Multiscale Representation for Vlasov Equations

Antonina N. Fedorova   Michael G. Zeitlin

anton@math.ipme.ru, zeitlin@math.ipme.ru

Abstract

We consider the applications of a numerical-analytical approach based on multi-scale variational wavelet technique to the systems with collective type behaviour described by some forms of Vlasov-Poisson/Maxwell equations. We calculate the exact fast convergent representations for solutions in framework of multiresolution decompositions via high-localized nonlinear eigenmodes (wavelet-like base functions), which correspond to internal hidden symmetry of underlying functional spaces. It allows to control process of self-organization during evolution from chaos to localization on the pure algebraical level of generalized dispersion relations.

1   Introduction

In this paper we consider the applications of numerical-analytical approach based on multi-scale variational wavelet technique to the systems with collective type behaviour described by some forms of Vlasov-Poisson/Maxwell equations [1], [2]. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by the systems of non-linear ordinary/partial differential/integral equations with or without some (functional) constraints. In periodic accelerators and transport systems at the high beam currents and charge densities the effects of the intense self-fields, which are produced by the beam space charge and currents, determinine (possible) equilibrium states, stability and transport properties according to underlying nonlinear dynamics [2]. The dynamics of such space-charge dominated high brightness beam systems can provide the understanding of the instability phenomena such as emittance growth, mismatch, halo formation related to the complicated behaviour of underlying hidden nonlinear modes outside of perturbative tori-like KAM regions. Our analysis is based on the variational-wavelet approach [3], [4], [7]–[23], which allows us to consider polynomial and rational type of nonlinearities. In some sense our approach is direct generaliztion of traditional nonlinear δF approach in which weighted Klimontovich representation

$$\delta f_j = a_j \sum_{i=1}^{N_j} w_{ji}\delta(x - x_{ji})\delta(p - p_{ji}) \tag{1}$$

or self-similar decompostion [2] like

$$\delta n_j = b_j \sum_{i=1}^{N_j} w_{ji}s(x - x_{ji}), \tag{2}$$

where $s(x - x_{ji})$ is a shape function of distributing particles on the grids in configuration space, are replaced by powerful technique from local nonlinear harmonic analysis, based

on underlying symmetries of functional space such as affine or more general. The solution has the multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying time/phase space scales. Starting from Vlasov-Poisson equations in part 2, we consider the approach based on multiscale variational-wavelet formulation in part 3. We give the explicit representation for all dynamical variables in the base of compactly supported wavelets or nonlinear eigenmodes. Our solutions are parametrized by solutions of a number of reduced algebraical problems one from which is nonlinear with the same degree of nonlinearity as initial problem and the others are the linear problems which correspond to the particular method of calculations inside concrete wavelet scheme. Because our approach started from variational formulation we can control evolution of instability on the pure algebraical level of reduced algebraical system of equations by means of generalized dispersion relations. It allows to control stability/unstability scenario of evolution in parameter space on pure algebraical level. In all these models numerical modeling demonstrates the appearance of coherent high-localized structures and as a result we may control process of localized pattern formation or self- organization of the (meta)stable long-living fluctuations (waveletons) from standard chaotic- like behaviour.

2   Vlasov-Poisson equations

Analysis based on the non-linear Vlasov equations leads to more clear understanding of collective effects and nonlinear beam dynamics of high intensity beam propagation in periodic-focusing and uniform-focusing transport systems. We consider the following form of equations (refs. [1], [2] for setup and designation):

$$
\left\{ \frac{\partial}{\partial s} + p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} - \left[k_x(s)x + \frac{\partial \psi}{\partial x} \right] \frac{\partial}{\partial p_x} - \right.
$$

$$
\left. \left[k_y(s)y + \frac{\partial \psi}{\partial y} \right] \frac{\partial}{\partial p_y} \right\} f_b(x, y, p_x, p_y, s) = 0, \tag{3}
$$

$$
\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\frac{2\pi K_b}{N_b} \int \mathrm{d}p_x \mathrm{d}p_y f_b, \tag{4}
$$

$$
\int \mathrm{d}x \mathrm{d}y \mathrm{d}p_x \mathrm{d}p_y f_b = N_b \tag{5}
$$

The corresponding Hamiltonian for transverse single-particle motion is given by

$$
H(x, y, p_x, p_y, s) = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}[k_x(s)x^2 \tag{6}
$$

$$
+ k_y(s)y^2] + H_1(x, y, p_x, p_y, s) + \psi(x, y, s),
$$

where H_1 is nonlinear (polynomial/rational) part of the full Hamiltonian and corresponding characteristic equations are:

$$
\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + k_x(s)x + \frac{\partial}{\partial x}\psi(x, y, s) \quad = \quad 0 \tag{7}
$$

$$
\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + k_y(s)y + \frac{\partial}{\partial y}\psi(x, y, s) \quad = \quad 0 \tag{8}
$$

