Variational Approach in Wavelet Framework to Polynomial Approximations of Nonlinear Accelerator Problems

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Abstract. In this paper we present applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. According to variational approach in the general case we have the solution as a multiresolution (multiscales) expansion in the base of compactly supported wavelet basis. We give extension of our results to the cases of periodic orbital particle motion and arbitrary variable coefficients. Then we consider more flexible variational method which is based on biorthogonal wavelet approach. Also we consider different variational approach, which is applied to each scale.

I INTRODUCTION

This is the first part of our two presentation in which we consider applications of methods from wavelet analysis to nonlinear accelerator physics problems. This is a continuation of results from [1]-[6], which is based on our approach to investigation of nonlinear problems – general, with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum, which are considered in the framework of local (nonlinear) Fourier analysis, or wavelet analysis. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases.

We consider application of multiresolution representation to general nonlinear dynamical system with polynomial type of nonlinearities. In part II we consider this very useful approximation in the cases of orbital motion in storage rings, a particle in the multipolar field, effects of insertion devices on beam dynamics, spin orbital motion. Starting in part III A from variational formulation of initial dynamical problem we construct via multiresolution analysis (part III B) explicit representation for all dynamical variables in the base of compactly supported

(Daubechies) wavelets. Our solutions (part III C) are parametrized by solutions of a number of reduced algebraical problems one from which is nonlinear with the same degree of nonlinearity and the rest are the linear problems which correspond to particular method of calculation of scalar products of functions from wavelet bases and their derivatives. Then we consider further extension of our previous results. In part V we consider modification of our construction to the periodic case, in part VI we consider generalization of our approach to variational formulation in the biorthogonal bases of compactly supported wavelets and in part VII to the case of variable coefficients. In part IV we consider the different variational approach which is based on ideas of para-products (A) and approximation for multiresolution approach, which gives us possibility for computations in each scale separately (B).

II PROBLEMS AND APPROXIMATIONS

Below we consider a number of examples of nonlinear accelerator physics problems which are from the formal mathematical point of view not more than nonlinear differential equations with polynomial nonlinearities and variable coefficients.

A Orbital Motion in Storage Rings

We consider as the main example the particle motion in storage rings in standard approach, which is based on consideration in [7]. Starting from Hamiltonian, which described classical dynamics in storage rings

$$\mathcal{H}(\vec{r},\vec{P},t)=c\{\pi^2+m_0^2c^2\}^{1/2}+e\phi \tag{1}$$

and using Serret–Frenet parametrization, we have the following Hamiltonian for orbital motion in machine coordinates:

$$\begin{aligned}\mathcal{H}(x,p_x,z,p_z,\sigma,p_\sigma;s)=p_\sigma-[1+f(p_\sigma]\cdot[1+K_x\cdot x+K_z\cdot z]\times\\ \left\{1-\frac{[p_x+H\cdot z]^2+[p_z-H\cdot x]^2}{[1+f(p_\sigma)]^2}\right\}^{1/2}\\ +\frac{1}{2}\cdot[1+K_x\cdot x+K_z\cdot z]^2-\frac{1}{2}\cdot g\cdot(z^2-x^2)-N\cdot xz\\ +\frac{\lambda}{6}\cdot(x^3-3xz^2)+\frac{\mu}{24}\cdot(z^4-6x^2z^2+x^4)\\ +\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\end{aligned} \tag{2}$$

Then, after standard manipulations with truncation of power series expansion of square root we arrive to the following approximated Hamiltonian for particle motion:

$$
\begin{aligned}
\mathcal{H} = {} & \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]} + p_\sigma - [1 + K_x\cdot x + K_z\cdot z] \\
& \cdot f(p_\sigma) + \frac{1}{2}\cdot[K_x^2 + g]\cdot x^2 + \frac{1}{2}\cdot[K_z^2 - g]\cdot z^2 - N\cdot xz + \\
& \frac{\lambda}{6}\cdot(x^3 - 3xz^2) + \frac{\mu}{24}\cdot(z^4 - 6x^2z^2 + x^4) \\
& + \frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right]
\end{aligned}
\tag{3}
$$

and the corresponding equations of motion:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{ds}}x &= \frac{\partial\mathcal{H}}{\partial p_x} = \frac{p_x + H\cdot z}{[1+f(p_\sigma)]}; \\
\frac{\mathrm{d}}{\mathrm{ds}}p_x &= -\frac{\partial\mathcal{H}}{\partial x} = \frac{[p_z - H\cdot x]}{[1+f(p_\sigma)]}\cdot H - [K_x^2 + g]\cdot x + N\cdot z + \\
&\qquad K_x\cdot f(p_\sigma) - \frac{\lambda}{2}\cdot(x^2 - z^2) - \frac{\mu}{6}(x^3 - 3xz^2); \\
\frac{\mathrm{d}}{\mathrm{ds}}z &= \frac{\partial\mathcal{H}}{\partial p_z} = \frac{p_z - H\cdot x}{[1+f(p_\sigma)]}; \\
\frac{\mathrm{d}}{\mathrm{ds}}p_z &= -\frac{\partial\mathcal{H}}{\partial z} = -\frac{[p_x + H\cdot z]}{[1+f(p_\sigma)]}\cdot H - [K_z^2 - g]\cdot z + N\cdot x + \\
&\qquad K_z\cdot f(p_\sigma) - \lambda\cdot xz - \frac{\mu}{6}(z^3 - 3x^2z); \\
\frac{\mathrm{d}}{\mathrm{ds}}\sigma &= \frac{\partial\mathcal{H}}{\partial p_\sigma} = 1 - [1 + K_x\cdot x + K_z\cdot z]\cdot f'(p_\sigma) - \\
&\qquad \frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma) \\
\frac{\mathrm{d}}{\mathrm{ds}}p_\sigma &= -\frac{\partial\mathcal{H}}{\partial\sigma} = \frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right]
\end{aligned}
\tag{4}
$$

