Symmetry, Hamiltonian Problems and Wavelets in Accelerator Physics

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Abstract. In this paper we consider applications of methods from wavelet analysis to nonlinear dynamical problems related to accelerator physics. In our approach we take into account underlying algebraical, geometrical and topological structures of corresponding problems.

I INTRODUCTION

This paper is the sequel of our first paper in this volume [1], in which we considered the applications of a number of analytical methods from nonlinear (local) Fourier analysis, or wavelet analysis, to nonlinear accelerator physics problems. This paper is the continuation of results from [2]–[7], which is based on our approach to investigation of nonlinear problems both general and with additional structures (Hamiltonian, symplectic or quasicomplex), chaotic, quasiclassical, quantum. Wavelet analysis is a relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (differential, integral, pseudodifferential) in such bases. In contrast with paper [1] in this paper we try to take into account before using power analytical approaches underlying algebraical, geometrical, topological structures related to kinematical, dynamical and hidden symmetry of physical problems. In this paper we give a review of a number of the corresponding problems and describe the key points of some possible methods by which we can find the full solutions of initial physical problem. We described a few concrete problems in [1, part II]. The most interesting case is the dynamics of spin-orbital motion [1, II D]. Related problems may be found in [8].

The content of this paper is not more than an attempt to extract the most complicated formal or mathematical or principal parts of the World of nonlinear accelerator physics, which is today beyond of mainstream in our opinion.

In part II we consider dynamical consequences of covariance properties regarding to relativity (kinematical) groups and continuous wavelet transform as a method for the solution of dynamical problems. In part II A we introduce the semidirect product structure, which allows us to consider from general point of view all relativity groups such as Euclidean, Galilei, Poincare. Then in part II B we consider the Lie-Poisson equations and obtain the manifestation of semiproduct structure of (kinematic) symmetry group on dynamical level. So, correct description of dynamics is a consequence of correct understanding of real symmetry of the concrete problem. In part II C we consider the technique for simplifications of dynamics related to semiproduct structure by using reduction to corresponding orbit structure. As result we have simplified Lie-Poisson equations. In part II D we consider the Lagrangian theory related to semiproduct structure and explicit form of variation principle and corresponding (semidirect) Euler-Poicare equations. In part II E we introduce a continuous wavelet transform and corresponding analytical technique which allow to consider covariant wavelet analysis. In part II F we consider in the particular case of affine Galilei group with the semiproduct structure also the corresponding orbit technique for constructing different types of invariant wavelet bases. In part III we consider instead of kinematical symmetry the dynamical symmetry. In part III A according to the orbit method and by using construction from the geometric quantization theory we construct the symplectic and Poisson structures associated with generalized wavelets by using metaplectic structure. We consider wavelet approach to the calculations of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems in part III B and for calculation of Arnold–Weinstein curves (closed loops) in Floer variational approach in part III C. In parts III D, III E we consider applications of very useful fast wavelet transform technique (part III F) to calculations in symplectic scale of spaces and to quasiclassical evolution dynamics. This method gives maximally sparse representation of (differential) operator that allows us to take into account contribution from each level of resolution. In part IV A we consider symplectic and Lagrangian structures for the case of discretization of flows by corresponding maps and in part IV B construction of corresponding solutions by applications of generalized wavelet approach which is based on generalization of multiresolution analysis for the case of maps.

II SEMIDIRECT PRODUCT, DYNAMICS, WAVELET REPRESENTATION

A Semidirect Product

Relativity groups such as Euclidean, Galilei or Poincare groups are the particular cases of semidirect product construction, which is very useful and simple general construction in the group theory [9]. We may consider as a basic example the Euclidean group $SE(3) = SO(3) \bowtie \mathbf{R}^3$, the semidirect product of rotations and translations. In general case we have $S = G \bowtie V$, where group G (Lie group or automorphisms group) acts on a vector space V and on its dual V^*. Let V be a vector space and G is the Lie group, which acts on the left by linear maps on V (G also acts on the left on its dual space V^*). The semidirect product $S = G \bowtie V$ is the Cartesian product $S = G \times V$ with group multiplication

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2), \tag{1}$$

where the action of $g \in G$ on $v \in V$ is denoted as gv. Of course, we can consider the corresponding definitions both in case of the right actions and in case, when G is a group of automorphisms of the vector space V. As we shall explain below both cases, Lie groups and automorphisms groups, are important for us.

So, the Lie algebra of S is the semidirect product Lie algebra, $s = \mathcal{G} \bowtie V$ with brackets

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1), \tag{2}$$

where the induced action of $\mathcal{G}$ by concatenation is denoted as $\xi_1 v_2$. Also we need expressions for adjoint and coadjoint actions for semidirect products. Let $(g, v) \in S = G \times V$, $\quad (\xi, u) \in s = \mathcal{G} \times V$, $(\mu, a) \in s^* = \mathcal{G}^* \times V^*$, $g\xi = Ad_g \xi$, $g\mu = Ad^*_{g^{-1}} \mu$, ga denotes the induced left action of g on a (the left action of G on V induces a left action on V^* — the inverse of the transpose of the action on V), $\rho_v : \mathcal{G} \to V$ is a linear map given by $\rho_v(\xi) = \xi v$, $\rho^*_v : V^* \to \mathcal{G}^*$ is its dual. Then these actions are given by simple concatenation:

$$\begin{aligned} (g, v)(\xi, u) &= (g\xi, gu - (g\xi)v), \\ (g, v)(\mu, a) &= (g\mu + \rho^*_v(ga), ga) \end{aligned} \tag{3}$$

Below we use the following notation: $\rho^*_v a = v \diamond a \in \mathcal{G}^*$ for $a \in V^*$, which is a bilinear operation in v and a. So, we have the coadjoint action:

$$(g, v)(\mu, a) = (g\mu + v \diamond (ga), ga). \tag{4}$$

Using concatenation notation for Lie algebra actions we have alternative definition of $v \diamond a \in \mathcal{G}^*$. For all $v \in V$, $a \in V^*$, $\eta \in \mathcal{G}$ we have

$$< \eta a, v >= - < v \diamond a, \eta > \tag{5}$$

B The Lie-Poisson Equations and Semiproduct Structure

Below we consider the manifestation of semiproduct structure of symmetry group on dynamical level. Let F, G be real valued functions on the dual space $\mathcal{G}^*$, $\mu \in \mathcal{G}^*$. Functional derivative of F at μ is the unique element $\delta F / \delta \mu \in \mathcal{G}$:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} [F(\mu + \epsilon \delta\mu) - F(\mu)] =< \delta\mu, \frac{\delta F}{\delta \mu} > \tag{6}$$

for all $\delta\mu \in \mathcal{G}^*$, $<,>$ is pairing between $\mathcal{G}^*$ and $\mathcal{G}$.