2

3   Multiscale representations

We obtain our multiscale/multiresolution representations for solutions of equations (3)-(8) via variational-wavelet approach. We decompose the solutions as

$$f_b(s, x, y, p_x, p_y) = \sum_{i=i_c}^{\infty} \oplus \delta^i f(s, x, y, p_x, p_y) \tag{9}$$

$$\psi(s, x, y) = \sum_{j=j_c}^{\infty} \oplus \delta^j \psi(s, x, y) \tag{10}$$

$$x(s) = \sum_{k=k_c}^{\infty} \oplus \delta^k x(s), \quad y(s) = \sum_{\ell=\ell_c}^{\infty} \oplus \delta^\ell y(s) \tag{11}$$

where set (i_c, j_c, k_c, ℓ_c) corresponds to the coarsest level of resolution (coarse graining) c in the full multiresolution decomposition [5], [6]:

$$V_c \subset V_{c+1} \subset V_{c+2} \subset \ldots \tag{12}$$

Introducing detail space W_j as the orthonormal complement of V_j with respect to V_{j+1} : $V_{j+1} = V_j \bigoplus W_j$, we have for f, ψ, x, $y \subset L^2(\mathbf{R})$ from decompositions (9)-(11) the full fine-grained representation:

$$L^2(\mathbf{R}) = \overline{V_c \bigoplus_{j=c}^{\infty} W_j}, \tag{13}$$

In some sense (9)-(11) can be considered as generalization of the old δF approach [1], [2].

Let L be an arbitrary (non) linear differential/integral operator with matrix dimension d, which acts on some set of functions $\Psi \equiv \Psi(s, x) = \left(\Psi^1(s, x), \ldots, \Psi^d(s, x) \right)$, $s, x \in \Omega \subset \mathbf{R}^{n+1}$ from $L^2(\Omega)$:

$$L\Psi \equiv L(R(s, x), s, x)\Psi(s, x) = 0, \tag{14}$$

(x are the generalized space coordinates or phase space coordinates, s is "time" coordinate). After some anzatzes [4] the main reduced problem may be formulated as the system
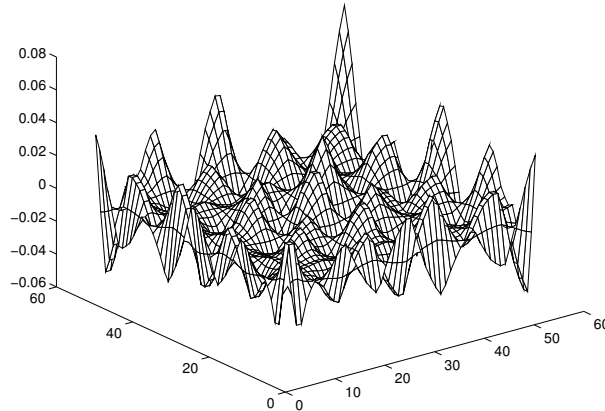

Figure 1: New Localized Modes

3

of ordinary differential equations

$$Q_i(f)\frac{\mathrm{d}f_i}{\mathrm{d}s} = P_i(f,s), \quad f = (f_1,...,f_n),$$

$$i = 1,\ldots,n, \quad \max_i deg\, P_i = p, \quad \max_i deg\, Q_i = q \tag{15}$$

or a set of such systems corresponding to each independent coordinate in phase space. They have the fixed initial (or boundary) conditions $f_i(0)$, where P_i, Q_i are not more than polynomial functions of dynamical variables f_j and have arbitrary dependence on time. As result we have the following reduced algebraical system of equations, the Generalized Dispersion Relations (GDR), on the set of unknown coefficients λ_i^k of localized eigenmode expansion (formula (17) below):

$$L(Q_{ij}, \lambda, \alpha_I) = M(P_{ij}, \lambda, \beta_J), \tag{16}$$

where operators L and M are algebraization of RHS and LHS of initial problem (15) and λ are unknowns of reduced system of algebraical equations (RSAE) (16). After solution of GDR, RSAE (16), we determine the coefficients of wavelet expansion and therefore obtain the solution of our initial problem. It should be noted that if we consider only truncated expansion with N terms then we have from (16) the system of $N \times n$ algebraical equations with degree $\ell = max\{p, q\}$ and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (rational) problem in the form

$$f_i(s) = f_i(0) + \sum_{k=1}^{N} \lambda_i^k f_k(s), \tag{17}$$

where coefficients λ_i^k are the roots of the corresponding reduced algebraical (polynomial) problem, GDR (16). Consequently, we have a parametrization of solution of initial problem by the solution of reduced algebraical problem (16). The obtained solutions are given in the form (17), where $f_k(t)$ are basis functions obtained via multiresolution expansions (9)-(11), (13) and represented by some compactly supported wavelets. As a result, the solution of equations (3)-(8) has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes, which corresponds to the full multiresolution expansion in all underlying scales (13) starting from coarsest one (polynomial tensor bases and proper norms are considered in [4]; $\mathbf{x} = (x, y, p_x, p_y)$):