Then we use series expansion of function $f(p_\sigma)$ from [7]:

$$
f(p_\sigma) = f(0) + f'(0)p_\sigma + f''(0)\frac{1}{2}p_\sigma^2 + \ldots = p_\sigma - \frac{1}{\gamma_0^2}\cdot\frac{1}{2}p_\sigma^2 + \ldots \tag{5}
$$

and the corresponding expansion of RHS of equations (4). In the following we take into account only an arbitrary polynomial (in terms of dynamical variables) expressions and neglecting all nonpolynomial types of expressions, i.e. we consider such approximations of RHS, which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variable s ("time" in our case, if we consider our system of equations as dynamical problem).

B Particle in the Multipolar Field

The magnetic vector potential of a magnet with $2n$ poles in Cartesian coordinates is

$$A = \sum_n K_n f_n(x, y), \tag{6}$$

where f_n is a homogeneous function of x and y of order n.

The real and imaginary parts of binomial expansion of

$$f_n(x, y) = (x + iy)^n \tag{7}$$

correspond to regular and skew multipoles. The cases $n = 2$ to $n = 5$ correspond to low-order multipoles: quadrupole, sextupole, octupole, decapole.

Then we have in particular case the following equations of motion for single particle in a circular magnetic lattice in the transverse plane (x, y) ([8] for designation):

$$\frac{\mathrm{d}^2\mathrm{x}}{\mathrm{ds}^2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right) x = \mathcal{R}e\left[\sum_{n\geq 2} \frac{k_n(s) + ij_n(s)}{n!} \cdot (x + iy)^n\right] \tag{8}$$
$$\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{ds}^2} + k_1(s)y = -\mathcal{J}m\left[\sum_{n\geq} \frac{k_n(s) + ij_n(s)}{n!} \cdot (x + iy)^n\right]$$

and the corresponding Hamiltonian:

$$H(x, p_x, y, p_y, s) = \frac{p_x^2 + p_y^2}{2} + \left(\frac{1}{\rho(s)^2} - k_1(s)\right) \cdot \frac{x^2}{2} + k_1(s)\frac{y^2}{2} \tag{9}$$
$$- \mathcal{R}e\left[\sum_{n\geq 2} \frac{k_n(s) + ij_n(s)}{(n+1)!} \cdot (x + iy)^{(n+1)}\right]$$

Then we may take into account arbitrary but finite number in expansion of RHS of Hamiltonian (9) and from our point of view the corresponding Hamiltonian equations of motions are also not more than nonlinear ordinary differential equations with polynomial nonlinearities and variable coefficients.

C Effects of Insertion Devices on Beam Dynamics

Assuming a sinusoidal field variation, we may consider according to [9] the analytical treatment of the effects of insertion devices on beam dynamics. One of the major detrimental aspects of the installation of insertion devices is the resulting reduction of dynamic aperture. Introduction of non-linearities leads to enhancement of the amplitude-dependent tune shifts and distortion of phase space. The nonlinear fields will produce significant effects at large betatron amplitudes.

The components of the insertion device magnetic field used for the derivation of equations of motion are as follows:

$$\begin{aligned} B_x &= \frac{k_x}{k_y} \cdot B_0 \sinh(k_x x) \sinh(k_y y) \cos(kz) \\ B_y &= B_0 \cosh(k_x x) \cosh(k_y y) \cos(kz) \\ B_z &= -\frac{k}{k_y} B_0 \cosh(k_x x) \sinh(k_y y) \sin(kz), \end{aligned} \tag{10}$$

with $k_x^2 + k_y^2 = k^2 = (2\pi/\lambda)^2$, where λ is the period length of the insertion device, B_0 is its magnetic field, ρ is the radius of the curvature in the field B_0. After a canonical transformation to change to betatron variables, the Hamiltonian is averaged over the period of the insertion device and hyperbolic functions are expanded to the fourth order in x and y (or arbitrary order).

Then we have the following Hamiltonian:

$$\begin{aligned} H = &\frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{4k^2\rho^2}[k_x^2 x^2 + k_y^2 y^2] \\ &+ \frac{1}{12k^2\rho^2}[k_x^4 x^4 + k_y^4 y^4 + 3k_x^2 k^2 x^2 y^2] \\ &- \frac{\sin(ks)}{2k\rho}[p_x(k_x^2 x^2 + k_y^2 y^2) - 2k_x p_y xy] \end{aligned} \tag{11}$$

We have in this case also nonlinear (polynomial with degree 3) dynamical system with variable (periodic) coefficients. As related cases we may consider wiggler and undulator magnets. We have in horizontal $x - s$ plane the following equations

$$\ddot{x} = -\dot{s}\frac{e}{m\gamma} B_z(s) \tag{12}$$

$$\ddot{s} = \dot{x}\frac{e}{m\gamma} B_z(s),$$

where magnetic field has periodic dependence on s and hyperbolic on z.

