff <u>Schouten–Nijenhuis</u> <u>bracket</u> <u>$[\eta,\eta]$</u> is zero.

Locally: $\eta_x = \sum_{i,j=1}^{m} \eta_{ij}(x) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}$ $\Longrightarrow$

$\{f,g\}(x) = \sum_{i,j=1}^{m} \eta_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$

<u>PM</u> may be singular!

$SM \subset PM$

Canonical Symplectic form or Lie-Poisson or Kirillov-Kostant-Souriau-Berezin symplectic structure (Geometric quantization) on Coadjoint orbits

Construct $\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle$

ω - nondegen. 2-form on the tangent spaces to orbits of coadjoint action

Closed, if $X^\# \Longrightarrow$ Jacobi $\Longrightarrow d\omega = 0$

Poisson structure on $\mathfrak{g}^*$

$$\{f,g\}(\xi) = \langle \xi, [df_\xi, dg_\xi] \rangle$$

$f, g \in C^\infty(\mathfrak{g}^*)$, $\xi \in \mathfrak{g}^*$.

$df_\xi : T_\xi \mathfrak{g}^* \cong \mathfrak{g}^* \to \mathbb{R}$

identif. with element of $\mathfrak{g} \cong \mathfrak{g}^{**}$

Objects Manifold. M, Symmetry: G

$\mathfrak{g}$: Lie alg of G, $\mathfrak{g}^*$: dual / co algebra

$\langle , \rangle$ - pairing

$\mathfrak{g} \sim T_e G$, $\mathfrak{g}^* \sim T_e^* G$

$H = H(g, p) \quad T^*G \to \mathbb{R}$

right invar. $H(gg^{-1}, pg^{-1}) = H(e, \mu)$, $\mu \in T_e^* G$

$$\boxed{h(\mu) \text{ on } \mathfrak{g}^*}$$

$L = (g, \dot{g}) : TG \to \mathbb{R}$

right invar. $L(gg^{-1}, \dot{g}g^{-1}) = L(e, \xi) = \ell(\xi)$, $\xi \in T_e G$

$$\boxed{\ell(\xi) \text{ on } \mathfrak{g}}$$

Lie - Poisson, Euler - Poincaré

Def. Lie - Poisson Hamiltonian System on $\mathfrak{g}^*$: $\mu \in \mathfrak{g}^*$, F, G - functionals of μ,

$\frac{\delta F}{\delta \mu}$ - functional derivative: $F: \mathfrak{g}^* \to \mathbb{R}$

$\frac{\delta F}{\delta \mu} \in \mathfrak{g}$: $DF(\mu) \cdot \nu = \langle \nu, \frac{\delta F}{\delta \mu} \rangle$, $\forall \nu \in \mathfrak{g}^*$

$\mathfrak{g}^{**} \sim \mathfrak{g}$, $DF(\mu) \in \mathfrak{g}^{**} \sim \mathfrak{g}$

PB

$$\boxed{\{F, G\}(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right\rangle}$$

$[\,,\,]$ - Lie bracket , $\langle\cdot,\cdot\rangle$ - pairing

right invar. $+$ (Berezin, Kirillov, Kostant,
left invar. $-$ Souriau)

Def. Lie - Poisson equations

$$\boxed{\frac{\partial \mu}{\partial t} + ad^*_{\frac{\delta H}{\delta \mu}} \mu = 0}$$

ad^* - coadjoint operator

$$\langle ad^*_\eta \nu, \xi \rangle \overset{\text{def}}{=} \langle \nu, ad_\eta \xi \rangle = \langle \nu, [\eta, \xi] \rangle$$

$\nu \in \mathfrak{g}^*$, $\eta, \xi \in \mathfrak{g}$

Non-singular Hamiltonian (invert Legendre transf.)