Define the $(\pm)$ Lie-Poisson brackets by

$$\{F,G\}_{\pm}(\mu) = \pm < \mu, [\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}] > \tag{7}$$

The Lie-Poisson equations, determined by

$$\dot{F} = \{F,H\} \tag{8}$$

read intrinsically

$$\dot{\mu} = \mp ad^*_{\partial H/\partial\mu}\mu. \tag{9}$$

For the left representation of G on V $\pm$ Lie-Poisson bracket of two functions $f,k: s^* \to \mathbf{R}$ is given by

$$\{f,k\}_{\pm}(\mu,a) = \pm < \mu, [\frac{\delta f}{\delta\mu}, \frac{\delta k}{\delta\mu}] > \pm < a, \frac{\delta f}{\delta\mu}\frac{\delta k}{\delta a} - \frac{\delta k}{\delta\mu}\frac{\delta f}{\delta a} >, \tag{10}$$

where $\delta f/\delta\mu \in \mathcal{G}$, $\delta f/\delta a \in V$ are the functional derivatives of f (6). The Hamiltonian vector field of $h : s^* \in \mathbf{R}$ has the expression

$$X_h(\mu,a) = \mp(ad^*_{\delta h/\delta\mu}\mu - \frac{\delta h}{\delta a} \diamond a, -\frac{\delta h}{\delta\mu}a). \tag{11}$$

Thus, Hamiltonian equations on the dual of a semidirect product are [9]:

$$\begin{aligned} \dot{\mu} &= \mp ad^*_{\delta h/\delta\mu}\mu \pm \frac{\delta h}{\delta a} \diamond a \\ \dot{a} &= \pm\frac{\delta h}{\delta\mu}a \end{aligned} \tag{12}$$

So, we can see the explicit difference between Poisson brackets (7) and (10) and the equations of motion (9) and (12), which come from the semiproduct structure.

C Reduction of Dynamics on Semiproduct

There is technique for reducing dynamics that is associated with the geometry of semidirect product reduction theorem[9]. Let us have a Hamiltonian on T^*G that is invariant under the isotropy G_{a_0} for $a_0 \in V^*$. The semidirect product reduction theorem states that reduction of T^*G by G_{a_0} gives reduced spaces that are simplectically diffeomorphic to coadjoint orbits in the dual of the Lie algebra of the semidirect product $(\mathcal{G} \bowtie V)^*$. If one reduces the semidirect group product $S = G \bowtie V$ in two stages, first by V and then by G one recovers this semidirect product reduction theorem. Thus, let $S = G \bowtie V$, choose $\sigma = (\mu, a) \in \mathcal{G}^* \times V^*$ and reduce T^*S by the action of S at σ giving the coadjoint orbit $\mathcal{O}_\sigma$ through $\sigma \in S^*$. There is a symplectic diffeomorphism between $\mathcal{O}_\sigma$ and the reduced space obtained be reducing T^*G by the subgroup G_a (the isotropy of G for its action on V^* at the point $a \in V^*$) at the point $\mu|\mathcal{G}_a$, where $\mathcal{G}_a$ is the Lie algebra of G_a.

Then we have the following procedure.

1. We start with a Hamiltonian H_{a_0} on T^*G that depends parametrically on a variable $a_0 \in V^*$.

2. The Hamiltonian regarded as a map: $T^*G \times V^* \to \mathbf{R}$ is assumed to be invariant on T^*G under the action of G on $T^*G \times V^*$.

3. The condition 2 is equivalent to the invariance of the function H defined on $T^*S = T^*G \times V \times V^*$ extended to be constant in the variable V under the action of the semidirect product.

4. By the semidirect product reduction theorem, the dynamics of H_{a_0} reduced by G_{a_0}, the isotropy group of a_0, is simplectically equivalent to Lie-Poisson dynamics on $s^* = \mathcal{G}^* \times V^*$.

5. This Lie-Poisson dynamics is given by equations (12) for the function $h(\mu, a) = H(\alpha_g, g^{-1}a)$, where $\mu = g^{-1}\alpha_g$.

D Lagrangian Theory, the Euler-Poincare Equations, Variational Approach on Semiproduct

Now we consider according to [9] Lagrangian side of a theory. This approach is based on variational principles with symmetry and is not dependent on Hamiltonian formulation, although it is demonstrated in [9] that this purely Lagrangian formulation is equivalent to the Hamiltonian formulation on duals of semidirect product (the corresponding Legendre transformation is a diffeomorphism).

We consider the case of the left representation and the left invariant Lagrangians (ℓ and L), which depend in additional on another parameter $a \in V^*$ (dynamical parameter), where V is representation space for the Lie group G and L has an invariance property related to both arguments. It should be noted that the resulting equations of motion, the Euler-Poincare equations, are not the Euler-Poincare equations for the semidirect product Lie algebra $\mathcal{G} \bowtie V^*$ or $\mathcal{G} \bowtie V$.

So, we have the following:

1. There is a left presentation of Lie group G on the vector space V and G acts in the natural way on the left on $TG \times V^* : h(v_g, a) = (hv_g, ha)$.

2. The function $L : TG \times V^* \in \mathbf{R}$ is the left G-invariant.

3. Let $a_0 \in V^*$, Lagrangian $L_{a_0} : TG \to \mathbf{R}$, $L_{a_0}(v_g) = L(v_g, a_0)$. L_{a_0} is left invariant under the lift to TG of the left action of G_{a_0} on G, where G_{a_0} is the isotropy group of a_0.

4. Left G-invariance of L permits us to define

$$\ell : \mathcal{G} \times V^* \to \mathbf{R} \tag{13}$$

by

$$\ell(g^{-1}v_g, g^{-1}a_0) = L(v_g, a_0). \tag{14}$$

This relation defines for any $\ell : \mathcal{G} \times V^* \to \mathbf{R}$ the left G-invariant function $L : TG \times V^* \to \mathbf{R}$.

5. For a curve $g(t) \in G$ let be

$$\xi(t) := g(t)^{-1}\dot{g}(t) \tag{15}$$

and define the curve $a(t)$ as the unique solution of the following linear differential equation with time dependent coefficients

$$\dot{a}(t) = -\xi(t)a(t), \tag{16}$$

with initial condition $a(0) = a_0$. The solution can be written as $a(t) = g(t)^{-1}a_0$.

Then we have four equivalent descriptions of the corresponding dynamics:

1. If a_0 is fixed then Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t)\mathrm{d}t = 0 \tag{17}$$

holds for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

2. $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

3. The constrained variational principle

$$\delta \int_{t_1}^{t_2} \ell(\xi(t), a(t))\mathrm{d}t = 0 \tag{18}$$

holds on $\mathcal{G} \times V^*$, using variations of ξ and a of the form $\delta\xi = \dot{\eta} + [\xi, \eta]$, $\delta a = -\eta a$, where $\eta(t) \in \mathcal{G}$ vanishes at the endpoints.

