NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 2. ORBITAL DYNAMICS IN GENERAL MULTIPOLAR FIELD

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Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this part we consider orbital motion in transverse plane for a single particle in a circular magnetic lattice in case when we take into account multipolar expansion up to an arbitrary finite number. We reduce initial dynamical problem to the finite number (equal to the number of n-poles) of standard algebraical problem and represent all dynamical variables via an expansion in the base of periodical wavelets.

1 INTRODUCTION

This is the second part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], which is based on our approach to investigation of nonlinear problems – general, with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum, which are considered in the framework of local (nonlinear) Fourier analysis, or wavelet analysis. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In this part we consider orbital motion in transverse plane for a single particle in a circular magnetic lattice in case when we take into account multipolar expansion up to an arbitrary finite number. We reduce initial dynamical problem to the finite number (equal to the number of n-poles) of standard algebraical problem and represent all dynamical variables as expansion in the base of periodical wavelet functions. Our consideration is based on generalization of variational wavelet approach from part 1. After introducing our starting points related to multiresolution in section 3, we consider methods which allow us to construct wavelet representation for solution in periodic case in section 4.

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2 PARTICLE IN THE MULTIPOLAR FIELD

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x,y), \tag{1}$$

where f_n is a homogeneous function of x and y of order n. The real and imaginary parts of binomial expansion of

$$f_n(x,y) = (x+iy)^n \tag{2}$$

correspond to regular and skew multipoles. The cases $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decapole. Then we have in particular case the following equations of motion for single particle in a circular magnetic lattice in the transverse plane (x,y) ([9] for designation):

$$\begin{aligned} &\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right)x = \\ &\mathcal{R}e\left[\sum_{n\geq 2}\frac{k_n(s)+ij_n(s)}{n!}\cdot(x+iy)^n\right], \\ &\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} + k_1(s)y = \\ &-\mathcal{J}m\left[\sum_{n\geq}\frac{k_n(s)+ij_n(s)}{n!}\cdot(x+iy)^n\right] \end{aligned} \tag{3}$$

and the corresponding Hamiltonian:

$$\begin{aligned} &H(x,p_x,y,p_y,s) = \frac{p_x^2+p_y^2}{2} + \\ &\left(\frac{1}{\rho(s)^2} - k_1(s)\right)\cdot\frac{x^2}{2} + k_1(s)\frac{y^2}{2} \\ &-\mathcal{R}e\left[\sum_{n\geq 2}\frac{k_n(s)+ij_n(s)}{(n+1)!}\cdot(x+iy)^{(n+1)}\right] \end{aligned} \tag{4}$$

Then we may take into account arbitrary but finite number in expansion of RHS of Hamiltonian (4) and from our point of view the corresponding Hamiltonian equations of motions are not more than nonlinear ordinary differential equations with polynomial nonlinearities and variable coefficients.

3 WAVELET FRAMEWORK

Our constructions are based on multiresolution approach. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a r-regular multiresolution analysis (MRA) of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \tag{5}$$

satisfying the following properties:

$$\bigcap_{j\in\mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^n),$$
$$f(x) \in V_j <=> f(2x) \in V_{j+1},$$
$$f(x) \in V_0 <=> f(x-k) \in V_0, \ \forall k \in \mathbf{Z}^n. \tag{6}$$

There exists a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k),\ k \in \mathbf{Z}^n\}$ forms a Riesz basis for V_0. The function φ is regular and localized: φ is C^{r-1}, $\varphi^{(r-1)}$ is almost everywhere differentiable and for almost every $x \in \mathbf{R}^n$, for every integer $\alpha \le r$ and for all integer p there exists constant C_p such that

$$\mid \partial^\alpha \varphi(x) \mid \le C_p(1+|x|)^{-p} \tag{7}$$

Let $\varphi(x)$ be a scaling function, $\psi(x)$ is a wavelet function and $\varphi_i(x) = \varphi(x-i)$. Scaling relations that define φ, ψ are

$$\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x), \tag{8}$$
$$\psi(x) = \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x+k). \tag{9}$$

Let indices ℓ, j represent translation and scaling, respectively and

$$\varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - \ell) \tag{10}$$

then the set $\{\varphi_{j,k}\}, k \in \mathbf{Z}^n$ forms a Riesz basis for V_j. The wavelet function ψ is used to encode the details between two successive levels of approximation. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}:

$$V_{j+1} = V_j \bigoplus W_j. \tag{11}$$

Then just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k). \tag{12}$$

All expansions which we used are based on the following properties:

$$\{\psi_{jk}\}, \quad j,k \in \mathbf{Z} \quad \text{is a Hilbertian basis of } L^2(\mathbf{R})$$
$$\{\varphi_{jk}\}_{j\ge 0, k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}),$$
$$L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j}, \tag{13}$$
$$\text{or} \qquad \{\varphi_{0,k}, \psi_{j,k}\}_{j\ge 0, k\in\mathbf{Z}}$$
$$\text{is an orthonormal basis for} L^2(\mathbf{R}).$$

Fig.1 and Fig.2 give the representation of some function and corresponding MRA on each level of resolution.