Lagrangian $L(\xi)$, $\xi \in \mathfrak{g}$

$$L(\xi) = \langle \mu, \xi \rangle - H(\mu)$$

Euler - Poincare Equations

$$\boxed{\frac{\partial}{\partial t} \frac{\partial L}{\partial \xi} + ad^*_\xi \frac{\partial L}{\partial \xi} = 0}$$

M, $G = \mathcal{D}iff(M)$, $\mathcal{L}G = Lie\{\mathcal{D}iff(M)\}$ — vector fields

Example $M \sim \mathbb{R}^n$, X, Y - vector fields

Jacobi - Lie brackets

$$[X, Y] = (X \cdot \nabla) Y - (Y \cdot \nabla) X$$

Variational principles

$\boxed{\text{Euler-Poincaré}}$ $\delta \int_{t_1}^{t_2} \mathcal{L}(\xi)\, dt = 0$

$\boxed{\text{Lie-Poisson}}$ $\delta \int_{t_1}^{t_2} \left(\langle \mu, \xi \rangle - H(\mu) \right) dt = 0$

$$\delta \xi = \dot{\eta} - [\xi, \eta], \quad \eta(t_1) = \eta(t_2) = 0$$

Coadjoint orbits

$$\mu(t) = Ad^*_{g^{-1}(t)}\, \mu(0)$$

$$g(t) = \exp\left(t \frac{\delta H}{\delta \mu}\right)$$

$$Ad^* : G \times g^* \mapsto g^*$$

$\boxed{\text{Coadjoint group action}}$ on Lie algebra

Ad^* - dual to Ad group action

$$Ad_g \xi = \frac{d}{d\tau}\Big|_{\cancel{t}=0} g \circ e^{\tau \xi} \circ g^{-1}, \quad \forall g \in G, \ \xi \in \mathfrak{g}$$

s.t. : $\langle \mu, Ad_g \xi \rangle = \langle Ad_g^* \mu, \xi \rangle$

$\boxed{\text{Coadjoint motion}}$ on

$\boxed{\text{Coadjoint orbits}}$

Coadjoint orbit $O(\mu)$, $\mu \in \mathfrak{g}^*$, $O(\mu) \subset \mathfrak{g}^*$

$$O(\mu) = G \cdot \mu = \{ Ad^*_{g^{-1}} \mu : g \in G \}$$

$$\mu(t) = Ad^*_{g^{-1}(t)} \mu(0)$$

pairing with $\eta \in \mathfrak{g}$

$$\langle \mu(t), \eta \rangle =$$

$$= \langle Ad^*_{g^{-1}(t)} \mu(0), \eta \rangle =$$

$$= \langle \mu(0), Ad_{g^{-1}(t)} \eta \rangle$$

where $Ad_{g^{-1}(t)} \eta = \frac{d}{d\tau}\Big|_{t=0} e^{-t \frac{\delta H}{\delta \mu}} \circ e^{\tau \eta} \circ e^{t \frac{\delta H}{\delta \mu}}$

$\frac{d}{dt}\Big|_{t=0}$

$$\langle \dot{\mu}(0), \eta \rangle = \langle \mu(0), \frac{d}{dt} Ad_{\exp(t \frac{\delta H}{\delta \mu})} \eta \Big|_{t=0} \rangle =$$

$$= - \langle \mu(0), ad_{\frac{\delta H}{\delta \mu}} \eta \rangle = - \langle ad^*_{\frac{\delta H}{\delta \mu}} \mu(0), \eta \rangle$$

where $ad_\xi \eta = \frac{d}{dt} Ad_{\exp(t\xi)} \eta \Big|_{t=0}$

System undergoing coadjoint orbits is a Lie - Poisson system (LP)

Trajectory LP starts on $\mathcal{O}$, then it stays in $\mathcal{O}$

Geometry of Lie Group generates dynamics

Lie derivative : invar. operator

Lie deriv. of a tensor field $T(q)$, $q \in \mathcal{O}$, along a vector field $X(q)$ on $\mathcal{O}$ is infinitesimal generator of group of diffeo. acting on $\mathcal{O}$

$$\mathcal{L}_X T = \xi_{T(G)} T = \frac{d}{dt}\Big|_{t=0} (e^{tX})^* T$$

If $T = \mathcal{Y} \in \mathrm{Vect}\{X\}$

$$g^{*}\mathcal{Y} = Ad_{g^{-1}}\mathcal{Y} \quad \text{(pull-back)}$$

$$\underline{\mathcal{L}_X \mathcal{Y}} = \frac{d}{dt}\Big|_{t=0} Ad_{\exp(-tX)}\mathcal{Y} =$$

$$= ad_{-X}\mathcal{Y} = \underline{[X, \mathcal{Y}]_{JL}}$$

Momentum map

$\underline{P}$ – Poisson manifold $(P, \{F(P)\}, \{F,G\}_{PB})$

$\underline{G}$ – Lie group, acts on P by Poisson maps $\{$preserved PB$\}$

Momentum map:

$J : \underline{\underline{P}} \to g^{*}$, such that

$$\{F(p), \langle J(p), \xi\rangle\} = \xi_{\underline{p}}[F(\underline{\underline{p}})],$$

$$\forall F \in \mathcal{F}(P), \ \forall \xi \in g$$

ξ_P, vector field, infinitesimal generator

$$\xi_P(p) = \frac{d}{dt}\Big|_{t=0} e^{t\xi}.p, \quad \forall p \in \underline{\underline{P}}$$

Liouville equation

$\boxed{\text{Lie - Poisson eq.}}$ on group of symplectomorphisms (canonical transf.) of N-particle phase space

$T^*Q \times \ldots T^*Q$

$\boxed{\text{Lie alg}\{\text{Sympl.}\}} = \{X_H\}$ - Hamiltonian vector fields ~ Isom. ~ $\boxed{\text{Poisson algebra of generating functions}}$

$H \in \mathcal{F}$ $\quad X_H(f) = \{f, H\}$

$\boxed{\text{Dual Lie algebra}}$ Isom $\boxed{\text{Space of densities}}$ distributions, $\rho \in \mathcal{F}^*$

Lie - Poisson Dynamics

$$\partial_t \rho + \{\rho, H\}_{PB} = 0 \quad , \rho(z_1, \ldots, z_n), \; z_i = (q_i, p_i)$$

BBGKY Hierarchy

$$f_n(z_1, \ldots, z_n) = \int \rho(z_1, \ldots, z_N)\, dz_{n+1} \ldots dz_N$$

$\boxed{\text{BBGKY moments}}$. Momentum maps

Lie - Poisson system (Marsden, Morrison, Weinstein)

Vlasov Equation

$f = f_1$ (first-order BBGKY moment)

$$\frac{\partial f}{\partial t} + \{f, H\} = 0 \quad \left(\left(\frac{\partial f}{\partial t}\right)_{coll}\right)$$

Lie-Poisson structure on Group canonical transf. of single-particle phasespace T^*Q.

$$\{F, G\}_V [f] = \iint f(q,p,t) \left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dq\,dp$$

(Vlasov) Moments : Chapman, Enskog

I) Statistical
II) kinetic
Moments

I). $$M_{n,\hat{n}}(t) = \iint p^n q^{\hat{n}} f(q,p,t)\, dq\,dp$$

Lie-Poisson

$$\{F, G\} = \sum_{\hat{m},m=0}^{\infty} \sum_{\hat{n},n=0}^{\infty} \left[\frac{\partial F}{\partial M_{\hat{m},m}} (\hat{m} m - \hat{n} n) \frac{\partial G}{\partial M_{\hat{n},n}}\right]$$

$$\times M_{\hat{m}+\hat{n}-1,\, m+n-1}$$

e.g. : $\underset{\text{emittance}}{\varepsilon^2} = M_{0,2} M_{2,0} - M_{1,1}^2$

II) (Vlasov) Moments : Kinetic Moments

$$A_n(q,p) = \int p^n f(q,p,t)\, dp$$

Kupershmidt - Manin : Benney long waves

Gibbons : Lie - Poisson from Vlasov

$$\{F,G\} = -\int A_{m+n-1} \left[n \frac{\delta F}{\delta A_n} \frac{\partial}{\partial q} \frac{\delta G}{\delta A_m} - m \frac{\delta G}{\delta A_m} \frac{\partial}{\partial q} \frac{\delta F}{\delta A_n} \right] dq$$

Again: <u>Vlasov - Lie - Poisson</u>

Hamilt. system on Group of canonical transf. of phase space T^*Q of config. space Q.