4. The Euler-Poincare equations hold on $\mathcal{G} \times V^*$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\ell}{\delta\xi} = ad^*_\xi \frac{\delta\ell}{\delta\xi} + \frac{\delta\ell}{\delta a} \diamond a \tag{19}$$

So, we may apply our wavelet methods either on the level of variational formulation (17) or on the level of Euler-Poincare equations (19).

E Continuous Wavelet Transform

Now we need take into account the Hamiltonian or Lagrangian structures related with systems (12) or (19). Therefore, we need to consider generalized wavelets, which allow us to consider the corresponding structures instead of compactly supported wavelet representation from paper [1].

In wavelet analysis the following three concepts are used now: 1). a square integrable representation U of a group G, 2). coherent states (CS) over G, 3). the wavelet transform associated to U. We consider now their unification [10], [11].

Let G be a locally compact group and U_a strongly continuous, irreducible, unitary representation of G on Hilbert space $\mathcal{H}$. Let H be a closed subgroup of G, $X = G/H$ with (quasi) invariant measure ν and $\sigma : X = G/H \to G$ is a Borel section in a principal bundle $G \to G/H$. Then we say that U is square integrable $mod(H, \sigma)$ if there exists a non-zero vector $\eta \in \mathcal{H}$ such that

$$0 < \int_X | < U(\sigma(x))\eta|\Phi > |^2 \mathrm{d}\nu(x) = < \Phi|A_\sigma\Phi > < \infty, \quad \forall\Phi \in \mathcal{H} \tag{20}$$

Given such a vector $\eta \in \mathcal{H}$ called admissible for (U, σ) we define the family of (covariant) coherent states or wavelets, indexed by points $x \in X$, as the orbit of η under G, though the representation U and the section σ [10], [11]

$$S_\sigma = \eta_{\sigma(x)} = U(\sigma(x))\eta|x \in X \tag{21}$$

So, coherent states or wavelets are simply the elements of the orbit under U of a fixed vector η in representation space. We have the following fundamental properties:

1. Overcompleteness:
   The set S_σ is total in $\mathcal{H} : (S_\sigma)^\perp = 0$

2. Resolution property:
   the square integrability condition (20) may be represented as a resolution relation:

   $$\int_X |\eta_\sigma(x) >< \eta_{\sigma(x)}|\mathrm{d}\nu(x) = A_\sigma, \tag{22}$$

   where A_σ is a bounded, positive operator with a densely defined inverse. Define the linear map

   $$W_\eta : \mathcal{H} \to L^2(X, \mathrm{d}\nu), (W_\eta\Phi)(x) = < \eta_{\sigma(x)}|\Phi > \tag{23}$$

   Then the range H_η of W_η is complete with respect to the scalar product $< \Phi|\Psi >_\eta = < \Phi|W_\eta A_\sigma^{-1} W_\eta^{-1}\Psi >$ and W_η is unitary operator from $\mathcal{H}$ onto $\mathcal{H}_\eta$. W_η is Continuous Wavelet Transform (CWT).

3. Reproducing kernel
   The orthogonal projection from $L^2(X, \mathrm{d}\nu)$ onto $\mathcal{H}_\eta$ is an integral operator K_σ and H_η is a reproducing kernel Hilbert space of functions:

$$\Phi(x) = \int_X K_\sigma(x,y)\Phi(y)\mathrm{d}\nu(y), \quad \forall\Phi \in \mathcal{H}_\eta. \tag{24}$$

   The kernel is given explicitly by $K_\sigma(x,y) =< \eta_{\sigma(x)}|A_\sigma^{-1}\eta_{\sigma(y)} >$, if $\eta_{\sigma(y)} \in D(A_\sigma^{-1})$, $\forall y \in X$. So, the function $\Phi \in L^2(X, \mathrm{d}\nu)$ is a wavelet transform (WT) iff it satisfies this reproducing relation.

4. Reconstruction formula.
   The WT W_η may be inverted on its range by the adjoint operator, $W_\eta^{-1} = W_\eta^*$ on $\mathcal{H}_\eta$ to obtain for $\eta_{\sigma(x)} \in D(A_\sigma^{-1})$, $\forall x \in X$

$$W_\eta^{-1}\Phi = \int_X \Phi(x)A_\sigma^{-1}\eta_{\sigma(x)}\mathrm{d}\nu(x), \quad \Phi \in \mathcal{H}_\eta. \tag{25}$$

   This is inverse WT.

If A_σ^{-1} is bounded then S_σ is called a frame, if $A_\sigma = \lambda I$ then S_σ is called a tight frame. This two cases are generalization of a simple case, when S_σ is an (ortho)basis.

The most simple cases of this construction are:
1. $H = \{e\}$. This is the standard construction of WT over a locally compact group. It should be noted that the square integrability of U is equivalent to U belonging to the discrete series. The most simple example is related to the affine $(ax + b)$ group and yields the usual one-dimensional wavelet analysis

$$[\pi(b,a)f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right). \tag{26}$$

For $G = SIM(2) = \mathbf{R}^2 \bowtie (\mathbf{R}_*^+ \times SO(2))$, the similitude group of the plane, we have the corresponding two-dimensional wavelets.

2. $H = H_\eta$, the isotropy (up to a phase) subgroup of η: this is the case of the Gilmore-Perelomov CS. Some cases of group G are:
a). Semisimple groups, such as SU(N), SU(N|M), SU(p,q), Sp(N,$\mathbf{R}$).
b). the Weyl-Heisenberg group G_{WH} which leads to the Gabor functions, i.e. canonical (oscillator)coherent states associated with windowed Fourier transform or Gabor transform (see also part III A):

$$[\pi(q,p,\varphi)f](x) = \exp(i\mu(\varphi - p(x-q))f(x-q) \tag{27}$$

In this case H is the center of G_{WH}. In both cases time-frequency plane corresponds to the phase space of group representation.
c). The similitude group SIM(n) of $\mathbf{R}^n (n \geq 3)$: for $H = SO(n-1)$ we have the axisymmetric n-dimensional wavelets.

d). Also we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group [11].
e). Relativity groups. In a nonrelativistic setup, the natural kinematical group is the (extended) Galilei group. Also we may adds independent space and time dilations and obtain affine Galilei group. If we restrict the dilations by the relation $a_0 = a^2$, where a_0, a are the time and space dilation we obtain the Galilei-Schrödinger group, invariance group of both Schrödinger and heat equations. We consider these examples in the next section. In the same way we may consider as kinematical group the Poincare group. When $a_0 = a$ we have affine Poincare or Weyl-Poincare group. Some useful generalization of that affinization construction we consider for the case of hidden metaplectic structure in section III A.

But the usual representation is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in our case) restores square – integrability: $G \longrightarrow$ homogeneous space.
Also, we have more general approach which allows to consider wavelets corresponding to more general groups and representations [12], [13].

Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (e) analogously.

F Bases for Solutions

We consider an important particular case of affine relativity group (relativity group combined with dilations) — affine Galilei group in n-dimensions. So, we have combination of Galilei group with independent space and time dilations: $G_{aff} = G_m \bowtie D_2$, where $D_2 = (\mathbf{R}_*^+)^2 \simeq \mathbf{R}^2$, G_m is extended Galilei group corresponding to mass parameter $m > 0$ (G_{aff} is noncentral extension of $G \bowtie D_2$ by $\mathbf{R}$, where G is usual Galilei group). Generic element of G_{aff} is $g = (\Phi, b_0, b; v; R, a_0, a)$, where $\Phi \in \mathbf{R}$ is the extension parameter in G_m, $b_0 \in \mathbf{R}$, $b \in \mathbf{R}^n$ are the time and space translations, $v \in \mathbf{R}^n$ is the boost parameter, $R \in SO(n)$ is a rotation and $a_0, a \in \mathbf{R}_*^+$ are time and space dilations. The actions of g on space-time is then $x \mapsto aRx + a_0vt + b$, $t \mapsto a_0t + b_0$, where $x = (x_1, x_2, ..., x_n)$. The group law is

$$\begin{aligned} gg' = (\Phi + \frac{a^2}{a_0}\Phi' + avRb' + \frac{1}{2}a_0v^2b_0', b_0 + a_0b_0', b + aRb' + a_0vb_0'; \qquad (28)\\ v + \frac{a}{a_0}Rv', RR'; a_aa_0', aa') \end{aligned}$$

It should be noted that D_2 acts nontrivially on G_m. Space-time wavelets associated to G_{aff} corresponds to unitary irreducible representation of spin zero. It may be obtained via orbit method. The Hilbert space is $\mathcal{H} = L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega)$, $k = (k_1, ..., k_n)$, where $\mathbf{R}^n \times \mathbf{R}$ may be identified with usual Minkowski space and we have for representation:

$$(U(g)\Psi)(k,\omega) = \sqrt{a_0 a^n}\mathrm{exp}i(m\Phi + kb - \omega b_0)\Psi(k',\omega'), \tag{29}$$

with $k' = aR^{-1}(k + mv)$, $\omega' = a_0(\omega - kv - \frac{1}{2}mv^2)$, $m' = (a^2/a_0)m$. Mass m is a coordinate in the dual of the Lie algebra and these relations are a part of coadjoint action of G_{aff}. This representation is unitary and irreducible but not square integrable. So, we need to consider reduction to the corresponding quotients $X = G/H$. We consider the case in which H={phase changes Φ and space dilations a}. Then the space $X = G/H$ is parametrized by points $\bar{x} = (b_0, b; v; R; a_0)$. There is a dense set of vectors $\eta \in \mathcal{H}$ admissible mod(H, σ_β), where σ_β is the corresponding section. We have a two-parameter family of functions β(dilations): $\beta(\bar{x}) = (\mu_0 + \lambda_) a_0)^{1/2}$, $\lambda_0, \mu_0 \in \mathbf{R}$. Then any admissible vector η generates a tight frame of Galilean wavelets

$$\eta_{\beta(\bar{x})}(k,\omega) = \sqrt{a_0(\mu_0 + \lambda_0 a_0)^{n/2}}\mathrm{e}^{i(kb-\omega b_0)}\eta(k',\omega'), \tag{30}$$

with $k' = (\mu_0 + \lambda_0 a)^{1/2}R^{-1}(k+mv)$, $\omega' = a_0(\omega - kv - mv^2/2)$. The simplest examples of admissible vectors (corresponding to usual Galilei case) are Gaussian vector: $\eta(k) \sim \exp(-k^2/2mu)$ and binomial vector: $\eta(k) \sim (1 + k^2/2mu)^{-\alpha/2}$, $\alpha > 1/2$, where u is a kind of internal energy. When we impose the relation $a_0 = a^2$ then we have the restriction to the Galilei-Schrödinger group $G_s = G_m \bowtie D_s$, where D_s is the one-dimensional subgroup of D_2. G_s is a natural invariance group of both the Schrödinger equation and the heat equation. The restriction to G_s of the representation (29) splits into the direct sum of two irreducible ones $U = U_+ \oplus U_-$ corresponding to the decomposition $L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\begin{aligned}\mathcal{H}_\pm &= L^2(D_\pm, \mathrm{d}k\mathrm{d}\omega \\ &= \{\psi \in L^2(\mathbf{R}^n \times \mathbf{R}, \mathrm{d}k\mathrm{d}\omega), \quad \psi(k,\omega) = 0 \quad for \quad \omega + k^2/2m = 0\}\end{aligned} \tag{31}$$

These two subspaces are the analogues of usual Hardy spaces on $\mathbf{R}$, i.e. the subspaces of (anti)progressive wavelets (see also below, part III A). The two representation $U_\pm$ are square integrable modulo the center. There is a dense set of admissible vectors η, and each of them generates a set of CS of Gilmore-Perelomov type. Typical wavelets of this kind are:
the Schrödinger-Marr wavelet:

$$\eta(x,t) = (i\partial_t + \frac{\triangle}{2m})\mathrm{e}^{-(x^2+t^2)/2} \tag{32}$$

the Schrödinger-Cauchy wavelet:

$$\psi(x,t) = (i\partial_t + \frac{\triangle}{2m})\frac{1}{(t+i)\prod_{j=1}^n(x_j + i)} \tag{33}$$

So, in the same way we can construct invariant bases with explicit manifestation of underlying symmetry for solving Hamiltonian (12) or Lagrangian (19) equations.

III SYMPLECTIC STRUCTURES, QUANTIZATION AND FAST WAVELET TRANSFORM

A Metaplectic Group and Representations

Let $Sp(n)$ be symplectic group, $Mp(n)$ be its unique two- fold covering – metaplectic group [14]. Let V be a symplectic vector space with symplectic form (,), then $R \oplus V$ is nilpotent Lie algebra - Heisenberg algebra:

$$[R, V] = 0, \quad [v, w] = (v, w) \in R, \quad [V, V] = R.$$

$Sp(V)$ is a group of automorphisms of Heisenberg algebra.

Let N be a group with Lie algebra $R \oplus V$, i.e. Heisenberg group. By Stone– von Neumann theorem Heisenberg group has unique irreducible unitary representation in which $1 \mapsto i$. Let us also consider the projective representation of simplectic group $Sp(V)$: $U_{g_1}U_{g_2} = c(g_1, g_2) \cdot U_{g_1 g_2}$, where c is a map: $Sp(V) \times Sp(V) \to S^1$, i.e. c is S^1-cocycle.

