NONLINEAR ACCELERATOR PROBLEMS VIA WAVELETS: 7. INVARIANT CALCULATIONS IN HAMILTON PROBLEMS

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Abstract

In this series of eight papers we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In this paper we consider invariant formulation of nonlinear (Lagrangian or Hamiltonian) dynamics on semidirect structure (relativity or dynamical groups) and corresponding invariant calculations via CWT.

1 INTRODUCTION

This is the seventh part of our eight presentations in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of our results from [1]-[8], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with parts 1–4 in parts 5–8 we try to take into account before using power analytical approaches underlying algebraical, geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems. We described a number of concrete problems in parts 1–4. The most interesting case is the dynamics of spin-orbital motion (part 4). In section 2 we consider dynamical consequences of covariance properties regarding to relativity (kinematical) groups and continuous wavelet transform (CWT) (in section 3) as a method for the solution of dynamical problems. We introduce the semidirect product structure, which allows us to consider from general point of view all relativity groups such as Euclidean, Galilei, Poincare. Then we consider the Lie-Poisson equations and obtain the manifestation of semiproduct structure of (kinematic) symmetry group on dynamical level. So, correct description of dynamics is a consequence of correct understanding of real symmetry of the concrete problem. We consider the Lagrangian theory related to semiproduct structure and explicit form of variation principle and corresponding (semidirect) Euler-Poincare equations. In section 3 we consider CWT and the corresponding analytical technique which allows to consider covariant wavelet analysis. In part 8 we consider in the particular case of affine Galilei group with the semiproduct structure the corresponding orbit technique for constructing different types of invariant wavelet bases.

2 DYNAMICS ON SEMIDIRECT PRODUCTS

Relativity groups such as Euclidean, Galilei or Poincare groups are the particular cases of semidirect product construction, which is very useful and simple general construction in the group theory [9]. We may consider as a basic example the Euclidean group $SE(3) = SO(3) \bowtie \mathbf{R}^3$, the semidirect product of rotations and translations. In general case we have $S = G \bowtie V$, where group G (Lie group or automorphisms group) acts on a vector space V and on its dual V^*. Let V be a vector space and G is the Lie group, which acts on the left by linear maps on V (G also acts on the left on its dual space V^*). The Lie algebra of S is the semidirect product Lie algebra, $s = \mathcal{G} \bowtie V$ with brackets $[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1)$, where the induced action of $\mathcal{G}$ by concatenation is denoted as $\xi_1 v_2$. Let $(g, v) \in S = G \times V$, $\quad (\xi, u) \in s = \mathcal{G} \times V$, $(\mu, a) \in s^* = \mathcal{G}^* \times V^*$, $g\xi = Ad_g \xi$, $g\mu = Ad^*_{g^{-1}}\mu$, ga denote the induced left action of g on a (the left action of G on V induces a left action on V^* — the inverse of the transpose of the action on V), $\rho_v : \mathcal{G} \to V$ is a linear map given by $\rho_v(\xi) = \xi v$, $\rho^*_v : V^* \to \mathcal{G}^*$ is its dual. Then adjoint and coadjoint actions are given by simple concatenation: $(g, v)(\xi, u) = (g\xi, gu - (g\xi)v)$, $(g, v)(\mu, a) = (g\mu + \rho^*_v(ga), ga)$. Also, let be $\rho^*_v a = v \diamond a \in \mathcal{G}^*$ for $a \in V^*$, which is a bilinear operation in v and a. So, we have the coadjoint action: $(g, v)(\mu, a) = (g\mu + v \diamond (ga), ga)$. Using concatenation notation for Lie algebra actions we have alternative definition of $v \diamond a \in \mathcal{G}^*$. For all $v \in V$, $a \in V^*$, $\eta \in \mathcal{G}$ we have $< \eta a, v >= - < v \diamond a, \eta >$.

Now we consider the manifestation of semiproduct structure of symmetry group on dynamical level. Let F, G be real valued functions on the dual space $\mathcal{G}^*$, $\mu \in \mathcal{G}^*$. Functional derivative of F at μ is the unique ele-

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ment $\delta F/\delta\mu \in \mathcal{G}$: $\lim_{\epsilon\to 0}[F(\mu+\epsilon\delta\mu)-F(\mu)]/\epsilon =< \delta\mu, \delta F/\delta\mu >$ for all $\delta\mu \in \mathcal{G}^*$, $<,>$ is pairing between $\mathcal{G}^*$ and $\mathcal{G}$. Define the $(\pm)$ Lie-Poisson brackets by $\{F,G\}_\pm(\mu) = \pm < \mu, [\delta F/\delta\mu, \delta G/\delta\mu] >$. The Lie-Poisson equations, determined by $\dot F = \{F,H\}$ or intrinsically $\dot\mu = \mp ad^*_{\partial H/\partial\mu}\mu$. For the left representation of G on V $\pm$ Lie-Poisson bracket of two functions $f,k : s^* \to \mathbf{R}$ is given by