Dynamics of Lie - Poisson on $\mathfrak{g}^*$ ($\mathfrak{g}$ - Lie alg of G).

$$G = Can(T^*Q), \quad \mathfrak{g} = \mathcal{X}_{can}(T^*Q)$$

Hamilt. Vect. fields

$$\mathcal{H}_{can.} \cong \mathcal{F}\{f(p,q)\} \quad \infty\text{-dim}$$

$$f \in \mathcal{F}^* \cong Den$$

<u>VLP</u> $$\{F,H\}[f] = \iint f(q,p) \left\{ \frac{\delta F}{\delta f}, \frac{\delta H}{\delta f} \right\} d^k q \, d^k p$$

$k = \dim Q$

$$F = f, \; h = \delta H / \delta f \implies \frac{\partial f}{\partial t} + \{f, h\} = 0$$

$$h = \frac{|p|^2}{2} + V(q)$$

<u>VP</u> $$H[f] = \iint f(q,p) \left(\frac{1}{2}|p|^2 + \Delta^{-1} \int f(q,p')\, d^k p' \right) d^k q \, d^k p$$

Δ^{-1} - Green function, convol.

Again, Kinetic moments : Kupershmidt–Manin

Def.

$$A_n(q,t) \overset{\text{def}}{=} \int_{T_q^*Q} (p\,dq)^n f(q,p,t) \underbrace{d^K q \wedge d^K p}_{d\mu^K} =$$

$$= \sum_{i_1 \ldots i_n = 1}^{K} \int_{T_q^*Q} p_{i_1} \ldots p_{i_n} dq^{i_1} \otimes \ldots \otimes dq^{i_n} f(q,p,t)\, d\mu^K =$$

$$= \sum_{i_1 \ldots i_n = 1}^{K} \big(A_n(q,t)\big)_{i_1 \ldots i_n} dq^{i_1} \otimes \ldots \otimes dq^{i_n} \otimes d^K q$$

Moment mapping: $p\,dq$ – canonical 1-form, $d^K q$ – measure on Q

So, it is $\boxed{\text{Proj}}$ of Vlasov distrib. onto the space of symmetric tensors

Chain rule: $$\frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \frac{\delta A_n}{\delta f} \frac{\delta F}{\delta A_n} = \sum_{n=0}^{\infty} p^n \frac{\delta F}{\delta A_n}$$

$$p^\sigma := p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3}$$

$$A_n(q,t) = \int p^n f(q,p,t)\, d^3 p \quad , \quad p^n = p^{\otimes n}$$

Moments $\simeq$ symmetric covariant tensor densities (volume elements)

Moments – functionals of Vlasov density f

operation: $\lrcorner$ contraction

$$\frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \frac{\delta F}{\delta A_n} \lrcorner \frac{\delta A_n}{\delta f} \overset{def}{=}$$

$$\sum_{n=0}^{\infty} \sum_{i_1 \ldots i_n = 1}^{K} \frac{\delta (A_n)_{i_1 \ldots i_n}}{\delta f} \frac{\delta F}{\delta (A_n)_{i_1 \ldots i_n}} =$$

$$= \sum_{n=0}^{\infty} \sum_{i_1 \ldots i_n = 1}^{K} p_{i_1} \ldots p_{i_n} \frac{\delta F}{\delta (A_n)_{i_1 \ldots i_n}} \overset{def}{=}$$

$$= \sum_{n=0}^{\infty} \frac{\delta F}{\delta A_n} \lrcorner p^n \quad , \quad \lrcorner \text{ - contraction}$$

Lie algebra Hom. (Isom) Symmetric Tensors $\rightarrow$ polynomials

Dual: Momentum map assoc. to moments

Insert chain rule into Vlasov bracket $\rightarrow$ Lie-Poisson bracket for moments

$$\{F, G\}[A] = -\sum_{n,m=0}^{\infty} \int A_{m+n-1}(q) \lrcorner \left[\frac{\delta F}{\delta A_n}, \frac{\delta G}{\delta A_m}\right] dq^3$$

where (Schouten concomitant)

$$\boxed{\left[\frac{\delta F}{\delta A_n}, \frac{\delta G}{\delta A_m}\right] = S\left(n\left(\frac{\delta F}{\delta A_n} \cdot \nabla\right) \otimes \frac{\delta G}{\delta A_m} - m\left(\frac{\delta G}{\delta A_m} \cdot \nabla\right) \otimes \frac{\delta F}{\delta A_n}\right)}$$

inherited from canonical PB

where $(A \cdot B)_{ij\ldots}^{kl\ldots} = A_{ij\ldots k} B^{kkl\ldots}$

$B \cdot A = (B \cdot A)_{jl\ldots}^{km\ldots} = B^{km\ldots i} A_{ijl}$

one index contraction between co(var)- and contra - variant tensors

This bracket - $\boxed{\text{Schouten concomitant}}$ or symmetric $\boxed{\text{Schouten bracket}}$