But this representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F = N' \bowtie Mp(V)$, $\quad \bowtie$ is semidirect product (we consider instead of $N = R \oplus V$ the $N' = S^1 \times V$, $\quad S^1 = (R/2\pi Z)$). Let V^* be dual to V, $G(V^*)$ be automorphism group of V^*.Then F is subgroup of $G(V^*)$, which consists of elements, which acts on V^* by affine transformations.
This is the key point!

Let $q_1, ..., q_n; p_1, ..., p_n$ be symplectic basis in V, $\alpha = pdq = \sum p_i dq_i$ and $d\alpha$ be symplectic form on V^*. Let M be fixed affine polarization, then for $a \in F$ the map $a \mapsto \Theta_a$ gives unitary representation of G: $\Theta_a : H(M) \to H(M)$

Explicitly we have for representation of N on H(M):

$$(\Theta_q f)^*(x) = e^{-iqx} f(x), \quad \Theta_p f(x) = f(x - p)$$

The representation of N on H(M) is irreducible. Let A_q, A_p be infinitesimal operators of this representation

$$A_q = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tq} - I], \quad A_p = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tp} - I],$$

$$\text{then} \qquad A_q f(x) = i(qx)f(x), \quad A_p f(x) = \sum p_j \frac{\partial f}{\partial x_j}(x)$$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group F is the algebra of all (nonhomogeneous) quadratic polynomials of (p,q) relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, ..., q_n,\ p_1, ..., p_n,\ q_i q_j, q_i p_j,\ p_i p_j, \quad i, j = 1, ..., n, \quad i \leq j,$

$$PB\ is \quad \{f,g\} = \sum \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \quad \text{and} \quad \{1,g\} = 0 \quad for \text{ all } g,$$

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i q_j, q_k\} = \delta_{ik} q_j, \quad \{p_i q_j, p_k\} = -\delta_{jk} p_i, \quad \{p_i p_j, p_k\} = 0,$$

$$\{p_i p_j, q_k\} = \delta_{ik} p_j + \delta_{jk} p_i, \quad \{q_i q_j, q_k\} = 0, \{q_i q_j, p_k\} = -\delta_{ik} q_j - \delta_{jk} q_i$$

so, we have the representation of basic elements $f \mapsto A_f : 1 \mapsto i, q_k \mapsto i x_k$,

$$p_l \mapsto \frac{\delta}{\delta x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2}\delta_{ij}, \qquad p_k p_l \mapsto \frac{1}{i}\frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto i x^k x^l$$

This gives the structure of the Poisson manifolds to representation of any (nilpotent) algebra or in other words to continuous wavelet transform.

The Segal-Bargman Representation. Let $z = 1/\sqrt{2} \cdot (p - iq), \quad \bar{z} = 1/\sqrt{2} \cdot (p + iq), \quad p = (p_1, ..., p_n), \quad F_n$ is the space of holomorphic functions of n complex variables with $(f, f) < \infty$, where

$$(f, g) = (2\pi)^{-n} \int f(z)\overline{g(z)} e^{-|z|^2} dp dq$$

Consider a map $U : H \to F_n$, where H is with real polarization, F_n is with complex polarization, then we have

$$(U\Psi)(a) = \int A(a,q)\Psi(q)dq, \qquad \text{where} \quad A(a,q) = \pi^{-n/4} e^{-1/2(a^2+q^2)+\sqrt{2}aq}$$

i.e. the Bargmann formula produce wavelets.We also have the representation of Heisenberg algebra on F_n :

$$U\frac{\partial}{\partial q_j}U^{-1} = \frac{1}{\sqrt{2}}\left(z_j - \frac{\partial}{\partial z_j}\right), \qquad U q_j U^{-1} = -\frac{i}{\sqrt{2}}\left(z_j + \frac{\partial}{\partial z_j}\right)$$

and also : $\omega = d\beta = dp \wedge dq$, where $\beta = i\bar{z}dz$.

Orbital Theory for Wavelets. Let coadjoint action be $< g \cdot f, Y > = < f, Ad(g)^{-1}Y >$, where $<, >$ is pairing $g \in G, \quad f \in g^*, \quad Y \in \mathcal{G}$. The orbit is $\mathcal{O}_f = G \cdot f \equiv G/G(f)$. Also, let A=A(M) be algebra of functions, V(M) is A-module of vector fields, A^p is A-module of p-forms. Vector fields on orbit is

$$\sigma(\mathcal{O}, X)_f(\phi) = \frac{d}{dt}(\phi(\exp tXf))\Big|_{t=0}$$

where $\phi \in A(\mathcal{O}), \quad f \in \mathcal{O}$. Then $\mathcal{O}_f$ are homogeneous symplectic manifolds with 2-form $\Omega(\sigma(\mathcal{O}, X)_f, \sigma(\mathcal{O}, Y)_f) = < f, [X, Y] >$, and $d\Omega = 0$. PB on $\mathcal{O}$ have the next form $\{\Psi_1, \Psi_2\} = p(\Psi_1)\Psi_2$ where p is $A^1(\mathcal{O}) \to V(\mathcal{O})$ with definition $\Omega(p(\alpha), X) = i(X)\alpha$. Here $\Psi_1, \Psi_2 \in A(\mathcal{O})$ and $A(\mathcal{O})$ is Lie algebra with bracket $\{,\}$. Now let N be a Heisenberg group. Consider adjoint and coadjoint representations in some particular case. $N = (z, t) \in C \times R, z = p + iq$; compositions in N are $(z, t) \cdot (z', t') = (z + z', t + t' + B(z, z'))$, where $B(z, z') = pq' - qp'$. Inverse element is $(-t, -z)$. Lie algebra n of N is $(\zeta,\tau)\in C\times R$ with bracket $[(\zeta,\tau),(\zeta',\tau')]=(0,B(\zeta,\zeta'))$. Centre is $\tilde{z}\in n$ and generated by (0,1); Z is a subgroup $\exp\tilde{z}$. Adjoint representation N on n is given by formula $Ad(z,t)(\zeta,\tau)=(\zeta,\tau+B(z,\zeta))$ Coadjoint: for $f\in n^*$, $\quad g=(z,t)$, $(g\cdot f)(\zeta,\zeta)=f(\zeta,\tau)-B(z,\zeta)f(0,1)$ then orbits for which $f|_{\tilde{z}}\neq 0$ are plane in n^* given by equation $f(0,1)=\mu$. If $X=(\zeta,0),\quad Y=(\zeta',0),\quad X,Y\in n$ then symplectic structure is

$$\Omega(\sigma(\mathcal{O},X)_f,\sigma(\mathcal{O},Y)_f)=<f,[X,Y]>=f(0,B(\zeta,\zeta'))\mu B(\zeta,\zeta')$$

Also we have for orbit $\mathcal{O}_\mu=N/Z$ and $\mathcal{O}_\mu$ is Hamiltonian G-space.