D Spin-Orbital Motion

Let us consider the system of equations for orbital motion

$$\frac{\mathrm{dq}}{\mathrm{dt}} = \frac{\partial H_{orb}}{\partial p}, \qquad \frac{\mathrm{dp}}{\mathrm{dt}} = -\frac{\partial H_{orb}}{\partial q} \tag{13}$$

and Thomas-BMT equation for classical spin vector (see [10] for designation)

$$\frac{\mathrm{ds}}{\mathrm{dt}} = w \times s \tag{14}$$

Here,

$$H_{orb} = c\sqrt{\pi^2 + m_0c^2} + e\Phi, \tag{15}$$

$$w = -\frac{e}{m_0\gamma c}\left((1+\gamma G)\vec{B} - \frac{G(\vec{\pi}\cdot\vec{B})\vec{\pi}}{m_0^2c^2(1+\gamma)} - \frac{1}{m_0 c}\left(G + \frac{1}{1+\gamma}\right)[\pi \times E]\right),$$

$q = (q_1, q_2, q_3)$, $p = (p_1, p_2, p_3)$ are canonical position and momentum, $s = (s_1, s_2, s_3)$ is the classical spin vector of length $\hbar/2$, $\pi = (\pi_1, \pi_2, \pi_3)$ is kinetic momentum vector. We may introduce in 9-dimensional phase space $z = (q, p, s)$ the Poisson brackets

$$\{f(z), g(z)\} = f_q g_p - f_p g_q + [f_s \times g_s]\cdot s \tag{16}$$

and the corresponding Hamiltonian equations:

$$\frac{\mathrm{dz}}{\mathrm{dt}} = \{z, H\}, \tag{17}$$

with Hamiltonian

$$H = H_{orb}(q, p, t) + w(q, p, t)\cdot s. \tag{18}$$

More explicitly we have

$$\begin{aligned}
\frac{\mathrm{dq}}{\mathrm{dt}} &= \frac{\partial H_{orb}}{\partial p} + \frac{\partial(w\cdot s)}{\partial p}\\
\frac{\mathrm{dp}}{\mathrm{dt}} &= -\frac{\partial H_{orb}}{\partial q} - \frac{\partial(w\cdot s)}{\partial q}\\
\frac{\mathrm{ds}}{\mathrm{dt}} &= [w \times s]
\end{aligned} \tag{19}$$

We will consider this dynamical system also in our second paper in this volume via invariant approach, based on consideration of Lie-Poison structures on semidirect products of groups.

But from the point of view which we used in this paper we may consider the similar approximations as in preceding examples and then we also arrive to some type of polynomial dynamics.

III POLYNOMIAL DYNAMICS

The first main part of our consideration is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. We have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one is nonlinear and the second is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform (FWT), Stationary Subdivision Schemes (SSS), the method of Connection Coefficients (CC).

A Variational Method

Our problems may be formulated as the systems of ordinary differential equations

$$\mathrm{dx_i/dt} = f_i(x_j, t), \quad (i, j = 1, ..., n) \tag{20}$$

with fixed initial conditions $x_i(0)$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \leq t \leq 1$. Let us consider a set of functions

$$\Phi_i(t) = x_i \mathrm{dy_i/dt} + f_i y_i \tag{21}$$

and a set of functionals

$$F_i(x) = \int_0^1 \Phi_i(t) dt - x_i y_i \mid_0^1, \tag{22}$$

where $y_i(t)(y_i(0) = 0)$ are dual variables. It is obvious that the initial system and the system

$$F_i(x) = 0 \tag{23}$$

are equivalent. In part VI we consider more general approach, which is based on possibility taking into account underlying symplectic structure and on more useful and flexible analytical approach, related to bilinear structure of initial functional.

Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \tag{24}$$

where because of initial conditions we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (24):

$$\sum_k \mu_{kr} \lambda_i^k - \gamma_i^r(\lambda_j) = 0 \tag{25}$$

Its coefficients are

$$\mu_{kr} = \int_0^1 \varphi_k'(t) \varphi_r(t) \mathrm{d}t, \quad \gamma_i^r = \int_0^1 f_i(x_j, t) \varphi_r(t) \mathrm{d}t. \tag{26}$$

Now, when we solve system (25) and determine unknown coefficients from formal expansion (24) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (24) with N terms then we have from (25) the system of $N \times n$ algebraical equations and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (polynomial) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^{N} \lambda_i^k X_k(t), \tag{27}$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical problem (25). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (25). The first main problem is a problem of computations of coefficients of reduced algebraical system. As we will see, these problems may be explicitly solved in wavelet approach.

Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (27), where $X_k(t)$ are basis functions and λ_k^i are roots of reduced system of equations. In our first wavelet case $X_k(t)$ are obtained via multiresolution expansions and represented by compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (25) with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method to give the reduction from differential to algebraical system of equations we need compute the objects γ_a^j and μ_{ji}, which are constructed from objects:

$$\begin{aligned} \sigma_i &\equiv \int_0^1 X_i(\tau)\mathrm{d}\tau, \\ \nu_{ij} &\equiv \int_0^1 X_i(\tau)X_j(\tau)\mathrm{d}\tau, \\ \mu_{ji} &\equiv \int X_i'(\tau)X_j(\tau)\mathrm{d}\tau, \\ \beta_{klj} &\equiv \int_0^1 X_k(\tau)X_l(\tau)X_j(\tau)\mathrm{d}\tau \end{aligned} \tag{28}$$

for the simplest case of Riccati systems, where degree of nonlinearity equals to two. For the general case of arbitrary n we have analogous to (28) iterated integrals with the degree of monomials in integrand which is one more bigger than degree of initial system.

B Wavelet Framework

Our constructions are based on multiresolution approach. Because affine group of translation and dilations is inside the approach, this method resembles the action of a microscope. We have contribution to final result from each scale of resolution from the whole infinite scale of spaces. More exactly, the closed subspace $V_j (j \in \mathbf{Z})$ corresponds to level j of resolution, or to scale j. We consider a r-regular multiresolution analysis of $L^2(\mathbf{R}^n)$ (of course, we may consider any different functional space) which is a sequence of increasing closed subspaces V_j:

$$...V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset ... \tag{29}$$

satisfying the following properties:

$$
\begin{aligned}
&\bigcap_{j\in\mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}^n),\\
&f(x)\in V_j <=> f(2x)\in V_{j+1},\\
&f(x)\in V_0 <=> f(x-k)\in V_0, \quad ,\forall k\in \mathbf{Z}^n.
\end{aligned} \tag{30}
$$

There exists a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x)=\varphi(x-k), k\in \mathbf{Z}^n\}$ forms a Riesz basis for V_0.