Kuperschmidt - Manin bracket for Benney

Lebedev, Gibbons

Relation to Bosonic Fock space

Moment algebra $= \{$ symmetric contra-variant tensor fields $\} \sim$ Fock (state) space represented by direct sum of symmetric powers of vector fields given by

Def. $\mathcal{J} \stackrel{def}{=} \bigoplus_{n=0}^{\infty} \left(\bigvee_{i=0}^{n} X(Q) \right)$ where

$\bigvee_{i=0}^{n} X \stackrel{def}{=} S\left(\bigotimes_{i=0}^{n} X \right) \stackrel{def}{=} \mathcal{J}_n$

Universal Enveloping Algebra of Diffeo group, Diff. Q ~ Enveloping algebra $\mathcal{U}(X)$ of vector fields $X(Q)$ on config. space Q.

Graded structure of enveloping algebra possesses PB structure

Reminder $U(g)$ - enveloping algebra for algebra $\mathfrak{g}$

$U(\mathfrak{g})$ - factor algebra of tensor algebra $T(g)$ on ideal J generated by $X \otimes Y - Y \otimes X - [X, Y]$, $X, Y \in \mathfrak{g}$

$T(\mathfrak{g}) = \bigoplus_{l=0}^{\infty} T^l(\mathfrak{g})$

$\psi : \mathfrak{g} \to U(\mathfrak{g})$ universal object in category of Mor ψ of Lie algebra $\mathfrak{g}$ into associative algebras with property $\psi([X, Y]) = \psi(X)\psi(Y) - \psi(Y)\psi(X)$

Filtrations in $U(\mathfrak{g})$ from grading in $T(\mathfrak{g})$

$$U_K(\mathfrak{g}) = \bigoplus_{l=0}^{K} T^l(\mathfrak{g}) \bmod J$$

PBW Construction ; Filtered Algebra $\to$ associated graded algebra

Schouten concomitants:

Kinetic Moment Algebra ~ Lie algebra of symbols of DO (ΨDO)

Diff. smls $\subset$ Ψ diff. smbls

No underlying Lie str.

$\Psi\mathcal{D}$ – well defined Poisson-Lie group

$\Downarrow$

Coadj. orbits for momentum dynamics requires complete Lie-Poisson group structure of $\Psi\mathcal{D}O$

Wick-ordered product, creation-annih. op., ...

Moment algebra

Graded structure, $\mathfrak{g} = \oplus_i \mathfrak{g}_i$

(****) with filtration $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m-1}$

$\mathfrak{g}_1 = \mathcal{X}$ – subalgebra of Vect fields

Largest subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathcal{X} \circledS \mathcal{F}$

ideal compressible fluids

fluid velocity density

Geodesic motion.

Moment equation as Lie - Poisson

$$\frac{\partial A_m}{\partial t} = - \sum_{m,n=0}^{\infty} ad^*_{\frac{\delta H}{\delta A_n}} A_{n+m-1} = \{A_m, H\}$$

ad^* : $\displaystyle\sum_{k,n=0}^{\infty} \langle ad^*_{\beta_n} A_k, \alpha_{k-n+1} \rangle \overset{def}{=}$

$\langle , \rangle$

$$\sum_{k,n=0}^{\infty} \langle A_k, [\beta_n, \alpha_{k-n+1}] \rangle$$

Schouten brackets : $[\beta_n, \alpha_m] = ad_{\beta_n} \alpha_m$

dim = 1 : $[\alpha_m, \beta_n] = m \alpha_n \partial_q \beta_n - n \beta_n \partial_q \alpha_m$

covar. tensor density , rank $k-n+1$

$$ad^*_{\beta_n} A_k = (k+1) A_k \partial_q \beta_n + n \beta_n \partial_q A_k$$

Kirillov, 1982

Lie - Poisson brackets on Symmetric Schouten algebra

$n = 1, k = 2$: fluid dynamics

Statistical Moments and their Lie-Poisson

Kinetic moment hierarchy requires kinetic equation on cotangent bundle, Statistical moments on Symplectic vector space

$$z = (q, p), \quad X^n(t) := \int z^n f(z,t)\, d^N z$$
$$\overline{\text{2dark}}$$
$$z^n \equiv \otimes^n z$$