$$\{f,k\}_\pm(\mu,a) = \pm < \mu, [\frac{\delta f}{\delta\mu}, \frac{\delta k}{\delta\mu}] > \tag{1}$$
$$\pm < a, \frac{\delta f}{\delta\mu}\frac{\delta k}{\delta a} - \frac{\delta k}{\delta\mu}\frac{\delta f}{\delta a} >,$$

where $\delta f/\delta\mu \in \mathcal{G}$, $\delta f/\delta a \in V$ are the functional derivatives of f. The Hamiltonian vector field of $h : s^* \in \mathbf{R}$ has the expression $X_h(\mu,a) = \mp(ad^*_{\delta h/\delta\mu}\mu - \delta h/\delta a \diamond a, -\delta h/\delta\mu a)$. Thus, Hamiltonian equations on the dual of a semidirect product are [9]:

$$\dot\mu = \mp ad^*_{\delta h/\delta\mu}\mu \pm \frac{\delta h}{\delta a}\diamond a, \quad \dot a = \pm\frac{\delta h}{\delta\mu}a \tag{2}$$

So, we can see the explicit contribution to the Poisson brackets and the equations of motion which come from the semiproduct structure.

Now we consider according to [9] Lagrangian side of a theory. This approach is based on variational principles with symmetry and is not dependent on Hamiltonian formulation, although it is demonstrated in [9] that this purely Lagrangian formulation is equivalent to the Hamiltonian formulation on duals of semidirect product (the corresponding Legendre transformation is a diffeomorphism). We consider the case of the left representation and the left invariant Lagrangians (ℓ and L), which depend in additional on another parameter $a \in V^*$ (dynamical parameter), where V is representation space for the Lie group G and L has an invariance property related to both arguments. It should be noted that the resulting equations of motion, the Euler-Poincare equations, are not the Euler-Poincare equations for the semidirect product Lie algebra $\mathcal{G} \bowtie V^*$ or $\mathcal{G} \bowtie V$. So, we have the following:

1. There is a left representation of Lie group G on the vector space V and G acts in the natural way on the left on $TG \times V^* : h(v_g, a) = (hv_g, ha)$. **2.** The function $L : TG \times V^* \in \mathbf{R}$ is the left G-invariant. **3.** Let $a_0 \in V^*$, Lagrangian $L_{a_0} : TG \to \mathbf{R}$, $L_{a_0}(v_g) = L(v_g, a_0)$. L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0. **4.** Left G-invariance of L permits us to define $\ell : \mathcal{G} \times V^* \to \mathbf{R}$ by $\ell(g^{-1}v_g, g^{-1}a_0) = L(v_g, a_0)$. This relation defines for any $\ell : \mathcal{G} \times V^* \to \mathbf{R}$ the left G-invariant function $L : TG \times V^* \to \mathbf{R}$. **5.** For a curve $g(t) \in G$ let be $\xi(t) := g(t)^{-1}\dot g(t)$ and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients $\dot a(t) = -\xi(t)a(t)$, with initial condition $a(0) = a_0$. The solution can be written as $a(t) = g(t)^{-1}a_0$.

Then we have four equivalent descriptions of the corresponding dynamics: **1.** If a_0 is fixed then Hamilton's variational principle $\delta\int_{t_1}^{t_2} L_{a_0}(g(t), \dot g(t)\mathrm{d}t = 0$ holds for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints. **2.** $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G. **3.** The constrained variational principle $\delta\int_{t_1}^{t_2}\ell(\xi(t), a(t))\mathrm{d}t = 0$ holds on $\mathcal{G} \times V^*$, using variations of ξ and a of the form $\delta\xi = \dot\eta + [\xi,\eta], \delta a = -\eta a$, where $\eta(t) \in \mathcal{G}$ vanishes at the endpoints. **4.** The Euler-Poincare equations hold on $\mathcal{G} \times V^*$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\ell}{\delta\xi} = ad^*_\xi\frac{\delta\ell}{\delta\xi} + \frac{\delta\ell}{\delta a}\diamond a \tag{3}$$

So, we may apply our wavelet methods either on the level of variational formulation or on the level of Euler-Poincare equations.