The function φ is regular and localized: φ is C^{r-1}, $\varphi^{(r-1)}$ is almost everywhere differentiable and for almost every $x\in\mathbf{R}^n$, for every integer $\alpha\le r$ and for all integer p there exists constant C_p such that

$$
\mid \partial^\alpha\varphi(x)\mid\le C_p(1+|x|)^{-p} \tag{31}
$$

Let $\varphi(x)$ be a scaling function, $\psi(x)$ is a wavelet function and $\varphi_i(x)=\varphi(x-i)$. Scaling relations that define φ,ψ are

$$
\varphi(x)=\sum_{k=0}^{N-1} a_k\varphi(2x-k)=\sum_{k=0}^{N-1} a_k\varphi_k(2x), \tag{32}
$$

$$
\psi(x)=\sum_{k=-1}^{N-2}(-1)^k a_{k+1}\varphi(2x+k). \tag{33}
$$

Let indices ℓ, j represent translation and scaling, respectively and

$$
\varphi_{jl}(x)=2^{j/2}\varphi(2^jx-\ell) \tag{34}
$$

then the set $\{\varphi_{j,k}\}, k\in\mathbf{Z}^n$ forms a Riesz basis for V_j. The wavelet function ψ is used to encode the details between two successive levels of approximation. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}:

$$
V_{j+1}=V_j\bigoplus W_j. \tag{35}
$$

Then just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$, where

$$
\psi_{jk}(x)=2^{j/2}\psi(2^jx-k). \tag{36}
$$

All expansions which we used are based on the following properties:

$$
\begin{aligned}
&\{\psi_{jk}\}, \quad j,k\in\mathbf{Z} \quad \text{is a Hilbertian basis of } L^2(\mathbf{R})\\
&\{\varphi_{jk}\}_{j\ge0,k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}),\\
&L^2(\mathbf{R})=\overline{V_0\bigoplus_{j=0}^{\infty} W_j},\\
&\text{or}\\
&\{\varphi_{0,k},\psi_{j,k}\}_{j\ge0,k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}).
\end{aligned} \tag{37}
$$

C Wavelet Computations

Now we give construction for computations of objects (28) in the wavelet case. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$.

Let be $f:\mathbf{R}\longrightarrow\mathbf{C}$ and the wavelet expansion is

$$f(x)=\sum_{\ell\in\mathbf{Z}}c_\ell\varphi_\ell(x)+\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}}c_{jk}\psi_{jk}(x) \tag{38}$$

If in formulae (38) $c_{jk}=0$ for $j\geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x)=\sum\limits_{\ell\in\mathbf{Z}}c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0,N-1]$ and $N/2$ vanishing moments. There exists $\lambda>0$ such that DN has λN continuous derivatives; for small $N,\lambda\geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi_\ell^n=\mathrm{d}^n\varphi_\ell(\mathrm{x})/\mathrm{dx}^n$. We consider computation of the wavelet - Galerkin integrals. Let $f^d(x)$ be d-derivative of function $f(x)$, then we have $f^d(x)=\sum_\ell c_l\varphi_\ell^d(x)$, and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$

$$\begin{aligned}\phi_\ell^d(x)&=\sum_m\lambda_m\varphi_m(x), \qquad (39)\\ \lambda_m&=\int\limits_{-\infty}^{\infty}\varphi_\ell^d(x)\varphi_m(x)\mathrm{dx},\end{aligned}$$

where λ_m are wavelet-Galerkin integrals. The coefficients λ_m are 2-term connection coefficients. In general we need to find ($d_i\geq 0$)

$$\Lambda^{d_1d_2\ldots d_n}_{\ell_1\ell_2\ldots\ell_n}=\int\limits_{-\infty}^{\infty}\prod\varphi_{\ell_i}^{d_i}(x)dx \tag{40}$$

For Riccati case we need to evaluate two and three connection coefficients

$$\Lambda_\ell^{d_1d_2}=\int_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi_\ell^{d_2}(x)dx,\quad \Lambda^{d_1d_2d_3}=\int\limits_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi_\ell^{d_2}(x)\varphi_m^{d_3}(x)dx \tag{41}$$

According to CC method [11] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0,N-1]$. For a fixed triple (d_1,d_2,d_3) only some $\Lambda_{\ell m}^{d_1d_2d_3}$ are nonzero: $2-N\leq\ell\leq N-2,\quad 2-N\leq m\leq N-2,\quad |\ell-m|\leq N-2$. There are $M=3N^2-9N+7$ such pairs (ℓ,m). Let $\Lambda^{d_1d_2d_3}$ be an M-vector, whose components are numbers $\Lambda_{\ell m}^{d_1d_2d_3}$. Then we have the first reduced algebraical system : Λ satisfy the system of equations ($d=d_1+d_2+d_3$)

$$A\Lambda^{d_1d_2d_3}=2^{1-d}\Lambda^{d_1d_2d_3},\qquad A_{\ell,m;q,r}=\sum_p a_p a_{q-2\ell+p}a_{r-2m+p} \tag{42}$$

By moment equations we have created a system of $M+d+1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second reduced algebraical system gives us the 2-term connection coefficients:

$$A\Lambda^{d_1d_2}=2^{1-d}\Lambda^{d_1d_2},\quad d=d_1+d_2,\quad A_{\ell,q}=\sum_p a_p a_{q-2\ell+p} \tag{43}$$

For nonquadratic case we have analogously additional linear problems for objects (40). Solving these linear problems we obtain the coefficients of nonlinear algebraical system (25) and after that we obtain the coefficients of wavelet expansion (27). As a result we obtained the explicit time solution of our problem in the base of compactly supported wavelets. We use for modelling D6, D8, D10 functions and programs RADAU and DOPRI for testing.