According to this approach we can construct by using methods of geometric quantization theory many "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it. Very useful particular spline–wavelet basis with uniform exponential control on stratified and nilpotent Lie groups was considered in [13].

B Applications to Melnikov Functions Approach

We give now some point of applications of wavelet methods from the preceding parts to Melnikov approach in the theory of homoclinic chaos in perturbed Hamiltonian systems for examples from [1].

In Hamiltonian form we have:

$$\dot{x}=J\cdot\nabla H(x)+\varepsilon g(x,\Theta),\quad \dot{\Theta}=\omega,\quad (x,\Theta)\in R^n\times T^m,$$

for $\varepsilon=0$ we have:

$$\dot{x}=J\cdot\nabla H(x),\quad \dot{\Theta}=\omega \tag{34}$$

For $\varepsilon=0$ we have homoclinic orbit $\bar{x}_0(t)$ to the hyperbolic fixed point x_0. For $\varepsilon\neq 0$ we have normally hyperbolic invariant torus T_ε and condition on transversally intersection of stable and unstable manifolds $W^s(T_\varepsilon)$ and $W^u(T_\varepsilon)$ in terms of Melnikov functions $M(\Theta)$ for $\bar{x}_0(t)$:

$$M(\Theta)=\int_{-\infty}^{\infty}\nabla H(\bar{x}_0(t))\wedge g(\bar{x}_0(t),\omega t+\Theta)dt$$

This condition has the next form:

$$M(\Theta_0)=0,\qquad \sum_{j=1}^{2}\omega_j\frac{\partial}{\partial\Theta_j}M(\Theta_0)\neq 0$$

According to the approach of Birkhoff-Smale-Wiggins we determined the region in parameter space in which we can observe the chaotic behaviour [4].

If we cannot solve equations (34) explicitly in time, then we use the wavelet approach from paper [1] for the computations of homoclinic (heteroclinic) loops as the wavelet solutions of system (34). For computations of quasiperiodic Melnikov functions

$$M^{m/n}(t_0) = \int_0^{mT} DH(x_\alpha(t)) \wedge g(x_\alpha(t), t + t_0)dt$$

we used periodization of wavelet construction from paper [1].
We also used symplectic Melnikov function approach in which we have:

$$M_i(z) = \lim_{j\to\infty} \int_{-T_j^*}^{T_j} \{h_i, \hat{h}\}_{\Psi(t,z)} dt$$

$$d_i(z, \varepsilon) = h_i(z_\varepsilon^u) - h_i(z_\varepsilon^s) = \varepsilon M_i(z) + O(\varepsilon^2)$$

where $\{,\}$ is the Poisson bracket, $d_i(z, \varepsilon)$ is the Melnikov distance. So, we need symplectic invariant wavelet expressions for Poisson brackets. The computations are produced according to invariant calculation of Poisson brackets, which is based on consideration in part III A and on operator representation from part III F (see below).

C Floer Approach for Closed Loops

Now we consider the generalization of wavelet variational approach to the symplectic invariant calculation of closed loops in Hamiltonian systems [15]. As we demonstrated in [4] we have the parametrization of our solution by some reduced algebraical problem but in contrast to the cases from paper [1], where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations. Now we consider a different approach.

Let(M, ω) be a compact symplectic manifold of dimension $2n$, ω is a closed 2-form (nondegenerate) on M which induces an isomorphism $T^*M \to TM$. Thus every smooth time-dependent Hamiltonian $H : \mathbf{R} \times M \to \mathbf{R}$ corresponds to a time-dependent Hamiltonian vector field $X_H : \mathbf{R} \times M \to TM$ defined by

$$\omega(X_H(t, x), \xi) = -d_x H(t, x)\xi \tag{35}$$

for $\xi \in T_x M$. Let H (and X_H) is periodic in time: $H(t + T, x) = H(t, x)$ and consider corresponding Hamiltonian differential equation on M:

$$\dot{x}(t) = X_H(t, x(t)) \tag{36}$$

The solutions $x(t)$ of (36) determine a 1-parameter family of diffeomorphisms $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_t(x(0)) = x(t)$. These diffeomorphisms are symplectic: $\omega =$

$\psi_t^*\omega$. Let $L = L_T M$ be the space of contractible loops in M which are represented by smooth curves $\gamma : \mathbf{R} \to M$ satisfying $\gamma(t+T) = \gamma(t)$. Then the contractible T-periodic solutions of (36) can be characterized as the critical points of the functional $S = S_T : L \to \mathbf{R}$:

$$S_T(\gamma) = -\int_D u^*\omega + \int_0^T H(t, \gamma(t))\mathrm{d}t, \tag{37}$$

where $D \subset \mathbf{C}$ be a closed unit disc and $u : D \to M$ is a smooth function, which on boundary agrees with γ, i.e. $u(\exp\{2\pi i\Theta\}) = \gamma(\Theta T)$. Because $[\omega]$, the cohomology class of ω, vanishes then $S_T(\gamma)$ is independent of choice of u. Tangent space $T_\gamma L$ is the space of vector fields $\xi \in C^\infty(\gamma^* TM)$ along γ satisfying $\xi(t+T) = \xi(t)$. Then we have for the 1-form $\mathrm{d}f : TL \to \mathbf{R}$

$$\mathrm{d}S_T(\gamma)\xi = \int_0^T (\omega(\dot\gamma, \xi) + \mathrm{d}H(t,\gamma)\xi)\mathrm{d}t \tag{38}$$

and the critical points of S are contractible loops in L which satisfy the Hamiltonian equation (36). Thus the critical points are precisely the required T-periodic solution of (36).

To describe the gradient of S we choose a on almost complex structure on M which is compatible with ω. This is an endomorphism $J \in C^\infty(\mathrm{End}(TM))$ satisfying $J^2 = -I$ such that

$$g(\xi, \eta) = \omega(\xi, J(x)\eta), \qquad \xi, \eta \in T_x M \tag{39}$$

defines a Riemannian metric on M. The Hamiltonian vector field is then represented by $X_H(t,x) = J(x)\nabla H(t,x)$, where ∇ denotes the gradient w.r.t. the x-variable using the metric (39). Moreover the gradient of S w.r.t. the induced metric on L is given by

$$\mathrm{grad}S(\gamma) = J(\gamma)\dot\gamma + \nabla H(t,\gamma), \qquad \gamma \in L \tag{40}$$

Studying the critical points of S is confronted with the well-known difficulty that the variational integral is neither bounded from below nor from above. Moreover, at every possible critical point the Hessian of f has an infinite dimensional positive and an infinite dimensional negative subspaces, so the standard Morse theory is not applicable. The additional problem is that the gradient vector field on the loop space L

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma = -\mathrm{grad}f(\gamma) \tag{41}$$

does not define a well posed Cauchy problem. But Floer [15] found a way to analyse the space $\mathcal{M}$ of bounded solutions consisting of the critical points together with their connecting orbits. He used a combination of variational approach and

