NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 8. INVARIANT BASES, LOOPS AND KAM

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Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this part we consider variational wavelet approach for loops, invariant bases on semidirect product, KAM calculation via FWT.

1 INTRODUCTION

This is the eighth part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with parts 1–4 in parts 5–8 we try to take into account before using power analytical approaches underlying algebraical, geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems. In section 2 we consider wavelet approach for calculation of Arnold–Weinstein curves (closed loops) in Floer variational approach. In section 3 we consider the applications of orbit technique for constructing different types of invariant wavelet bases in the particular case of affine Galilei group with the semiproduct structure. In section 4 we consider applications of very useful fast wavelet transform (FWT) technique (part 6) to calculations in KAM theory (symplectic scale of spaces). This method gives maximally sparse representation of (differential) operator that allows us to take into account contribution from each level of resolution.

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2 FLOER APPROACH FOR CLOSED LOOPS

Now we consider the generalization of wavelet variational approach to the symplectic invariant calculation of closed loops in Hamiltonian systems [9]. As we demonstrated in [3]–[4] we have the parametrization of our solution by some reduced algebraical problem but in contrast to the cases from parts 1–4, where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations. Now we consider a different approach. Let(M, ω) be a compact symplectic manifold of dimension $2n$, ω is a closed 2-form (nondegenerate) on M which induces an isomorphism $T^*M \to TM$. Thus every smooth time-dependent Hamiltonian $H : \mathbf{R} \times M \to \mathbf{R}$ corresponds to a time-dependent Hamiltonian vector field $X_H : \mathbf{R} \times M \to TM$ defined by $\omega(X_H(t, x), \xi) = -d_x H(t, x)\xi$ for $\xi \in T_x M$. Let H (and X_H) is periodic in time: $H(t + T, x) = H(t, x)$ and consider corresponding Hamiltonian differential equation on M: $\dot{x}(t) = X_H(t, x(t))$ The solutions $x(t)$ determine a 1-parameter family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_t(x(0)) = x(t)$. These diffeomorphisms are symplectic: $\omega = \psi_t^* \omega$. Let $L = L_T M$ be the space of contractible loops in M which are represented by smooth curves $\gamma : \mathbf{R} \to M$ satisfying $\gamma(t + T) = \gamma(t)$. Then the contractible T-periodic solutions can be characterized as the critical points of the functional $S = S_T : L \to \mathbf{R}$:

$$S_T(\gamma) = -\int_D u^*\omega + \int_0^T H(t, \gamma(t))\mathrm{d}t, \qquad (1)$$

where $D \subset \mathbf{C}$ be a closed unit disc and $u : D \to M$ is a smooth function, which on boundary agrees with γ, i.e. $u(\exp\{2\pi i\Theta\}) = \gamma(\Theta T)$. Because $[\omega]$, the cohomology class of ω, vanishes then $S_T(\gamma)$ is independent of choice of u. Tangent space $T_\gamma L$ is the space of vector fields $\xi \in C^\infty(\gamma^* TM)$ along γ satisfying $\xi(t + T) = \xi(t)$. Then we have for the 1-form $\mathrm{d}f : TL \to \mathbf{R}$

$$\mathrm{d}S_T(\gamma)\xi = \int_0^T (\omega(\dot{\gamma}, \xi) + \mathrm{d}H(t, \gamma)\xi)\mathrm{d}t \qquad (2)$$

and the critical points of S are contractible loops in L which satisfy the Hamiltonian equations. Thus the critical points are precisely the required T-periodic solutions.

To describe the gradient of S we choose a on almost complex structure on M which is compatible with ω. This is an endomorphism $J \in C^\infty(\mathrm{End}(TM))$ satisfying $J^2 = -I$ such that $g(\xi,\eta) = \omega(\xi, J(x)\eta),\ \xi,\eta \in T_xM$ defines a Riemannian metric on M. The Hamiltonian vector field is then represented by $X_H(t,x) = J(x)\nabla H(t,x)$, where ∇ denotes the gradient w.r.t. the x-variable using the metric. Moreover the gradient of S w.r.t. the induced metric on L is given by $\mathrm{grad}S(\gamma) = J(\gamma)\dot\gamma + \nabla H(t,\gamma),\ \gamma \in L$. Studying the critical points of S is confronted with the well-known difficulty that the variational integral is neither bounded from below nor from above. Moreover, at every possible critical point the Hessian of f has an infinite dimensional positive and an infinite dimensional negative subspaces, so the standard Morse theory is not applicable. The additional problem is that the gradient vector field on the loop space L: $\mathrm{d}\gamma/\mathrm{d}s = -\mathrm{grad}f(\gamma)$ does not define a well posed Cauchy problem. But Floer [9] found a way to analyse the space $\mathcal{M}$ of bounded solutions consisting of the critical points together with their connecting orbits. He used a combination of variational approach and Gromov's elliptic technique. A gradient flow line of f is a smooth solution $u : \mathbf{R} \to M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0, \tag{3}$$