In the following we consider extension of this approach to the case of periodic boundary conditions, the case of presence of arbitrary variable coefficients and more flexible biorthogonal wavelet approach.

IV EVALUATION OF NONLINEARITIES SCALE BY SCALE

A Para-product and Decoupling between Scales

But before we consider two different schemes of modification of our variational approach in the case when we consider different scales separately. For this reason we need to compute errors of approximations. The main problems come of course from nonlinear terms. We follow approach from [12].

Let P_j be projection operators on the subspaces $V_j, j\in\mathbf{Z}$:

$$\begin{aligned}P_j\ &:\ L^2(\mathbf{R})\to V_j \\ (P_jf)(x)&=\sum_k <f,\varphi_{j,k}>\varphi_{j,k}(x)\end{aligned} \tag{44}$$

and Q_j are projection operators on the subspaces W_j:

$$Q_j=P_{j-1}-P_j \tag{45}$$

So, for $u\in L^2(\mathbf{R})$ we have $u_j=P_ju$ and $u_j\in V_j$, where $\{V_j\}, j\in\mathbf{Z}$ is a multiresolution analysis of $L^2(\mathbf{R})$. It is obviously that we can represent u_0^2 in the following form:

$$u_0^2=2\sum_{j=1}^n(P_ju)(Q_ju)+\sum_{j=1}^n(Q_ju)(Q_ju)+u_n^2 \tag{46}$$

In this formula there is no interaction between different scales. We may consider each term of (46) as a bilinear mappings:

$$M^j_{VW} : V_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \tag{47}$$

$$M^j_{WW} : W_j \times W_j \to L^2(\mathbf{R}) = V_j \oplus_{j' \geq j} W_{j'} \tag{48}$$

For numerical purposes we need formula (46) with a finite number of scales, but when we consider limits $j \to \infty$ we have

$$u^2 = \sum_{j \in \mathbf{Z}} (2P_j u + Q_j u)(Q_j u), \tag{49}$$

which is para-product of Bony, Coifman and Meyer.

Now we need to expand (46) into the wavelet bases. To expand each term in (46) into wavelet basis, we need to consider the integrals of the products of the basis functions, e.g.

$$M^{j,j'}_{WWW}(k, k', \ell) = \int_{-\infty}^{\infty} \psi^j_k(x) \psi^j_{k'}(x) \psi^{j'}_{\ell}(x) \mathrm{dx}, \tag{50}$$

where $j' > j$ and

$$\psi^j_k(x) = 2^{-j/2} \psi(2^{-j} x - k) \tag{51}$$

are the basis functions. If we consider compactly supported wavelets then

$$M^{j,j'}_{WWW}(k, k', \ell) \equiv 0 \quad \text{for} \quad |k - k'| > k_0, \tag{52}$$

where k_0 depends on the overlap of the supports of the basis functions and

$$|M^r_{WWW}(k - k', 2^r k - \ell)| \leq C \cdot 2^{-r\lambda M} \tag{53}$$

Let us define j_0 as the distance between scales such that for a given ε all the coefficients in (53) with labels $r = j - j'$, $r > j_0$ have absolute values less than ε. For the purposes of computing with accuracy ε we replace the mappings in (47), (48) by

$$M^j_{VW} : V_j \times W_j \to V_j \oplus_{j \leq j' \leq j_0} W_{j'} \tag{54}$$

$$M^j_{WW} : W_j \times W_j \to V_j \oplus_{J \leq j' \leq j_0} W_{j'} \tag{55}$$

Since

$$V_j \oplus_{j \leq j' \leq j_0} W_{j'} = V_{j_0 - 1} \tag{56}$$

and

$$V_j \subset V_{j_0-1}, \qquad W_j \subset V_{j_0-1} \tag{57}$$

we may consider bilinear mappings (54), (55) on $V_{j_0-1} \times V_{j_0-1}$. For the evaluation of (54), (55) as mappings $V_{j_0-1} \times V_{j_0-1} \to V_{j_0-1}$ we need significantly fewer coefficients than for mappings (54), (55). It is enough to consider only coefficients

$$M(k, k', \ell) = 2^{-j/2} \int_{\infty}^{\infty} \varphi(x-k)\varphi(x-k')\varphi(x-\ell)\mathrm{dx}, \tag{58}$$

where $\varphi(x)$ is scale function. Also we have

$$M(k, k', \ell) = 2^{-j/2} M_0(k-\ell, k'-\ell), \tag{59}$$

where

$$M_0(p, q) = \int \varphi(x-p)\varphi(x-q)\varphi(x)\mathrm{dx} \tag{60}$$

Now as in section (3C) we may derive and solve a system of linear equations to find $M_0(p, q)$.

B Non-regular Approximation

We use wavelet function $\psi(x)$, which has k vanishing moments $\int x^k\psi(x)\mathrm{dx} = 0$, or equivalently $x^k = \sum c_\ell\varphi_\ell(x)$ for each k, $0 \le k \le K$.