Gromov's elliptic technique. A gradient flow line of f is a smooth solution $u : \mathbf{R} \rightarrow M$ of the partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0, \tag{42}$$

which satisfies $u(s,t+T) = u(s,t)$. The key point is to consider (42) not as the flow on the loop space but as an elliptic boundary value problem. It should be noted that (42) is a generalization of equation for Gromov's pseudoholomorphic curves (correspond to the case $\nabla H = 0$ in (42)). Let $\mathcal{M}_T = \mathcal{M}_T(H,J)$ the space of bounded solutions of (42), i.e. the space of smooth functions $u : \mathbf{C}/iT\mathbf{Z} \rightarrow M$, which are contractible, solve equation (42) and have finite energy flow:

$$\Phi_T(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^T (|\frac{\partial u}{\partial s}|^2 + |\frac{\partial u}{\partial t} - X_H(t,u)|^2)\mathrm{d}t\mathrm{d}s \quad < \infty. \tag{43}$$

For every $u \in M_T$ there exists a pair x, y of contractible T-periodic solutions of (36), such that u is a connecting orbit from y to x:

$$\lim_{s\rightarrow-\infty} u(s,t) = y(t), \qquad \lim_{s\rightarrow+\infty} = x(t) \tag{44}$$

Then the approach from [1], which we may apply or on the level of standard boundary problem (42) or on the level of variational approach (43) and representation of operators (in our case, J and ∇) according to part III F (see below) lead us to wavelet representation of closed loops.

D Quasiclassical Evolution

Let us consider classical and quantum dynamics in phase space $\Omega = R^{2m}$ with coordinates (x,ξ) and generated by Hamiltonian $\mathcal{H}(x,\xi) \in C^\infty(\Omega; R)$. If $\Phi_t^{\mathcal{H}} : \Omega \longrightarrow \Omega$ is (classical) flow then time evolution of any bounded classical observable or symbol $b(x,\xi) \in C^\infty(\Omega, R)$ is given by $b_t(x,\xi) = b(\Phi_t^{\mathcal{H}}(x,\xi))$. Let $H = Op^W(\mathcal{H})$ and $B = Op^W(b)$ are the self-adjoint operators or quantum observables in $L^2(R^n)$, representing the Weyl quantization of the symbols $\mathcal{H}, b$ [14]

$$(Bu)(x) = \frac{1}{(2\pi\hbar)^n}\int_{R^{2n}} b\left(\frac{x+y}{2},\xi\right)\cdot e^{i<(x-y),\xi>/\hbar}u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $u \in S(R^n)$ and $B_t = e^{iHt/\hbar}Be^{-iHt/\hbar}$ be the Heisenberg observable or quantum evolution of the observable B under unitary group generated by H. B_t solves the Heisenberg equation of motion

$$\dot{B}_t = \frac{i}{\hbar}[H, B_t].$$

Let $b_t(x,\xi;\hbar)$ is a symbol of B_t then we have the following equation for it

$$\dot{b}_t = \{\mathcal{H}, b_t\}_M, \tag{45}$$

with initial condition $b_0(x, \xi, \hbar) = b(x, \xi)$. Here $\{f, g\}_M(x, \xi)$ is the Moyal brackets of the observables $f, g \in C^\infty(R^{2n})$, $\{f, g\}_M(x, \xi) = f\sharp g - g\sharp f$, where $f\sharp g$ is the symbol of the operator product and is presented by the composition of the symbols f, g

$$(f\sharp g)(x, \xi) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{R^{4n}} e^{-i<r,\rho>/\hbar + i<\omega,\tau>/\hbar} \cdot f(x+\omega, \rho+\xi) \cdot$$
$$g(x+r, \tau+\xi)\mathrm{d}\rho\mathrm{d}\tau\mathrm{d}r\mathrm{d}\omega.$$

For our problems it is useful that $\{f, g\}_M$ admits the formal expansion in powers of $\hbar$:

$$\{f, g\}_M(x, \xi) \sim \{f, g\} + 2^{-j} \sum_{|\alpha+\beta|=j\geq 1} (-1)^{|\beta|} \cdot (\partial_\xi^\alpha f D_x^\beta g) \cdot (\partial_\xi^\beta g D_x^\alpha f),$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D_x = -i\hbar\partial_x$. So, evolution (45) for symbol $b_t(x, \xi; \hbar)$ is

$$\dot{b}_t = \{\mathcal{H}, b_t\} + \frac{1}{2^j} \sum_{|\alpha|+\beta|=j\geq 1} (-1)^{|\beta|} \cdot \hbar^j (\partial_\xi^\alpha \mathcal{H} D_x^\beta b_t) \cdot (\partial_\xi^\beta b_t D_x^\alpha \mathcal{H}). \tag{46}$$

At $\hbar = 0$ this equation transforms to classical Liouville equation

$$\dot{b}_t = \{\mathcal{H}, b_t\}. \tag{47}$$

Equation (46) plays a key role in many quantum (semiclassical) problem. We note only the problem of relation between quantum and classical evolutions or how long the evolution of the quantum observables is determined by the corresponding classical one [14]. Our approach to solution of systems (46), (47) is based on our technique from [1]-[7] and very useful linear parametrization for differential operators which we present in section III F.

E SYMPLECTIC HILBERT SCALES VIA WAVELETS

We can solve many important dynamical problems such that KAM perturbations, spread of energy to higher modes, weak turbulence, growths of solutions of Hamiltonian equations only if we consider scales of spaces instead of one functional space. For Hamiltonian system and their perturbations for which we need take into account underlying symplectic structure we need to consider symplectic scales of spaces. So, if $\dot{u}(t) = J\nabla K(u(t))$ is Hamiltonian equation we need wavelet description of symplectic or quasicomplex structure on the level of functional spaces. It is very important that according to [16] Hilbert basis is in the same time a Darboux basis to corresponding symplectic structure. We need to provide Hilbert scale $\{Z_s\}$

with symplectic structure [16], [17]. All what we need is the following. J is a linear operator, $J : Z_\infty \to Z_\infty$, $J(Z_\infty) = Z_\infty$, where $Z_\infty = \cap Z_s$. J determines an isomorphism of scale $\{Z_s\}$ of order $d_J \ge 0$. The operator J with domain of definition Z_∞ is antisymmetric in Z: $< Jz_1, z_2 >_Z = - < z_1, Jz_2 >_Z, z_1, z_2 \in Z_\infty$. Then the triple

$$\{Z, \{Z_s | s \in R\}, \quad \alpha =< \bar{J} dz, dz >\}$$

is symplectic Hilbert scale. So, we may consider any dynamical Hamiltonian problem on functional level. As an example, for KdV equation we have

$$Z_s = \{u(x) \in H^s(T^1) | \int_0^{2\pi} u(x)\mathrm{dx} = 0\}, \ \mathrm{s} \in \mathrm{R}, \quad \mathrm{J} = \partial/\partial\mathrm{x},$$

