NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 6. REPRESENTATIONS AND QUASICLASSICS VIA FWT

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Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this part we consider application of FWT to metaplectic representation(quantum and chaotical problems) and quasiclassics.

1 INTRODUCTION

This is the sixth part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with parts 1–4 in parts 5–8 we try to take into account before using power analytical approaches underlying algebraical, geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems. In part 2 according to the orbit method and by using construction from the geometric quantization theory we construct the symplectic and Poisson structures associated with generalized wavelets by using metaplectic structure. In part 3 we consider applications of very useful fast wavelet transform technique (FWT) (part 4) to calculations in quasiclassical evolution dynamics. This method gives maximally sparse representation of (differential) operator that allows us to take into account contribution from each level of resolution.

2 METAPLECTIC GROUP AND REPRESENTATIONS

Let $Sp(n)$ be symplectic group, $Mp(n)$ be its unique two-fold covering – metaplectic group [9]. Let V be a symplectic vector space with symplectic form (,), then $R\oplus V$ is nilpotent Lie algebra - Heisenberg algebra: $[R,V]=0,\ [v,w]=(v,w)\in R,\ [V,V]=R$. $Sp(V)$ is a group of automorphisms of Heisenberg algebra. Let N be a group with Lie algebra $R\oplus V$, i.e. Heisenberg group. By Stone–von Neumann theorem Heisenberg group has unique irreducible unitary representation in which $1\mapsto i$. Let us also consider the projective representation of symplectic group $Sp(V)$: $U_{g_1}U_{g_2}=c(g_1,g_2)\cdot U_{g_1g_2}$, where c is a map: $Sp(V)\times Sp(V)\to S^1$, i.e. c is S^1-cocycle. But this representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F=N'\bowtie Mp(V),\quad \bowtie$ is semidirect product (we consider instead of $N=R\oplus V$ the $N'=S^1\times V,\quad S^1=(R/2\pi Z)$). Let V^* be dual to V, $G(V^*)$ be automorphism group of V^*.Then F is subgroup of $G(V^*)$, which consists of elements, which acts on V^* by affine transformations. This is the key point! Let $q_1,...,q_n;p_1,...,p_n$ be symplectic basis in V, $\alpha=pdq=\sum p_idq_i$ and $d\alpha$ be symplectic form on V^*. Let M be fixed affine polarization, then for $a\in F$ the map $a\mapsto\Theta_a$ gives unitary representation of G: $\Theta_a: H(M)\to H(M)$. Explicitly we have for representation of N on H(M): $(\Theta_qf)^*(x)=e^{-iqx}f(x),\ \Theta_pf(x)=f(x-p)$ The representation of N on H(M) is irreducible. Let A_q, A_p be infinitesimal operators of this representation

$$A_q=\lim_{t\to 0}\frac{1}{t}[\Theta_{-tq}-I],\quad A_p=\lim_{t\to 0}\frac{1}{t}[\Theta_{-tp}-I],$$

then $A_qf(x)=i(qx)f(x),\ A_pf(x)=\sum p_j\dfrac{\partial f}{\partial x_j}(x)$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group F is the algebra of all (nonhomogeneous) quadratic polynomials of (p,q) relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, ..., q_n,\ p_1, ..., p_n,\ q_iq_j, q_ip_j,\ p_ip_j,\quad i,j=1,...,n,\quad i\le j$,

$$PB\ is\quad \{f,g\}=\sum\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

and $\{1,g\}=0\quad for$ all g,

$\{p_i,q_j\}=\delta_{ij},\quad \{p_iq_j,q_k\}=\delta_{ik}q_j,$

$\{p_iq_j,p_k\}=-\delta_{jk}p_i,\quad \{p_ip_j,p_k\}=0,$

$\{p_ip_j,q_k\}=\delta_{ik}p_j+\delta_{jk}p_i,\quad \{q_iq_j,q_k\}=0,$

$\{q_iq_j,p_k\}=-\delta_{ik}q_j-\delta_{jk}q_i$

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so, we have the representation of basic elements $f \mapsto A_f$: $1 \mapsto i, q_k \mapsto ix_k$,

$$p_l \mapsto \frac{\partial}{\partial x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2}\delta_{ij},$$
$$p_k p_l \mapsto \frac{1}{i}\frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto i x^k x^l$$

This gives the structure of the Poisson manifolds to representation of any (nilpotent) algebra or in other words to continuous wavelet transform. According to this approach we can construct by using methods of geometric quantization theory many "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it. Then we may produce symplectic invariant wavelet calculations for PB or commutators which we may use in quantization procedure or in chaotic dynamics (part 8) via operator representation from section 4.