Let P_j again be orthogonal projector on space V_j. By tree algorithm we have for any $u \in L^2(\mathbf{R})$ and $\ell \in \mathbf{Z}$, that the wavelet coefficients of $P_\ell(u)$, i.e. the set $\{< u, \psi_{j,k} >, j \le \ell - 1, k \in \mathbf{Z}\}$ can be compute using hierarchic algorithms from the set of scaling coefficients in V_ℓ, i.e. the set $\{< u, \varphi_{\ell,k} >, k \in \mathbf{Z}\}$ [13]. Because for scaling function φ we have in general only $\int \varphi(x)\mathrm{dx} = 1$, therefore we have for any function $u \in L^2(\mathbf{R})$:

$$\lim_{j\to\infty, k2^{-j}\to x} \mid 2^{j/2} < u, \varphi_{j,k} > -u(x) \mid = 0 \tag{61}$$

If the integer $n(\varphi)$ is the largest one such that

$$\int x^\alpha \varphi(x)\mathrm{dx} = 0 \qquad \text{for} \qquad 1 \le \alpha \le \mathrm{n} \tag{62}$$

then if $u \in C^{(n+1)}$ with $u^{(n+1)}$ bounded we have for $j \to \infty$ uniformly in k:

$$\mid 2^{j/2} < u, \varphi_{j,k} > -u(k2^{-j}) \mid = O(2^{-j(n+1)}). \tag{63}$$

Such scaling functions with zero moments are very useful for us from the point of view of time-frequency localization, because we have for Fourier component

$\hat{\Phi}(\omega)$ of them, that exists some $C(\varphi) \in \mathbf{R}$, such that for $\omega \to 0$ $\quad \hat{\Phi}(\omega) = 1 + C(\varphi) \mid \omega \mid^{2r+2}$ (remember, that we consider r-regular multiresolution analysis). Using such type of scaling functions lead to superconvergence properties for general Galerkin approximation [13]. Now we need some estimates in each scale for non-linear terms of type $u \mapsto f(u) = f \circ u$, where f is C^∞ (in previous and future parts we consider only truncated Taylor series action). Let us consider non regular space of approximation $\widetilde{V}$ of the form

$$\widetilde{V} = V_q \oplus \sum_{q \le j \le p-1} \widetilde{W_j}, \tag{64}$$

with $\widetilde{W_j} \subset W_j$. We need efficient and precise estimate of $f \circ u$ on $\widetilde{V}$. Let us set for $q \in \mathbf{Z}$ and $u \in L^2(\mathbf{R})$

$$\prod f_q(u) = 2^{-q/2} \sum_{k \in \mathbf{Z}} f(2^{q/2} < u, \varphi_{q,k} >) \cdot \varphi_{q,k} \tag{65}$$

We have the following important for us estimation (uniformly in q) for $u, f(u) \in H^{(n+1)}$ [13]:

$$\|P_q\left(f(u)\right) - \prod f_q(u)\|_{L^2} = O\left(2^{-(n+1)q}\right) \tag{66}$$

For non regular spaces (64) we set

$$\prod f_{\widetilde{V}}(u) = \prod f_q(u) + \sum_{\ell = q, p-1} P_{\widetilde{W_j}} \prod f_{\ell+1}(u) \tag{67}$$

Then we have the following estimate:

$$\|P_{\widetilde{V}}\left(f(u)\right) - \prod f_{\widetilde{V}}(u)\|_{L^2} = O(2^{-(n+1)q}) \tag{68}$$

uniformly in q and $\widetilde{V}$ (64).

This estimate depends on q, not p, i.e. on the scale of the coarse grid, not on the finest grid used in definition of $\widetilde{V}$. We have for total error

$$\|f(u) - \prod f_{\widetilde{V}}(u)\| = \|f(u) - P_{\widetilde{V}}(f(u))\|_{L^2} + \|P_{\widetilde{V}}(f(u) - \prod f_{\widetilde{V}}(u))\|_{L^2} \tag{69}$$

and since the projection error in $\widetilde{V}$: $\|f(u) - P_{\widetilde{V}}\left(f(u)\right)\|_{L^2}$ is much smaller than the projection error in V_q we have the improvement (68) of (66). In our concrete calculations and estimates it is very useful to consider our approximations in the particular case of c-structured space:

$$\widetilde{V} = V_q + \sum_{j=q}^{p-1} span\{\psi_{j,k}, k \in [2^{(j-1)} - c, 2^{(j-1)} + c] \mod 2^j\} \tag{70}$$

V VARIATIONAL WAVELET APPROACH FOR PERIODIC TRAJECTORIES

We start with extension of our approach to the case of periodic trajectories. The equations of motion corresponding to Hamiltonians (from part II) may also be formulated as a particular case of the general system of ordinary differential equations $dx_i/dt = f_i(x_j, t)$, $(i, j = 1, ..., n)$, $0 \leq t \leq 1$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time but with periodic boundary conditions. According to our variational approach we have the solution in the following form

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t), \qquad x_i(0) = x_i(1), \tag{71}$$

where λ_i^k are again the roots of reduced algebraical systems of equations with the same degree of nonlinearity and $\varphi_k(t)$ corresponds to useful type of wavelet bases (frames). It should be noted that coefficients of reduced algebraical system are the solutions of additional linear problem and also depend on particular type of wavelet construction and type of bases.

