NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 5. MAPS AND DISCRETIZATION VIA WAVELETS

A. Fedorova, M. Zeitlin, IPME, RAS, St. Petersburg, Russia * †

Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this part we consider the applications of discrete wavelet analysis technique to maps which come from discretization of continuous nonlinear polynomial problems in accelerator physics. Our main point is generalization of wavelet analysis which can be applied for both discrete and continuous cases. We give explicit multiresolution representation for solutions of discrete problems, which is correct discretization of our representation of solutions of the corresponding continuous cases.

1 INTRODUCTION

This is the fifth part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with parts 1–4 in parts 5–8 we try to take into account before using power analytical approaches underlying algebraical, geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems.

In this paper we consider the applications of discrete wavelet analysis technique to maps which come from discretization of continuous nonlinear polynomial problems in accelerator physics. Our main point is generalization of wavelet analysis which can be applied for both discrete and continuous cases. We give explicit multiresolution representation for solutions of discrete problems, which is correct discretization of our representation of solutions of the corresponding continuous cases.

In part 2 we consider symplectic and Lagrangian structures for the case of discretization of flows by corresponding maps and in part 3 construction of corresponding solutions by applications of generalized wavelet approach which is based on generalization of multiresolution analysis for the case of maps.

* e-mail: zeitlin@math.ipme.ru

† http://www.ipme.ru/zeitlin.html; http://www.ipme.nw.ru/zeitlin.html

2 VESELOV-MARSDEN DISCRETIZATION

Discrete variational principles lead to evolution dynamics analogous to the Euler-Lagrange equations [9]. Let Q be a configuration space, then a discrete Lagrangian is a map $L: Q\times Q\to\mathbf{R}$. usually L is obtained by approximating the given Lagrangian. For $N\in N_+$ the action sum is the map $S: Q^{N+1}\to\mathbf{R}$ defined by

$$S=\sum_{k=0}^{N-1}L(q_{k+1},q_k), \qquad (1)$$

where $q_k\in Q$, $k\geq 0$. The action sum is the discrete analog of the action integral in continuous case. Extremizing S over $q_1,...,q_{N-1}$ with fixing q_0,q_N we have the discrete Euler-Lagrange equations (DEL):

$$D_2L(q_{k+1},q_k)+D_1(q_k,q_{q-1})=0, \qquad (2)$$

for $k=1,...,N-1$.

Let

$$\Phi: Q\times Q\to Q\times Q \qquad (3)$$

and

$$\Phi(q_k,q_{k-1})=(q_{k+1},q_k) \qquad (4)$$

is a discrete function (map), then we have for DEL:

$$D_2L\circ\Phi+D_1L=0 \qquad (5)$$

or in coordinates q^i on Q we have DEL

$$\frac{\partial L}{\partial q^i_k}\circ\Phi(q_{k+1},q_k)+\frac{\partial L}{\partial q^i_{k+1}}(q_{k+1},q_k)=0. \qquad (6)$$

It is very important that the map Φ exactly preserves the symplectic form ω:

$$\omega=\frac{\partial^2 L}{\partial q^i_k\partial q^j_{k+1}}(q_{k+1},q_k)\mathrm{d}q^i_k\wedge\mathrm{d}q^j_{k+1} \qquad (7)$$

3 GENERALIZED WAVELET APPROACH

Our approach to solutions of equations (6) is based on applications of general and very efficient methods developed by A. Harten [10], who produced a "General Framework" for multiresolution representation of discrete data.

It is based on consideration of basic operators, decimation and prediction, which connect adjacent resolution levels. These operators are constructed from two basic blocks: the discretization and reconstruction operators. The former obtains discrete information from a given continuous functions (flows), and the latter produces an approximation to those functions, from discrete values, in the same function space to which the original function belongs.

A "new scale" is defined as the information on a given resolution level which cannot be predicted from discrete information at lower levels. If the discretization and reconstruction are local operators, the concept of "new scale" is also local.

The scale coefficients are directly related to the prediction errors, and thus to the reconstruction procedure. If scale coefficients are small at a certain location on a given scale, it means that the reconstruction procedure on that scale gives a proper approximation of the original function at that particular location.