Unified Framework

Stat. moments : Symmetric covar. tensors on phase space = symplectic vector space V^N, $N = 2K$,

$z = z^i e_i \in V$

Moment PB : insert chain rule into Vlasov structure

$$\frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \left(\frac{\partial F}{\partial X^n}\right)_{i_1 \ldots i_n} \left(\frac{\delta X^n}{\delta f}\right)^{i_1 \ldots i_n} \stackrel{def}{=}$$

$$= \sum_{n=0}^{\infty} \frac{\partial F}{\partial X^n} \lrcorner \frac{\delta X^n}{\delta f} = \sum_{n=0}^{\infty} \frac{\partial F}{\partial X^n} \lrcorner z^n$$

Vlasov structure

$$\{F, G\}[f] = \iint f(z) \left\{ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right\} d^N z =$$

$$= \iint f(z) \left[J \lrcorner \left(\frac{\partial}{\partial z} \frac{\delta F}{\delta f} \otimes \frac{\partial}{\partial z} \frac{\delta G}{\delta f} \right) \right] d^N z$$

J – non-degen. 2-form ($N \times N$ – antisymm. matrix, max rank)

$\Downarrow$ Chain rule

$$\frac{\partial}{\partial z} \frac{\delta F}{\delta f} = \sum_n n \frac{\partial F}{\partial X^n} \lrcorner z^{n-1} \Longrightarrow$$

Moment Poisson structure is

$$\{F, G\}(X) = \sum_{n,m=0}^{\infty} X^{n+m-2} \lrcorner \left[\frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right] \stackrel{def}{=}$$

$$\sum_{n,m=0}^{\infty} \left\langle X^{n+m-2}, \left[\frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right] \right\rangle, \text{ where}$$

$$(*)(*) \quad \boxed{\left[\frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m} \right] := nm\, S\left(\frac{\partial F}{\partial X^n} \cdot J \cdot \frac{\partial G}{\partial X^m} \right)}$$

$J^{ij} = -J^{ji}$ | contravar., antisymm.

Moment Lie bracket

Isom. between symmetric tensors and polynomials produces momentum map associated with moments.

Lie-Poisson brackets for statistical moments is inherited from Vlasov Lie-Poisson structure.

In contrast to the Schouten concomitant (p.22)
(*) for kinetic moments, Lie bracket for statistical moments (**) still involves symplectic matrix J (Poisson tensor). Remember Poisson Manifold! Bivector!

So, Dynamics of Statistical Moments depends explicitly on original symplectic structure.

$\Longrightarrow$ Casimirs

(**) involves symmetric tensors – covariant
(*) Schouten concomitant – contravariant

As for kinetic moments, Lie algebra of statistical moments, by grading

$$\mathfrak{g} := \bigoplus_{n=0}^{\infty}\left(\bigvee_{i=0}^{n} V^*\right) =: \bigoplus_{i=0}^{\infty} \mathfrak{g}_i \qquad (***)$$

with filtration

$$[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m-2}$$

Symm. matrices $\mathfrak{g}_2 = V^* \vee V^* = Sym^*(N)$ — (

subalgebra ($Sym^*(N)$) — covar. symm. matrices)

<u>Largest subalgebra</u>: $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathbb{R} \oplus V^* \oplus Sym^*(N)$

Analogously to <u>Lie algebra of statistical moments</u>, <u>kinetic moments</u>, carry a bosonic <u>Fock space structure</u>.