This linear problem is our second reduced algebraical problem. We need to find in general situation objects

$$\Lambda^{d_1 d_2 \dots d_n}_{\ell_1 \ell_2 \dots \ell_n} = \int_{-\infty}^{\infty} \prod \varphi^{d_i}_{\ell_i}(x) \mathrm{dx}, \tag{72}$$

but now in the case of periodic boundary conditions. Now we consider the procedure of their calculations in case of periodic boundary conditions in the base of periodic wavelet functions on the interval [0,1] and corresponding expansion (71) inside our variational approach. Periodization procedure gives us

$$\begin{aligned} \hat\varphi_{j,k}(x) &\equiv \sum_{\ell \in Z} \varphi_{j,k}(x - \ell) \qquad (73) \\ \hat\psi_{j,k}(x) &= \sum_{\ell \in Z} \psi_{j,k}(x - \ell) \end{aligned}$$

So, $\hat\varphi, \hat\psi$ are periodic functions on the interval [0,1]. Because $\varphi_{j,k} = \varphi_{j,k'}$ if $k = k' \mathrm{mod}(2^{\mathrm{j}})$, we may consider only $0 \leq k \leq 2^j$ and as consequence our multiresolution has the form $\bigcup_{j \geq 0} \hat V_j = L^2[0,1]$ with $\hat V_j = \mathrm{span}\{\hat\varphi_{\mathrm{j,k}}\}^{\mathrm{2j-1}}_{\mathrm{k=0}}$ [14]. Integration by parts and periodicity gives useful relations between objects (72) in particular quadratic case $(d = d_1 + d_2)$:

$$\Lambda^{d_1, d_2}_{k_1, k_2} = (-1)^{d_1} \Lambda^{0, d_2 + d_1}_{k_1, k_2}, \Lambda^{0,d}_{k_1,k_2} = \Lambda^{0,d}_{0,k_2-k_1} \equiv \Lambda^d_{k_2-k_1} \tag{74}$$

So, any 2-tuple can be represent by Λ^d_k. Then our second additional linear problem is reduced to the eigenvalue problem for $\{\Lambda^d_k\}_{0 \leq k \leq 2^j}$ by creating a system of 2^j

homogeneous relations in Λ_k^d and inhomogeneous equations. So, if we have dilation equation in the form $\varphi(x) = \sqrt{2}\sum_{k\in Z} h_k\varphi(2x-k)$, then we have the following homogeneous relations

$$\Lambda_k^d = 2^d \sum_{m=0}^{N-1}\sum_{\ell=0}^{N-1} h_m h_\ell \Lambda_{\ell+2k-m}^d, \tag{75}$$

or in such form $A\lambda^d = 2^d\lambda^d$, where $\lambda^d = \{\Lambda_k^d\}_{0\le k\le 2^j}$. Inhomogeneous equations are:

$$\sum_\ell M_\ell^d \Lambda_\ell^d = d!2^{-j/2}, \tag{76}$$

where objects $M_\ell^d(|\ell|\le N-2)$ can be computed by recursive procedure

$$M_\ell^d = 2^{-j(2d+1)/2}\tilde{M}_\ell^d, \quad \tilde{M}_\ell^k =< x^k, \varphi_{0,\ell} >= \sum_{j=0}^k \binom{k}{j} n^{k-j} M_0^j, \quad \tilde{M}_0^\ell = 1. \tag{77}$$

So, we reduced our last problem to standard linear algebraical problem. Then we use the same methods as in part III C. As a result we obtained for closed trajectories of orbital dynamics described by Hamiltonians from part II the explicit time solution (71) in the base of periodized wavelets (73).

VI VARIATIONAL APPROACH IN BIORTHOGONAL WAVELET BASES

Now we consider further generalization of our variational wavelet approach. In [1]-[3] we consider different types of variational principles which give us weak solutions of our nonlinear problems.

Before we consider the generalization of our wavelet variational approach to the symplectic invariant calculation of closed loops in Hamiltonian systems [3]. We also have the parametrization of our solution by some reduced algebraical problem but in contrast to the general case where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations [3]. But because integrand of variational functionals is represented by bilinear form (scalar product) it seems more reasonable to consider wavelet constructions [15] which take into account all advantages of this structure.

The action functional for loops in the phase space is [16]

$$F(\gamma) = \int_\gamma pdq - \int_0^1 H(t,\gamma(t))dt \tag{78}$$

The critical points of F are those loops γ, which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. By the way, all critical points of F are the saddle points of infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, (M,ω) is symplectic manifolds, $H: M\to R$, H is Hamiltonian, X_H is unique Hamiltonian vector field defined by

$$\omega(X_H(x),v)=-dH(x)(v),\quad v\in T_xM,\quad x\in M, \tag{79}$$

where ω is the symplectic structure. A T-periodic solution $x(t)$ of the Hamiltonian equations

$$\dot x = X_H(x)\quad \text{on } M \tag{80}$$

is a solution, satisfying the boundary conditions $x(T)=x(0), T>0$. Let us consider the loop space $\Omega=C^\infty(S^1,R^{2n})$, where $S^1=R/\mathbf{Z}$, of smooth loops in R^{2n}. Let us define a function $\Phi:\Omega\to R$ by setting

$$\Phi(x)=\int_0^1\frac{1}{2}<-J\dot x, x>dt-\int_0^1 H(x(t))dt,\quad x\in\Omega \tag{81}$$

The critical points of Φ are the periodic solutions of $\dot x = X_H(x)$. Computing the derivative at $x\in\Omega$ in the direction of $y\in\Omega$, we find

$$\Phi'(x)(y)=\frac{d}{d\epsilon}\Phi(x+\epsilon y)|_{\epsilon=0}=\int_0^1<-J\dot x-\bigtriangledown H(x),y>dt \tag{82}$$

Consequently, $\Phi'(x)(y)=0$ for all $y\in\Omega$ iff the loop x satisfies the equation

$$-J\dot x(t)-\bigtriangledown H(x(t))=0, \tag{83}$$

i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0)=x(1)$, i.e. periodic of period 1. Periodic loops may be represented by their Fourier series:

$$x(t)=\sum_{k\in\mathbf{Z}} e^{k2\pi Jt}x_k,\quad x_k\in R^{2k}, \tag{84}$$

where J is quasicomplex structure. We give relations between quasicomplex structure and wavelets in our second paper in this volume (see also [3]). But now we need to take into account underlying bilinear structure via wavelets.