3 QUASICLASSICAL EVOLUTION

Let us consider classical and quantum dynamics in phase space $\Omega = R^{2m}$ with coordinates (x,ξ) and generated by Hamiltonian $\mathcal{H}(x,\xi) \in C^\infty(\Omega; R)$. If $\Phi_t^{\mathcal{H}} : \Omega \longrightarrow \Omega$ is (classical) flow then time evolution of any bounded classical observable or symbol $b(x,\xi) \in C^\infty(\Omega, R)$ is given by $b_t(x,\xi) = b(\Phi_t^{\mathcal{H}}(x,\xi))$. Let $H = Op^W(\mathcal{H})$ and $B = Op^W(b)$ are the self-adjoint operators or quantum observables in $L^2(R^n)$, representing the Weyl quantization of the symbols $\mathcal{H}, b$ [9]

$$(Bu)(x) = \frac{1}{(2\pi\hbar)^n}\int_{R^{2n}} b\left(\frac{x+y}{2},\xi\right)\cdot$$
$$e^{i<(x-y),\xi>/\hbar}u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $u \in S(R^n)$ and $B_t = e^{iHt/\hbar}Be^{-iHt/\hbar}$ be the Heisenberg observable or quantum evolution of the observable B under unitary group generated by H. B_t solves the Heisenberg equation of motion $\dot{B}_t = (i/\hbar)[H, B_t]$. Let $b_t(x,\xi;\hbar)$ is a symbol of B_t then we have the following equation for it

$$\dot{b}_t = \{\mathcal{H}, b_t\}_M, \tag{1}$$

with the initial condition $b_0(x,\xi,\hbar) = b(x,\xi)$. Here $\{f,g\}_M(x,\xi)$ is the Moyal brackets of the observables $f,g \in C^\infty(R^{2n})$, $\{f,g\}_M(x,\xi) = f\sharp g - g\sharp f$, where $f\sharp g$ is the symbol of the operator product and is presented by the composition of the symbols f,g

$$(f\sharp g)(x,\xi) = \frac{1}{(2\pi\hbar)^{n/2}}\int_{R^{4n}} e^{-i<r,\rho>/\hbar + i<\omega,\tau>/\hbar}$$
$$\cdot f(x+\omega,\rho+\xi)\cdot g(x+r,\tau+\xi)\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}r\mathrm{d}\omega.$$

For our problems it is useful that $\{f,g\}_M$ admits the formal expansion in powers of $\hbar$:

$$\{f,g\}_M(x,\xi) \sim \{f,g\} + 2^{-j}\cdot$$
$$\sum_{|\alpha+\beta|=j\geq 1}(-1)^{|\beta|}\cdot(\partial_\xi^\alpha f D_x^\beta g)\cdot(\partial_\xi^\beta g D_x^\alpha f),$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D_x = -i\hbar\partial_x$. So, evolution (1) for symbol $b_t(x,\xi;\hbar)$ is

$$\dot{b}_t = \{\mathcal{H}, b_t\} + \frac{1}{2^j}\sum_{|\alpha|+\beta|=j\geq 1}(-1)^{|\beta|}\cdot \tag{2}$$
$$\hbar^j(\partial_\xi^\alpha \mathcal{H} D_x^\beta b_t)\cdot(\partial_\xi^\beta b_t D_x^\alpha \mathcal{H}).$$

At $\hbar = 0$ this equation transforms to classical Liouville equation

$$\dot{b}_t = \{\mathcal{H}, b_t\}. \tag{3}$$

Equation (2) plays a key role in many quantum (semiclassical) problem. We note only the problem of relation between quantum and classical evolutions or how long the evolution of the quantum observables is determined by the corresponding classical one [9]. Our approach to solution of systems (2), (3) is based on our technique from [1]-[8] and very useful linear parametrization for differential operators which we present in the next section.