This approach may be considered as some generalization of standard wavelet analysis approach. It allows to consider multiresolution decomposition when usual approach is impossible (δ-functions case). We demonstrated the discretization of Dirac function by wavelet packets on Fig. 1 and Fig. 2.

Let F be a linear space of mappings

$$F \subset \{f | f : X \to Y\}, \tag{8}$$

where X, Y are linear spaces. Let also D_k be a linear operator

$$\begin{aligned} &D_k : f \to \{v^k\}, \quad v^k = D_k f, \\ &v^k = \{v_i^k\}, \quad v_i^k \in Y. \end{aligned} \tag{9}$$

This sequence corresponds to k level discretization of X. Let

$$D_k(F) = V^k = \mathrm{span}\{\eta_i^k\} \tag{10}$$

and the coordinates of $v^k \in V^k$ in this basis are $\hat{v}^k = \{\hat{v}_i^k\}$, $\hat{v}^k \in S^k$:

$$v^k = \sum_i \hat{v}_i^k \eta_i^k, \tag{11}$$

D_k is a discretization operator. Main goal is to design a multiresolution scheme (MR) [10] that applies to all sequences $s \in S^L$, but corresponds for those sequences $\hat{v}^L \in S^L$, which are obtained by the discretization (8).

Since D_k maps F onto V^k then for any $v^k \subset V^k$ there is at least one f in F such that $D_k f = v^k$. Such correspondence from $f \in F$ to $v^k \in V^k$ is reconstruction and the corresponding operator is the reconstruction operator R_k:

$$R_k : V_k \to F, \qquad D_k R_k = I_k, \tag{12}$$

where I_k is the identity operator in V^k (R^k is right inverse of D^k in V^k).

Given a sequence of discretization $\{D_k\}$ and sequence of the corresponding reconstruction operators $\{R_k\}$, we define the operators D_k^{k-1} and P_{k-1}^k

$$\begin{aligned} D_k^{k-1} &= D_{k-1} R_k : V_k \to V_{k-1} \\ P_{k-1}^k &= D_k R_{k-1} : V_{k-1} \to V_k \end{aligned} \tag{13}$$

If the set D_k in nested [10], then

$$D_k^{k-1} P_{k-1}^k = I_{k-1} \tag{14}$$

and we have for any $f \in F$ and any $p \in F$ for which the reconstruction R_{k-1} is exact:

$$\begin{aligned} D_k^{k-1}(D_k f) &= D_{k-1} f \\ P_{k-1}^k(D_{k-1} p) &= D_k p \end{aligned} \tag{15}$$

Let us consider any $v^L \in V^L$, Then there is $f \in F$ such that

$$v^L = D_L f, \tag{16}$$

and it follows from (15) that the process of successive decimation [10]

$$v^{k-1} = D_k^{k-1} v^k, \qquad k = L, ..., 1 \tag{17}$$

yields for all k

$$v^k = D_k f \tag{18}$$

Thus the problem of prediction, which is associated with the corresponding MR scheme, can be stated as a problem of approximation: knowing $D_{k-1} f$, $f \in F$, find a "good approximation" for $D_k f$. It is very important that each space V^L has a multiresolution basis

$$\bar{B}_M = \{\bar{\phi}_i^{0,L}\}_i, \{\{\bar{\psi}_j^{k,L}\}_j\}_{k=1}^L \tag{19}$$

and that any $v^L \in V^L$ can be written as

$$v^L = \sum_i \hat{v}_i^0 \bar{\phi}_i^{0,L} + \sum_{k=1}^L \sum_j d_j^k \bar{\psi}_j^{k,L}, \tag{20}$$

where $\{d_j^k\}$ are the k scale coefficients of the associated MR, $\{\hat{v}_i^0\}$ is defined by (11) with $k = 0$. If $\{D_k\}$ is a nested sequence of discretization [10] and $\{R_k\}$ is any corresponding sequence of linear reconstruction operators, then we have from (20) for $v^L = D_L f$ applying R_L:

$$R_L D_L f = \sum_i \hat{f}_i^0 \phi_i^{0,L} + \sum_{k=1}^L \sum_j d_j^k \psi_j^{k,L}, \tag{21}$$

where

$$\begin{aligned} &\phi_i^{0,L} = R_L \bar{\phi}_i^{0,L} \in F, \quad \psi_j^{k,L} = R_L \bar{\psi}_j^{k,L} \in F, \\ &D_0 f = \sum \hat{f}_i^0 \eta_i^0. \end{aligned} \tag{22}$$