which satisfies $u(s,t+T) = u(s,t)$. The key point is to consider (3) not as the flow on the loop space but as an elliptic boundary value problem. It should be noted that (3) is a generalization of equation for Gromov's pseudoholomorphic curves (correspond to the case $\nabla H = 0$ in (3)). Let $\mathcal{M}_T = \mathcal{M}_T(H,J)$ the space of bounded solutions of (3), i.e. the space of smooth functions $u : \mathbf{C}/iT\mathbf{Z} \to M$, which are contractible, solve equation (3) and have finite energy flow:

$$\Phi_T(u) = \frac{1}{2}\int\int_0^T \left(|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_H(t,u)|^2\right)\mathrm{d}t\mathrm{d}s. \tag{4}$$

For every $u \in M_T$ there exists a pair x, y of contractible T-periodic solutions, such that u is a connecting orbit from y to x: $\lim_{s\to-\infty} u(s,t) = y(t),\ \lim_{s\to+\infty} = x(t)$. Then our approach from preceding parts, which we may apply or on the level of standard boundary problem or on the level of variational approach and representation of operators (in our case, J and ∇) according to part 6(FWT technique) lead us to wavelet representation of closed loops.

3 CONTINUOUS WAVELET TRANSFORM. BASES FOR SOLUTIONS.

When we take into account the Hamiltonian or Lagrangian structures from part 7 we need to consider generalized wavelets, which allow us to consider the corresponding structures instead of compactly supported wavelet representation from parts 1–4. We consider an important particular case of constructions from part 7: affine relativity group (relativity group combined with dilations) — affine Galilei group in n-dimensions. So, we have combination of Galilei group with independent space and time dilations: $G_{aff} = G_m \bowtie D_2$, where $D_2 = (\mathbf{R}_*^+)^2 \simeq \mathbf{R}^2$, G_m is extended Galilei group corresponding to mass parameter $m > 0$ (G_{aff} is noncentral extension of $G \bowtie D_2$ by $\mathbf{R}$, where G is usual Galilei group). Generic element of G_{aff} is $g = (\Phi, b_0, b; v; R, a_0, a)$, where $\Phi \in \mathbf{R}$ is the extension parameter in G_m, $b_0 \in \mathbf{R}$, $b \in \mathbf{R}^n$ are the time and space translations, $v \in \mathbf{R}^n$ is the boost parameter, $R \in SO(n)$ is a rotation and $a_0, a \in \mathbf{R}_*^+$ are time and space dilations. The actions of g on space-time is then $x \mapsto aRx + a_0vt + b$, $t \mapsto a_0t + b_0$, where $x = (x_1, x_2, ..., x_n)$. It should be noted that D_2 acts nontrivially on G_m. Space-time wavelets associated to G_{aff} corresponds to unitary irreducible representation of spin zero. It may be obtained via orbit method. The Hilbert space is $\mathcal{H} = L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega)$, $k = (k_1, ..., k_n)$, where $\mathbf{R}^n \times \mathbf{R}$ may be identified with usual Minkowski space and we have for representation:

$$(U(g)\Psi)(k,\omega) = \sqrt{a_0a^n}\exp i(m\Phi + kb - \omega b_0)\Psi(k',\omega'), \tag{5}$$

with $k' = aR^{-1}(k + mv)$, $\omega' = a_0(\omega - kv - \frac{1}{2}mv^2)$, $m' = (a^2/a_0)m$. Mass m is a coordinate in the dual of the Lie algebra and these relations are a part of coadjoint action of G_{aff}. This representation is unitary and irreducible but not square integrable. So, we need to consider reduction to the corresponding quotients $X = G/H$. We consider the case in which H=$\{$phase changes Φ and space dilations $a\}$. Then the space $X = G/H$ is parametrized by points $\bar{x} = (b_0, b; v; R; a_0)$. There is a dense set of vectors $\eta \in \mathcal{H}$ admissible $\mathrm{mod}(H, \sigma_\beta)$, where σ_β is the corresponding section. We have a two-parameter family of functions β(dilations): $\beta(\bar{x}) = (\mu_0 + \lambda_) a_0)^{1/2}$, $\lambda_0, \mu_0 \in \mathbf{R}$. Then any admissible vector η generates a tight frame of Galilean wavelets

$$\eta_{\beta(\bar{x})}(k,\omega) = \sqrt{a_0(\mu_0 + \lambda_0 a_0)^{n/2}}\mathrm{e}^{i(kb-\omega b_0)}\eta(k',\omega'), \tag{6}$$