4 FAST WAVELET TRANSFORM FOR DIFFERENTIAL OPERATORS

Let us consider multiresolution representation $\dots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2}\dots$ (see our other papers from this series for details of wavelet machinery). Let T be an operator $T : L^2(R) \rightarrow L^2(R)$, with the kernel $K(x,y)$ and $P_j : L^2(R) \rightarrow V_j$ $(j \in Z)$ is projection operators on the subspace V_j corresponding to j level of resolution: $(P_jf)(x) = \sum_k < f,\varphi_{j,k} > \varphi_{j,k}(x)$. Let $Q_j = P_{j-1} - P_j$ is the projection operator on the subspace W_j then we have the following "microscopic or telescopic" representation of operator T which takes into account contributions from each level of resolution from different scales starting with coarsest and ending to finest scales [10]: $T = \sum_{j\in Z}(Q_jTQ_j + Q_jTP_j + P_jTQ_j)$. We remember that this is a result of presence of affine group inside this construction. The non-standard form of operator representation [10] is a representation of an operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j\in Z}$, acting on the subspaces V_j and W_j: $A_j : W_j \rightarrow W_j, B_j : V_j \rightarrow W_j, \Gamma_j : W_j \rightarrow V_j$, where operators $\{A_j, B_j, \Gamma_j\}_{j\in Z}$ are defined as $A_j = Q_jTQ_j, \quad B_j = Q_jTP_j, \quad \Gamma_j = P_jTQ_j$. The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix},$$

where $T_j = P_jTP_j$ and T_j works on $V_j : V_j \rightarrow V_j$. It should be noted that operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}. The operators A_j, B_j, Γ_j, T_j are represented by matrices $\alpha^j, \beta^j, \gamma^j, s^j$

$$\alpha^j_{k,k'} = \int\int K(x,y)\psi_{j,k}(x)\psi_{j,k'}(y)\mathrm{d}x\mathrm{d}y$$

$$
\begin{aligned}
\beta^j_{k,k'} &= \int\int K(x,y)\psi_{j,k}(x)\varphi_{j,k'}(y)\mathrm{d}x\mathrm{d}y \qquad (4)\\
\gamma^j_{k,k'} &= \int\int K(x,y)\varphi_{j,k}(x)\psi_{j,k'}(y)\mathrm{d}x\mathrm{d}y\\
s^j_{k,k'} &= \int\int K(x,y)\varphi_{j,k}(x)\varphi_{j,k'}(y)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

We may compute the non-standard representations of operator $\mathrm{d}/\mathrm{d}x$ in the wavelet bases by solving a small system of linear algebraical equations. So, we have for objects (4)

$$
\begin{aligned}
\alpha^j_{i,\ell} &= 2^{-j}\int\psi(2^{-j}x-i)\psi'(2^{-j}-\ell)2^{-j}\mathrm{d}x\\
&= 2^{-j}\alpha_{i-\ell}\\
\beta^j_{i,\ell} &= 2^{-j}\int\psi(2^{-j}x-i)\varphi'(2^{-j}x-\ell)2^{-j}\mathrm{d}x\\
&= 2^{-j}\beta_{i-\ell}\\
\gamma^j_{i,\ell} &= 2^{-j}\int\varphi(2^{-j}x-i)\psi'(2^{-j}x-\ell)2^{-j}\mathrm{d}x\\
&= 2^{-j}\gamma_{i-\ell},
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_\ell &= \int\psi(x-\ell)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\mathrm{d}x\\
\beta_\ell &= \int\psi(x-\ell)\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\mathrm{d}x\\
\gamma_\ell &= \int\varphi(x-\ell)\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)\mathrm{d}x
\end{aligned}
$$

then by using refinement equations we have in terms of filters (h_k, g_k):

$$
\begin{aligned}
\alpha_j &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} g_k g_{k'} r_{2i+k-k'},\\
\beta_j &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} g_k h_{k'} r_{2i+k-k'},\\
\gamma_i &= 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} h_k g_{k'} r_{2i+k-k'},
\end{aligned}
$$