<u>Moment equations</u> via <u>coadjoint operator</u>, ad^*

$$\frac{dX^m}{dt} = -\sum_{n=0}^{\infty} ad^*_{\frac{\partial H}{\partial X^n}} X^{n+m-2} =$$

$$= -m \sum_{n=0}^{\infty} n\, S\left(\left(\frac{\partial H}{\partial X^n}\cdot J\right) \lrcorner X^{m+n-2}\right)$$

$$(\dot{X}^m)^{i_1 \ldots i_m} = -m \sum_{n=0}^{\infty} n\, S\left(\left(\frac{\partial H}{\partial X^n}\right)_{j_1 \ldots j_{n-1} k} J^{k i_m} \left(X^{m+n-2}\right)^{i_1 \ldots i_{m-1}\, j_1 \ldots j_{n-1}}\right)$$

Truncation of Moment Hierarchies

Channel : Levi decomp. theorem

For a Moment Hamiltonian not depending on the first-order moment $\left(\frac{\partial H}{\partial X^1}=0\right)$ the moment hierarchy can be truncated at any order
$\Longrightarrow$ truncated Lie-Poisson.

So, truncated equations at order K for hierarchy of statistical moments

$$\dot{X}^n = ad^*_{h_m} X^{m+n-2}, \quad h_m = \frac{\partial H}{\partial X^m}$$

$$\dot{X}^1 = ad^*_{h_2} X^1 + ad^*_{h_3} X^2 + \ldots + ad^*_{h_K} X^{K-1}$$

$$\dot{X}^2 = ad^*_{h_2} X^2 + ad^*_{h_3} X^3 + \ldots + ad^*_{h_K} X^K$$

$$\dot{X}^3 = ad^*_{h_2} X^3 + \ldots + ad^*_{h_{K-1}} X^K$$

$$\vdots$$

$$\dot{X}^{K-1} = ad^*_{h_2} X^{K-1} + ad^*_{h_3} X^K$$

$$\dot{X}^K = ad^*_{h_2} X^K$$

Eq. for X^1 – decoupled, eqs. for: $X^2 \ldots X^K$.

One verifies that Lie – Poisson bracket for truncated moment system is given by

$$\{F,G\}(X) = \sum_{n=2}^{K} \sum_{m=2}^{K-n+2} \left\langle X^{n+m-2}, \left[\frac{\partial F}{\partial X^n}, \frac{\partial G}{\partial X^m}\right]\right\rangle$$

(~~**~~) notations

Truncated structure is completely determined by Lie algebra filtrations (***) and does not depend on the particular expression of Lie bracket, which was not used in deriving truncated system.

For kinetic moments:

$$\partial_t A_0 = -ad^*_{h_1} A^0 - ad^*_{h_2} A_1 - \ldots - ad^*_{h_K} A_{K-1}$$

$$\partial_t A_1 = -ad^*_{h_1} A_1 - ad^*_{h_2} A_2 - \ldots - ad^*_{h_K} A_K$$

$$\partial_t A_2 = -ad^*_{h_1} A_2 - \ldots - ad^*_{h_{K-1}} A_K$$

$$\ldots$$

$$\partial_t A_{K-1} = -ad^*_{h_1} A_{K-1} - ad^*_{h_2} A_K$$

$$\partial_t A_K = -ad^*_{h_1} A_K$$

where $h_m = \delta H / \delta A_m$, $h_0 \equiv 0$

Zero-th moment eq. decouples: remaining $A_1, \ldots, A_K$.

Brackets for these equations

$$\{F, G\}(A) = -\sum_{n=1}^{K} \sum_{m=1}^{K-n+1} \left\langle A_{n+m-1}, \left[\frac{\delta F}{\delta A_n}, \frac{\delta G}{\delta A_m}\right]\right\rangle \quad (*)$$

Truncated structure is uniquely determined by Lie algebra filtration (****).

Similar approach to BBGKY moments : <u>FZ : Fusion</u>

<u>Filtrations</u>

<u>Geodesic Vlasov Equation</u>

Moment hier. equiv. description of Vlasov eq. : <u>geometrical closures</u> of kinetic system (ideal fluid closure for kinetic moments).

Vlasov : Lie-Poisson on Lie algebra of group $Can(T^*Q)$

Lie-Poisson structure of reflected in moment dynamics.

<u>Folklor</u>: Physical system with geometrical behaviour — <u>geodesic flow</u> on <u>Lie groups</u> wrt metric ~ kinetic energy

Rigid body motions, geodesic motions on $SO(3)$.

Euler eq. for ideal fluids — geodesic motion on volume-preserving Diffeo. $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^3)$, 3D flow domain,

Arnold, 1965/6. $[L, A]$ - pair!

Euler-Poincare Diff. Eq.: geodesic motion on the full Diffeo. Group $\mathrm{Diff}(\mathbb{R}^n)$.