$$\Psi(s, \mathbf{x}) = \sum_{(i,j) \in Z^2} a_{ij} \mathbf{U}^i \otimes V^j(s, \mathbf{x}), \tag{18}$$

$$V^j(s) = V_N^{j,slow}(s) + \sum_{l \geq N} V_l^j(\omega_l s), \quad \omega_l \sim 2^l$$

$$\mathbf{U}^i(\mathbf{x}) = \mathbf{U}_M^{i,slow}(\mathbf{x}) + \sum_{m \geq M} \mathbf{U}_m^i(k_m \mathbf{x}), \quad k_m \sim 2^m,$$

Decomposition (18) gives us expansion into the slow part $\Psi_{N,M}^{slow}$ and fast oscillating parts for arbitrary N, M. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first terms in the RHS of formulas (18) correspond on the global level of function space decomposition to resolution space and the second ones to detail space. It should be noted that such representations give the best possible localization properties in the corresponding (phase)space/time coordinates. In contrast with different approaches formulas (18) do
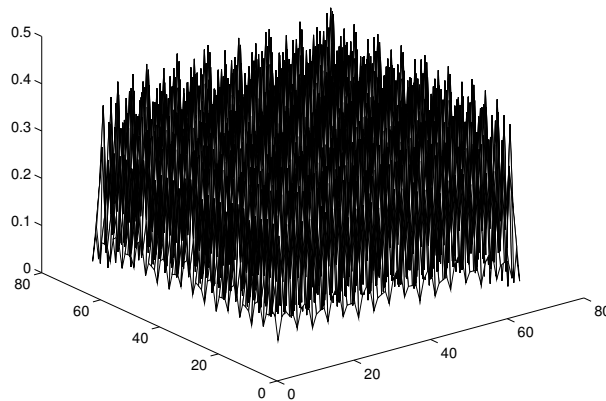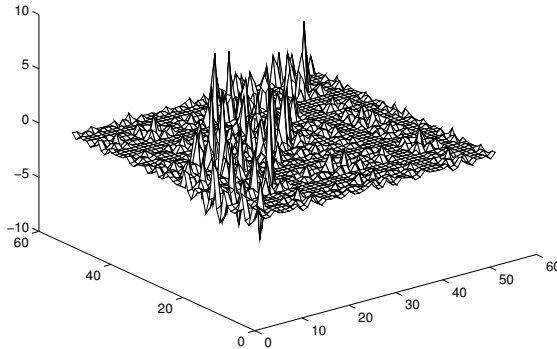
Figure 2: Chaotic-like Pattern

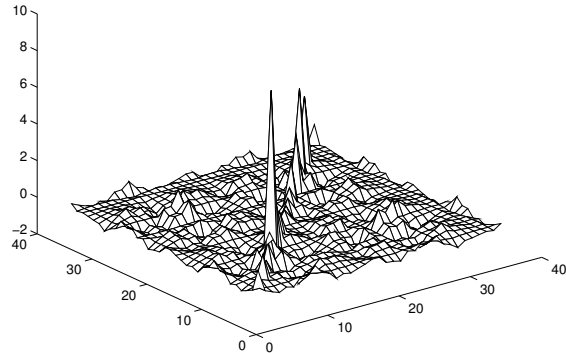Figure 3: On the Route to Self-organization:
From Chaos to Localization

Figure 4: Localized Pattern: Waveleton

not use perturbation technique or linearization procedures. So, by using wavelet bases with their good (phase) space/time localization properties we can describe high-localized (coherent) structures in spatially-extended stochastic systems with collective behaviour. Modelling demonstrates the appearance of (meta)stable localized pattern formation from high-localized coherent base structures or chaotic behaviour [3]. On Fig. 1 we present contribution to the full expansion from the coarsest level of resolution (base fundamental localized non-linear eigenmodes) of decomposition (18). Figs. 2-4 show the representation for the controllable (via GDR (16)) process of self-organization during the evolution of pattern formation: from chaotic-like pattern (Fig. 2) via partly localized pattern (Fig. 3) to our main goal, localized pattern or waveleton (Fig. 4). It is a universal model for long-living (meta)stable state/fluctuation useful in many applications, in both beam physics and plasma physics. GDR (16) provide a way for state control on the algebraic level. According to this approach, it is possible on the formal level, in principle, to control ensemble behaviour and to realize the localization of energy (confinement state) inside waveleton configurations created from a few fundamental physical modes only. Definitely, it is a good pre-image for fusion modeling in plasma.

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Antonina N. Fedorova, Michael G. Zeitlin
Mathematical Methods in Mechanics Group, IPME RAS, V.O. Bolshoj pr., 61, 199178, St. Petersburg, Russia
http://www.ipme.ru/zeitlin.html, http:/mp.ipme.ru/zeitlin.html