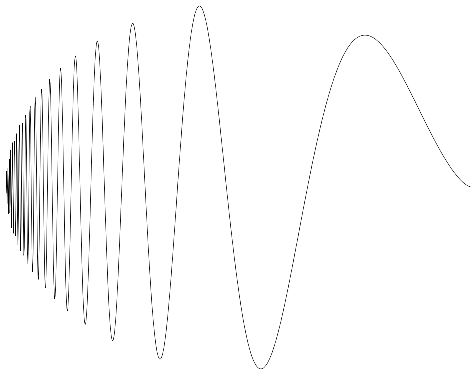

Figure 1: Analyzed function.

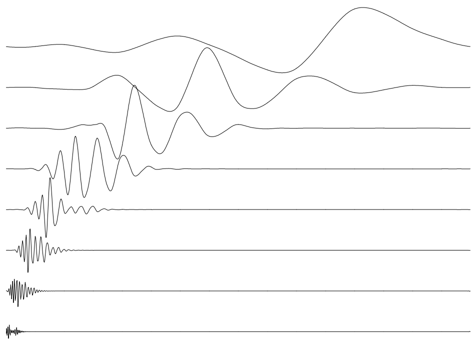

Figure 2: MRA representation.

4 VARIATIONAL WAVELET APPROACH FOR PERIODIC TRAJECTORIES

We start with extension of our approach from part 1 to the case of periodic trajectories. The equations of motion corresponding to Hamiltonian (4) may also be formulated as a particular case of the general system of ordinary differential equations $dx_i/dt = f_i(x_j, t)$, $(i, j = 1, ..., n)$, $0 \le t \le 1$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time but with periodic boundary conditions. According to our variational approach from part 1 we have the solution in the following form

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t), \qquad x_i(0) = x_i(1), \quad (14)$$

where λ_i^k are again the roots of reduced algebraical systems of equations with the same degree of nonlinearity and $\varphi_k(t)$ corresponds to useful type of wavelet bases (frames). It should be noted that coefficients of reduced algebraical system are the solutions of additional linear problem and also depend on particular type of wavelet construction and type of bases. This linear problem is our second reduced algebraical problem. We need to find in general situation objects

$$\Lambda^{d_1 d_2 \ldots d_n}_{\ell_1 \ell_2 \ldots \ell_n} = \int_{-\infty}^{\infty} \prod \varphi^{d_i}_{\ell_i}(x) \mathrm{d}x, \qquad (15)$$

but now in the case of periodic boundary conditions. Now we consider the procedure of their calculations in case of periodic boundary conditions in the base of periodic wavelet functions on the interval [0,1] and corresponding expansion (14) inside our variational approach. Periodization procedure gives us

$$\hat\varphi_{j,k}(x) \equiv \sum_{\ell \in Z} \varphi_{j,k}(x-\ell) \qquad (16)$$

$$\hat\psi_{j,k}(x) = \sum_{\ell \in Z} \psi_{j,k}(x-\ell)$$

So, $\hat\varphi, \hat\psi$ are periodic functions on the interval [0,1]. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k' \mathrm{mod}(2^j)$, we may consider only $0 \le k \le 2^j$ and as consequence our multiresolution has the form $\bigcup_{j \ge 0} \hat V_j = L^2[0,1]$ with $\hat V_j = \mathrm{span}\{\hat\varphi_{j,k}\}^{2j-1}_{k=0}$ [10]. Integration by parts and periodicity gives useful relations between objects (15) in particular quadratic case $(d = d_1 + d_2)$:

$$\Lambda^{d_1,d_2}_{k_1,k_2} = (-1)^{d_1} \Lambda^{0,d_2+d_1}_{k_1,k_2}, \ \Lambda^{0,d}_{k_1,k_2} = \Lambda^{0,d}_{0,k_2-k_1} \equiv \Lambda^d_{k_2-k_1}$$

So, any 2-tuple can be represent by Λ^d_k. Then our second additional linear problem is reduced to the eigenvalue problem for $\{\Lambda^d_k\}_{0 \le k \le 2^j}$ by creating a system of 2^j homogeneous relations in Λ^d_k and inhomogeneous equations. So, if we have dilation equation in the form $\varphi(x) = \sqrt{2} \sum_{k \in Z} h_k \varphi(2x-k)$, then we have the following homogeneous relations

$$\Lambda^d_k = 2^d \sum_{m=0}^{N-1} \sum_{\ell=0}^{N-1} h_m h_\ell \Lambda^d_{\ell+2k-m}, \qquad (17)$$

or in such form $A\lambda^d = 2^d \lambda^d$, where $\lambda^d = \{\Lambda^d_k\}_{0 \le k \le 2^j}$. Inhomogeneous equations are:

$$\sum_\ell M^d_\ell \Lambda^d_\ell = d! 2^{-j/2}, \qquad (18)$$

where objects $M^d_\ell(|\ell| \le N-2)$ can be computed by recursive procedure

$$M^d_\ell = 2^{-j(2d+1)/2} \tilde M^d_\ell, \qquad (19)$$

$$\tilde M^k_\ell = < x^k, \varphi_{0,\ell} > = \sum_{j=0}^{k} \binom{k}{j} n^{k-j} M^j_0, \quad \tilde M^\ell_0 = 1.$$

So, we reduced our last problem to standard linear algebraical problem. Then we use the same methods as in part 1. As a result we obtained for closed trajectories of orbital dynamics described by Hamiltonian (4) the explicit time solution (14) in the base of periodized wavelets (16).

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