Completely integrable!

Camassa-Holm eq: $\mathbb{R}$, E.-P.

Symplectic Hydrodynamic: Arnold-Khesin

Geodesic Vlasov Equation $\equiv$

Euler-Poincare on canonical transf.

Motive: geodesic equations for kinetic moments include CH

Geodesic moment eq. recover 2-component CH.

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

$$L(u) = \frac{1}{2}\int u(1-\partial_x^2)u\,dx$$

$$m = \frac{\delta L}{\delta u} = u - u_{xx}, \quad u = (1-\partial_x^2)^{-1} m$$

$$H(m) = \frac{1}{2}\int m\,(1-\partial_x)^{-1} m\,dx$$

Lie-Poisson: $m_t = -\mathcal{L}_u m = -u m_x - 2 m u_x$

<u>Symplectic Manifold</u> P, $N = 2K$, $z \in P$

<u>Euler-Poincaré Vlasov Hamiltonian</u>

$$H[f] = \frac{1}{2}\iint f(z)\, G(z, z')\, f(z')\, d^N z\, d^N z' \equiv$$

$$\frac{1}{2}\|f\|_G^2 \quad \text{on } \mathrm{Den}(P).$$

ker G defined appr. norm

Moments: $P = T^*Q$ or $P = V$ (symplectic)

<u>Geodesic Vlasov</u>

$$\frac{\partial f}{\partial t} = -\left\{f, \frac{\partial H}{\partial f}\right\} = -\{f, G * f\}$$

$\simeq$ Euler's vorticity equation in 2D when $G = (-\Delta)^{-1}$

Important remark:

Taking moments (BBGKY, kinetic, statistical) of Lie-Poisson kinetic equation is always <u>momentum map</u>.

Moreover, closures adopted to obtain Vlasov from BBGKY, fluid theory from kinetic moments and beam optics from statistical moments are also momentum maps arising from particular subgroup of symmetry group of initial system.

$$\boxed{\text{All moment approx. in kinetic theory are momentum maps}}$$

Marsden-Weinstein, Arnold, Holm.

Example

Moments constitute momentum map (p. 16)

Vlasov eq. : $\frac{\partial f}{\partial t} + \mathcal{L}_{X_H} f = 0$

$$H = \frac{\delta \mathcal{H}}{\delta f}$$

$$\frac{\delta F}{\delta f} = \sum_{n=0}^{\infty} \frac{\delta A_n}{\delta f} \frac{\delta F}{\delta A_n} = \sum_{n=0}^{\infty} p^n \frac{\delta F}{\delta A_n}$$

$\forall F(f)$

$$\boxed{\frac{\partial f}{\partial t} + \left\{ f, \sum_n p^n \frac{\delta H}{\delta A_n} \right\} = 0}$$

determined by symplectic Lie algebra action on its dual

Isom. : polynomials ~ symm. tensors

$\{,\}$: arising from moment Lie algebra action on phase space densities, so that

$$\beta \cdot f = -\{ f, p^n \beta_n \}$$

β – whole sequence of symmetric ~~covariant~~ contravariant tensor fields, $\beta := \{\beta_n\}_{n \in \mathbb{N}}$

Define map $J : \mathcal{F}^*(T^*Q) \to \mathfrak{g}^*$

($\mathfrak{g}$ – Lie algebra of contravar. tensor fields)

J : {Vlasov distributions} $\to$ {moments}

Direct verification shows that this is a momentum map

$$\{ F(f), \langle J(f), \beta \rangle \}_V =$$

$$= \iint f(q,p) \left\{ \frac{\delta F}{\delta f}, \frac{\delta}{\delta f} \int \left(\int p^n f(q,p)\, dp \right) \beta_n(q)\, dq \right\} dq\, dp =$$

$$= \iint f(q,p) \left\{ \frac{\delta F}{\delta f}, p^n \beta_n(q) \right\} dq\, dp =$$

$$= -\iint \{ f(q,p), p^n \beta_n(q) \} \frac{\delta F}{\delta f}\, dq\, dp =$$

$$= \beta_P [F(f)] \qquad , \; P = \mathcal{F}^*(T^*Q)$$

So, it's invariant calculations.

Symmetry!