3 CONTINUOUS WAVELET TRANSFORM

Now we need take into account the Hamiltonian or Lagrangian structures related with systems (2) or (3). Therefore, we need to consider generalized wavelets, which allow us to consider the corresponding structures instead of compactly supported wavelet representation from parts 1-4. In wavelet analysis the following three concepts are used now: 1). a square integrable representation U of a group G, 2). coherent states (CS) over G, 3). the wavelet transform associated to U. We consider now their unification [10]-[12]. Let G be a locally compact group and U_a strongly continuous, irreducible, unitary representation of G on Hilbert space $\mathcal{H}$. Let H be a closed subgroup of G, $X = G/H$ with (quasi) invariant measure ν and $\sigma : X = G/H \to G$ is a Borel section in a principal bundle $G \to G/H$. Then we say that U is square integrable $mod(H,\sigma)$ if there exists a non-zero vector $\eta \in \mathcal{H}$ such that $0 < \int_X | < U(\sigma(x))\eta|\Phi > |^2\mathrm{d}\nu(x) =< \Phi|A_\sigma\Phi > < \infty, \forall\Phi \in \mathcal{H}$. Given such a vector $\eta \in \mathcal{H}$ called admissible for (U,σ) we define the family of (covariant) coherent states or wavelets, indexed by points $x \in X$, as the orbit of η under G, though the representation U and the section σ [10]-[12]: $S_\sigma = \eta_{\sigma(x)} = U(\sigma(x))\eta | x \in X$. So, coherent states or wavelets are simply the elements of the orbit under U of a fixed vector η in representation space. We have the following fundamental properties: **1.**Overcompleteness: the set S_σ is total in $\mathcal{H} : (S_\sigma)^\perp = 0$. **2.** Resolution property: the square integrability condition may be represented as a resolution relation: $\int_X |\eta_\sigma(x) >< \eta_{\sigma(x)}|\mathrm{d}\nu(x) = A_\sigma$, where A_σ is a bounded, positive operator with a densely defined inverse. Define the linear map $W_\eta : \mathcal{H} \to L^2(X, \mathrm{d}\nu), (W_\eta\Phi)(x) =< \eta_{\sigma(x)}|\Phi >$. Then the range H_η of W_η is complete with respect to the scalar product $< \Phi|\Psi >_\eta =< \Phi|W_\eta A_\sigma^{-1}W_\eta^{-1}\Psi >$ and W_η is unitary operator from $\mathcal{H}$ onto $\mathcal{H}_\eta$. W_η is Continuous Wavelet Transform (CWT). **3.** Reproducing kernel. The orthogonal projection from $L^2(X, \mathrm{d}\nu)$ onto $\mathcal{H}_\eta$ is an integral operator K_σ and H_η is a reproducing kernel Hilbert space of functions:

$\Phi(x) = \int_X K_\sigma(x,y)\Phi(y)\mathrm{d}\nu(y), \quad \forall\Phi \in \mathcal{H}_\eta$. The kernel is given explicitly by $K_\sigma(x,y) =< \eta_{\sigma(x)} A_\sigma^{-1}\eta_{\sigma(y)} >$, if $\eta_{\sigma(y)} \in D(A_\sigma^{-1})$, $\forall y \in X$. So, the function $\Phi \in L^2(X, \mathrm{d}\nu)$ is a wavelet transform (WT) iff it satisfies this reproducing relation. **4.** Reconstruction formula. The WT W_η may be inverted on its range by the adjoint operator, $W_\eta^{-1} = W_\eta^*$ on $\mathcal{H}_\eta$ to obtain for $\eta_{\sigma(x)} \in D(A_\sigma^{-1}), \forall x \in X$ $W_\eta^{-1}\Phi = \int_X \Phi(x) A_\sigma^{-1}\eta_{\sigma(x)}\mathrm{d}\nu(x)$, $\Phi \in \mathcal{H}_\eta$. This is inverse WT. If A_σ^{-1} is bounded then S_σ is called a frame, if $A_\sigma = \lambda I$ then S_σ is called a tight frame. This two cases are generalization of a simple case, when S_σ is an (ortho)basis.

The most simple cases of this construction are:
1. $H = \{e\}$. This is the standard construction of WT over a locally compact group. It should be noted that the square integrability of U is equivalent to U belonging to the discrete series. The most simple example is related to the affine $(ax+b)$ group and yields the usual one-dimensional wavelet analysis $[\pi(b,a)f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right)$. For $G = SIM(2) = \mathbf{R}^2 \bowtie (\mathbf{R}_*^+ \times SO(2))$, the similitude group of the plane, we have the corresponding two-dimensional wavelets. **2.** $H = H_\eta$, the isotropy (up to a phase) subgroup of η: this is the case of the Gilmore-Perelomov CS. Some cases of group G are: **a).** Semisimple groups, such as SU(N), SU(N|M), SU(p,q), Sp(N,**R**). **b).** the Weyl-Heisenberg group G_{WH} which leads to the Gabor functions, i.e. canonical (oscillator)coherent states associated with windowed Fourier transform or Gabor transform (see also part 6): $[\pi(q,p,\varphi)f](x) = \exp(i\mu(\varphi - p(x-q))f(x-q)$. In this case H is the center of G_{WH}. In both cases time-frequency plane corresponds to the phase space of group representation. **c).** The similitude group SIM(n) of $\mathbf{R}^n\,(n \geq 3)$: for $H = SO(n-1)$ we have the axisymmetric n-dimensional wavelets. **d).** Also we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group [12]. **e).** Relativity groups. In a nonrelativistic setup, the natural kinematical group is the (extended) Galilei group. Also we may adds independent space and time dilations and obtain affine Galilei group. If we restrict the dilations by the relation $a_0 = a^2$, where a_0, a are the time and space dilation we obtain the Galilei-Schrodinger group, invariance group of both Schrodinger and heat equations. We consider these examples in the next section. In the same way we may consider as kinematical group the Poincare group. When $a_0 = a$ we have affine Poincare or Weyl-Poincare group. Some useful generalization of that affinization construction we consider for the case of hidden metaplectic structure in part 6. But the usual representation is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in our case) restores square – integrability: $G \longrightarrow$ homogeneous space. Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (e) analogously. In next part we consider construction of invariant bases.

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