where $r_\ell = \int\varphi(x-\ell)\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)\mathrm{d}x, \ell\in Z$. Therefore, the representation of d/dx is completely determined by the coefficients r_ℓ or by representation of d/dx only on the subspace V_0. The coefficients $r_\ell, \ell\in Z$ satisfy the following system of linear algebraical equations

$$
r_\ell = 2\left[r_{2l}+\frac{1}{2}\sum_{k=1}^{L/2}a_{2k-1}(r_{2\ell-2k+1}+r_{2\ell+2k-1})\right]
$$

and $\sum_\ell \ell r_\ell = -1$, where $a_{2k-1} = 2\sum_{i=0}^{L-2k} h_i h_{i+2k-1}$, $k=1,\ldots,L/2$ are the autocorrelation coefficients of the filter H. If a number of vanishing moments $M\geq 2$ then this linear system of equations has a unique solution with finite number of non-zero r_ℓ, $r_\ell\neq 0$ for $-L+2\leq\ell\leq L-2, r_\ell=-r_{-\ell}$. For the representation of operator d^n/dx^n we have the similar reduced linear system of equations. Then finally we have for action of operator $T_j(T_j: V_j\to V_j)$ on sufficiently smooth function f:

$$
(T_jf)(x) = \sum_{k\in Z}\left(2^{-j}\sum_\ell r_\ell f_{j,k-\ell}\right)\varphi_{j,k}(x),
$$

where $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x-k)$ is wavelet basis and

$$
f_{j,k-1} = 2^{-j/2}\int f(x)\varphi(2^{-j}x-k+\ell)\mathrm{d}x
$$

are wavelet coefficients. So, we have simple linear parametrization of matrix representation of our differential operator in wavelet basis and of the action of this operator on arbitrary vector in our functional space. Then we may use such representation in all preceding sections.

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5 REFERENCES

[1] Fedorova, A.N., Zeitlin, M.G. 'Wavelets in Optimization and Approximations', *Math. and Comp. in Simulation*, **46**, 527-534 (1998).

[2] Fedorova, A.N., Zeitlin, M.G., 'Wavelet Approach to Polynomial Mechanical Problems', New Applications of Nonlinear and Chaotic Dynamics in Mechanics, Kluwer, 101-108, 1998.

[3] Fedorova, A.N., Zeitlin, M.G., 'Wavelet Approach to Mechanical Problems. Symplectic Group, Symplectic Topology and Symplectic Scales', New Applications of Nonlinear and Chaotic Dynamics in Mechanics, Kluwer, 31-40, 1998.

[4] Fedorova, A.N., Zeitlin, M.G 'Nonlinear Dynamics of Accelerator via Wavelet Approach', AIP Conf. Proc., vol. **405**, 87-102, 1997, Los Alamos preprint, physics/9710035.

[5] Fedorova, A.N., Zeitlin, M.G, Parsa, Z., 'Wavelet Approach to Accelerator Problems', parts 1-3, Proc. PAC97, vol. **2**, 1502-1504, 1505-1507, 1508-1510, IEEE, 1998.

[6] Fedorova, A.N., Zeitlin, M.G, Parsa, Z., 'Nonlinear Effects in Accelerator Physics: from Scale to Scale via Wavelets', 'Wavelet Approach to Hamiltonian, Chaotic and Quantum Calculations in Accelerator Physics', Proc. EPAC'98, 930-932, 933-935, Institute of Physics, 1998.

[7] Fedorova, A.N., Zeitlin, M.G., Parsa, Z., 'Variational Approach in Wavelet Framework to Polynomial Approximations of Nonlinear Accelerator Problems', AIP Conf. Proc., vol. **468**, 48-68, 1999.
Los Alamos preprint, physics/9902062.

[8] Fedorova, A.N., Zeitlin, M.G., Parsa, Z., 'Symmetry, Hamiltonian Problems and Wavelets in Accelerator Physics', AIP Conf.Proc., vol. **468**, 69-93, 1999.
Los Alamos preprint, physics/9902063.

[9] G.B. Folland, 'Harmonic Analysis in Phase Space', Princeton, 1989.

[10] G. Beylkin, R.R. Coifman, V. Rokhlin, Comm. Pure and Appl. Math., **44**, 141-183, 1991.