We started with two hierarchical sequences of approximations spaces [15]:

$$\dots V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\dots,\qquad \dots\widetilde V_{-2}\subset\widetilde V_{-1}\subset\widetilde V_0\subset\widetilde V_1\subset\widetilde V_2\dots, \tag{85}$$

and as usually, W_0 is complement to V_0 in V_1, but now not necessarily orthogonal complement. New orthogonality conditions have now the following form:

$$\widetilde W_0\perp V_0,\qquad W_0\perp\widetilde V_0,\qquad V_j\perp\widetilde W_j,\qquad \widetilde V_j\perp W_j \tag{86}$$

translates of ψ span W_0, translates of $\tilde{\psi}$ span $\widetilde{\mathrm{W}}_0$. Biorthogonality conditions are

$$< \psi_{jk}, \tilde{\psi}_{j'k'} >= \int_{-\infty}^{\infty} \psi_{jk}(x)\tilde{\psi}_{j'k'}(x)\mathrm{dx} = \delta_{\mathrm{kk'}}\delta_{\mathrm{jj'}}, \tag{87}$$

where $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. Functions $\varphi(x), \tilde{\varphi}(x-k)$ form dual pair:

$$< \varphi(x-k), \tilde{\varphi}(x-\ell) >= \delta_{kl}, \quad < \varphi(x-k), \tilde{\psi}(x-\ell) >= 0 \quad \text{for} \quad \forall k,\ \forall \ell. \tag{88}$$

Functions $\varphi, \tilde{\varphi}$ generate a multiresolution analysis. $\varphi(x-k)$, $\psi(x-k)$ are synthesis functions, $\tilde{\varphi}(x-\ell)$, $\tilde{\psi}(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining $V_j + W_j = V_{j+1}$, $\widetilde{V}_j + \widetilde{W}_j = \widetilde{V}_{j+1}$. These are direct sums but not orthogonal sums.

So, our representation for solution has now the form

$$f(t) = \sum_{j,k} \tilde{b}_{jk}\psi_{jk}(t), \tag{89}$$

where synthesis wavelets are used to synthesize the function. But $\tilde{b}_{jk}$ come from inner products with analysis wavelets. Biorthogonality yields

$$\tilde{b}_{\ell m} = \int f(t)\tilde{\psi}_{\ell m}(t)\mathrm{dt}. \tag{90}$$

So, now we can introduce this more complicated construction into our variational approach. We have modification only on the level of computing coefficients of reduced nonlinear algebraical system. This new construction is more flexible. Biorthogonal point of view is more stable under the action of large class of operators while orthogonal (one scale for multiresolution) is fragile, all computations are much more simpler and we accelerate the rate of convergence. In all types of Hamiltonian calculation, which are based on some bilinear structures (symplectic or Poissonian structures, bilinear form of integrand in variational integral) this framework leads to greater success.

VII VARIABLE COEFFICIENTS

In the case when we have situation when our problem is described a system of nonlinear (polynomial)differential equations, we need to consider extension of our previous approach which can take into account any type of variable coefficients (periodic, regular or singular). We can produce such approach if we add in our construction additional refinement equation, which encoded all information about variable coefficients [17]. According to our variational approach we need to compute integrals of the form

$$\int_D b_{ij}(t)(\varphi_1)^{d_1}(2^m t - k_1)(\varphi_2)^{d_2}(2^m t - k_2)\mathrm{dx}, \tag{91}$$

where now $b_{ij}(t)$ are arbitrary functions of time, where trial functions φ_1, φ_2 satisfy a refinement equations:

$$\varphi_i(t) = \sum_{k\in\mathbf{Z}} a_{ik}\varphi_i(2t - k) \tag{92}$$

If we consider all computations in the class of compactly supported wavelets then only a finite number of coefficients do not vanish. To approximate the non-constant coefficients, we need choose a different refinable function φ_3 along with some local approximation scheme

$$(B_\ell f)(x) := \sum_{\alpha\in\mathbf{Z}} F_{\ell,k}(f)\varphi_3(2^\ell t - k), \tag{93}$$

where $F_{\ell,k}$ are suitable functionals supported in a small neighborhood of $2^{-\ell}k$ and then replace b_{ij} in (91) by $B_\ell b_{ij}(t)$. In particular case one can take a characteristic function and can thus approximate non-smooth coefficients locally. To guarantee sufficient accuracy of the resulting approximation to (91) it is important to have the flexibility of choosing φ_3 different from φ_1, φ_2. In the case when D is some domain, we can write

$$b_{ij}(t)\mid_D = \sum_{0\le k\le 2^\ell} b_{ij}(t)\chi_D(2^\ell t - k), \tag{94}$$

where χ_D is characteristic function of D. So, if we take $\varphi_4 = \chi_D$, which is again a refinable function, then the problem of computation of (91) is reduced to the problem of calculation of integral

$$\begin{aligned} &H(k_1, k_2, k_3, k_4) = H(k) = \\ &\int_{\mathbf{R}^s} \varphi_4(2^j t - k_1)\varphi_3(2^\ell t - k_2)\varphi_1^{d_1}(2^r t - k_3)\varphi_2^{d_2}(2^s t - k_4)\mathrm{dx} \end{aligned} \tag{95}$$

The key point is that these integrals also satisfy some sort of refinement equation:

$$2^{-|\mu|}H(k) = \sum_{\ell\in\mathbf{Z}} b_{2k-\ell}H(\ell), \qquad \mu = d_1 + d_2. \tag{96}$$

This equation can be interpreted as the problem of computing an eigenvector. Thus, we reduced the problem of extension of our method to the case of variable coefficients to the same standard algebraical problem as in the preceding sections. So, the general scheme is the same one and we have only one more additional linear algebraic problem by which we in the same way can parameterize the solutions of corresponding problem.

Extended version and related results may be found in [1]-[6].

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