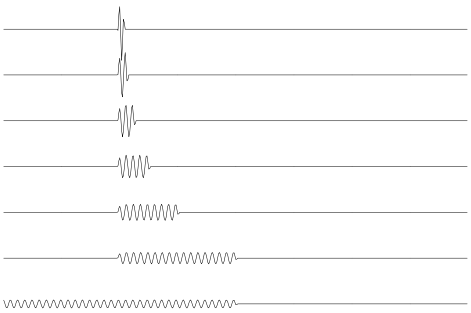

Figure 1: Wavelet packets.

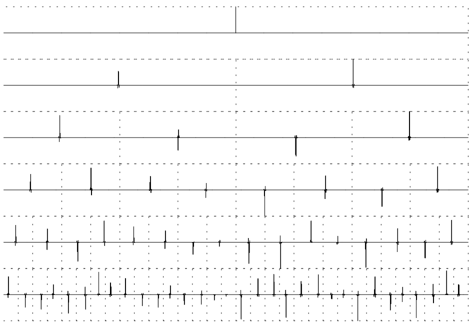

Figure 2: The discretization of Dirac function.

When $L \to \infty$ we have sufficient conditions which ensure that the limiting process $L \to \infty$ in (21, 22) yields a multiresolution basis for F. Then, according to (19), (20) we have very useful representation for solutions of equations (6) or for different maps construction in the form which are a counterparts for discrete (difference) cases of constructions from parts 1-4.

We are very grateful to M. Cornacchia (SLAC), W. Herrmannsfeldt (SLAC) Mrs. J. Kono (LBL) and M. Laraneta (UCLA) for their permanent encouragement.

4 REFERENCES

[1] Fedorova, A.N., Zeitlin, M.G. 'Wavelets in Optimization and Approximations', *Math. and Comp. in Simulation*, **46**, 527-534 (1998).

[2] Fedorova, A.N., Zeitlin, M.G., 'Wavelet Approach to Polynomial Mechanical Problems', New Applications of Nonlinear and Chaotic Dynamics in Mechanics, Kluwer, 101-108, 1998.

[3] Fedorova, A.N., Zeitlin, M.G., 'Wavelet Approach to Mechanical Problems. Symplectic Group, Symplectic Topology and Symplectic Scales', New Applications of Nonlinear and Chaotic Dynamics in Mechanics, Kluwer, 31-40, 1998.

[4] Fedorova, A.N., Zeitlin, M.G 'Nonlinear Dynamics of Accelerator via Wavelet Approach', AIP Conf. Proc., vol. **405**, 87-102, 1997, Los Alamos preprint, physics/9710035.

[5] Fedorova, A.N., Zeitlin, M.G, Parsa, Z., 'Wavelet Approach to Accelerator Problems', parts 1-3, Proc. PAC97, vol. **2**, 1502-1504, 1505-1507, 1508-1510, IEEE, 1998.

[6] Fedorova, A.N., Zeitlin, M.G, Parsa, Z., 'Nonlinear Effects in Accelerator Physics: from Scale to Scale via Wavelets', 'Wavelet Approach to Hamiltonian, Chaotic and Quantum Calculations in Accelerator Physics', Proc. EPAC'98, 930-932, 933-935, Institute of Physics, 1998.

[7] Fedorova, A.N., Zeitlin, M.G., Parsa, Z., 'Variational Approach in Wavelet Framework to Polynomial Approximations of Nonlinear Accelerator Problems', AIP Conf. Proc., vol. **468**, 48-68, 1999.
Los Alamos preprint, physics/9902062.

[8] Fedorova, A.N., Zeitlin, M.G., Parsa, Z., 'Symmetry, Hamiltonian Problems and Wavelets in Accelerator Physics', AIP Conf.Proc., vol. **468**, 69-93, 1999.
Los Alamos preprint, physics/9902063.

[9] Marsden, J.E., Park City Lectures on Mechanics, Dynamics and Symmetry, CALTECH, 1998.

[10] Harten, A., *SIAM J. Numer. Anal.*, **31**, 1191-1218, 1994.