with $k' = (\mu_0 + \lambda_0 a)^{1/2}R^{-1}(k + mv)$, $\omega' = a_0(\omega - kv - mv^2/2)$. The simplest examples of admissible vectors (corresponding to usual Galilei case) are Gaussian vector: $\eta(k) \sim \exp(-k^2/2mu)$ and binomial vector: $\eta(k) \sim (1 + k^2/2mu)^{-\alpha/2}$, $\alpha > 1/2$, where u is a kind of internal energy. When we impose the relation $a_0 = a^2$ then we have the restriction to the Galilei-Schrodinger group $G_s = G_m \bowtie D_s$, where D_s is the one-dimensional subgroup of D_2. G_s is a natural invariance group of both the Schrodinger equation and the heat equation. The restriction to G_s of the representation (29) splits into the direct sum of two irreducible ones $U = U_+ \oplus U_-$ corresponding to the decomposition $L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm = L^2(D_\pm, \mathrm{d}k\mathrm{d}\omega = \{\psi \in L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega),\ \psi(k,\omega) = 0\ for\ \omega + k^2/2m = 0\}$. These two subspaces are the analogues of usual Hardy spaces on $\mathbf{R}$, i.e. the subspaces of (anti)progressive wavelets (see also below, part III A). The two representation $U_\pm$ are square integrable modulo the center. There is a dense set of admissible vectors η, and each of them generates a set of CS of Gilmore-Perelomov type. Typical wavelets of this kind are: the Schrodinger-Marr wavelet: $\eta(x,t) = (i\partial_t + \triangle/2m)\mathrm{e}^{-(x^2+t^2)/2}$, the Schrodinger-Cauchy wavelet: $\psi(x,t) = (i\partial_t + \triangle/2m) \times (t+i)\prod_{j=1}^n (x_j+i)^{-1}$. So, in the same way we can construct different invariant bases with explicit manifestation of underlying symmetry for solving Hamiltonian or Lagrangian equations.

4 SYMPLECTIC HILBERT SCALES VIA WAVELETS

We can solve many important dynamical problems such that KAM perturbations, spread of energy to higher modes, weak turbulence, growths of solutions of Hamiltonian equations only if we consider scales of spaces instead of one functional space. For Hamiltonian system and their perturbations for which we need take into account underlying symplectic structure we need to consider symplectic scales of spaces. So, if $\dot u(t) = J\nabla K(u(t))$ is Hamiltonian equation we need wavelet description of symplectic or quasicomplex structure on the level of functional spaces. It is very important that according to [12] Hilbert basis is in the same time a Darboux basis to corresponding symplectic structure. We need to provide Hilbert scale $\{Z_s\}$ with symplectic structure [12]. All what we need is the following. J is a linear operator, $J: Z_\infty \to Z_\infty$, $J(Z_\infty) = Z_\infty$, where $Z_\infty = \cap Z_s$. J determines an isomorphism of scale $\{Z_s\}$ of order $d_J \geq 0$. The operator J with domain of definition Z_∞ is antisymmetric in Z: $< Jz_1, z_2 >_Z = - < z_1, Jz_2 >_Z, z_1, z_2 \in Z_\infty$. Then the triple $\{Z, \{Z_s | s \in R\},\ \alpha =< \bar J dz, dz >\}$ is symplectic Hilbert scale. So, we may consider any dynamical Hamiltonian problem on functional level. As an example, for KdV equation we have $Z_s = \{u(x) \in H^s(T^1) | \int_0^{2\pi} u(x)\mathrm{d}x = 0\}$, $s \in R$, $J = \partial/\partial x$, is isomorphism of the scale of order one, $\bar J = -(J)^{-1}$ is isomorphism of order -1. According to [13] general functional spaces and scales of spaces such as Holder–Zygmund, Triebel–Lizorkin and Sobolev can be characterized through wavelet coefficients or wavelet transforms. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular [13]. Most important for us example is the scale of Sobolev spaces. Let $H_k(R^n)$ is the Hilbert space of all distributions with finite norm $\|s\|^2_{H_k(R^n)} = \int \mathrm{d}\xi (1+|\xi|^2)^{k/2} |\hat s(\xi)|^2$. Let us consider wavelet transform

$$W_g f(b,a) = \int_{R^n} \mathrm{d}x \frac{1}{a^n} \bar g\left(\frac{x-b}{a}\right) f(x),$$

$b \in R^n$, $a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e. $C_{g,g} = \int_0^\infty (\mathrm{d}a/a)|\hat g(ak)|^2 < \infty$. Then there is a $c \geq 1$ such that

$$c^{-1}\|s\|^2_{H_k(R^n)} \leq \int_{H^n} \frac{\mathrm{d}b\mathrm{d}a}{a}(1+a^{-2\gamma})| \times W_g s(b,a)|^2 \leq c\|s\|^2_{H_k(R^n)}.$$

This shows that localization of the wavelet coefficients at small scale is linked to local regularity. So, we need representation for differential operator (J in our case) in wavelet basis. We consider it by means of the methods from part 6.

We are very grateful to M. Cornacchia (SLAC), W. Herrmannsfeldt (SLAC) Mrs. J. Kono (LBL) and M. Laraneta (UCLA) for their permanent encouragement.

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