J is isomorphism of the scale of order one, $\bar{J} = -(J)^{-1}$ is isomorphism of order -1. According to [18] general functional spaces and scales of spaces such as Holder–Zygmund, Triebel–Lizorkin and Sobolev can be characterized through wavelet coefficients or wavelet transforms. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular [18]. Most important for us example is the scale of Sobolev spaces. Let $H_k(R^n)$ is the Hilbert space of all distributions with finite norm

$$\|s\|^2_{H_k(R^n)} = \int \mathrm{d}\xi(1 + |\xi|^2)^{\mathrm{k}/2} |\hat{\mathrm{s}}(\xi)|^2.$$

Let us consider wavelet transform

$$W_g f(b, a) = \int_{R^n} \mathrm{dx} \frac{1}{\mathrm{a^n}} \bar{\mathrm{g}} \left(\frac{\mathrm{x} - \mathrm{b}}{\mathrm{a}} \right) \mathrm{f(x)},$$

$b \in R^n, \quad a > 0$, w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{\mathrm{da}}{a} |\hat{g}(\bar{a}k)|^2 < \infty.$$

Then there is a $c \ge 1$ such that

$$c^{-1} \|s\|^2_{H_k(R^n)} \le \int_{H^n} \frac{\mathrm{dbda}}{a} (1 + a^{-2\gamma}) |W_g s(b, a)|^2 \le c \|s\|^2_{H_k(R^n)}.$$

This shows that localization of the wavelet coefficients at small scale is linked to local regularity.

So, we need representation for differential operator (J in our case) in wavelet basis. We consider it in the next section.

F FAST WAVELET TRANSFORM FOR DIFFERENTIAL OPERATORS

Let us consider multiresolution representation $\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \ldots$ (see our other paper from this proceedings for details of wavelet machinery).

Let T be an operator $T : L^2(R) \to L^2(R)$, with the kernel $K(x,y)$ and $P_j : L^2(R) \to V_j$ $(j \in Z)$ is projection operators on the subspace V_j corresponding to j level of resolution:

$$(P_j f)(x) = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}(x).$$

Let $Q_j = P_{j-1} - P_j$ is the projection operator on the subspace W_j then we have the following "microscopic or telescopic" representation of operator T which takes into account contributions from each level of resolution from different scales starting with coarsest and ending to finest scales:

$$T = \sum_{j \in Z} (Q_j T Q_j + Q_j T P_j + P_j T Q_j).$$

We remember that this is a result of presence of affine group inside this construction. The non-standard form of operator representation [19] is a representation of an operator T as a chain of triples $T = \{A_j, B_j, \Gamma_j\}_{j \in Z}$, acting on the subspaces V_j and W_j:

$$A_j : W_j \to W_j, B_j : V_j \to W_j, \Gamma_j : W_j \to V_j,$$

where operators $\{A_j, B_j, \Gamma_j\}_{j \in Z}$ are defined as

$$A_j = Q_j T Q_j, \quad B_j = Q_j T P_j, \quad \Gamma_j = P_j T Q_j.$$

The operator T admits a recursive definition via

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix},$$

where $T_j = P_j T P_j$ and T_j works on $V_j : V_j \to V_j$. It should be noted that operator A_j describes interaction on the scale j independently from other scales, operators B_j, Γ_j describe interaction between the scale j and all coarser scales, the operator T_j is an "averaged" version of T_{j-1}.

The operators A_j, B_j, Γ_j, T_j are represented by matrices $\alpha^j, \beta^j, \gamma^j, s^j$

$$\begin{aligned}
\alpha^j_{k,k'} &= \int\int K(x,y)\psi_{j,k}(x)\psi_{j,k'}(y)\mathrm{dxdy} \\
\beta^j_{k,k'} &= \int\int K(x,y)\psi_{j,k}(x)\varphi_{j,k'}(y)\mathrm{dxdy} \\
\gamma^j_{k,k'} &= \int\int K(x,y)\varphi_{j,k}(x)\psi_{j,k'}(y)\mathrm{dxdy} \\
s^j_{k,k'} &= \int\int K(x,y)\varphi_{j,k}(x)\varphi_{j,k'}(y)\mathrm{dxdy}
\end{aligned} \tag{48}$$

We may compute the non-standard representations of operator d/dx in the wavelet bases by solving a small system of linear algebraical equations. So, we have for objects (48)

$$\alpha^j_{i,\ell} = 2^{-j}\int \psi(2^{-j}x - i)\psi'(2^{-j} - \ell)2^{-j}\mathrm{dx} = 2^{-\mathrm{j}}\alpha_{\mathrm{i}-\ell}$$
$$\beta^j_{i,\ell} = 2^{-j}\int \psi(2^{-j}x - i)\varphi'(2^{-j}x - \ell)2^{-j}\mathrm{dx} = 2^{-\mathrm{j}}\beta_{\mathrm{i}-\ell}$$
$$\gamma^j_{i,\ell} = 2^{-j}\int \varphi(2^{-j}x - i)\psi'(2^{-j}x - \ell)2^{-j}\mathrm{dx} = 2^{-\mathrm{j}}\gamma_{\mathrm{i}-\ell},$$

where

$$\alpha_\ell = \int \psi(x-\ell)\frac{\mathrm{d}}{\mathrm{dx}}\psi(x)\mathrm{dx}$$
$$\beta_\ell = \int \psi(x-\ell)\frac{\mathrm{d}}{\mathrm{dx}}\varphi(x)\mathrm{dx}$$
$$\gamma_\ell = \int \varphi(x-\ell)\frac{\mathrm{d}}{\mathrm{dx}}\psi(x)\mathrm{dx}$$

then by using refinement equations

$$\varphi(x) = \sqrt{2}\sum_{k=0}^{L-1} h_k\varphi(2x-k),$$
$$\psi(x) = \sqrt{2}\sum_{k=0}^{L-1} g_k\varphi(2x-k),$$

$g_k = (-1)^k h_{L-k-1}, \quad k = 0,\ldots,L-1$ we have in terms of filters (h_k, g_k):

$$\alpha_j = 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} g_k g_{k'} r_{2i+k-k'},$$
$$\beta_j = 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} g_k h_{k'} r_{2i+k-k'},$$
$$\gamma_i = 2\sum_{k=0}^{L-1}\sum_{k'=0}^{L-1} h_k g_{k'} r_{2i+k-k'},$$

where

$$r_\ell = \int \varphi(x-\ell)\frac{\mathrm{d}}{\mathrm{dx}}\varphi(x)\mathrm{dx}, \ell\in \mathrm{Z}.$$

Therefore, the representation of d/dx is completely determined by the coefficients r_ℓ or by representation of d/dx only on the subspace V_0. The coefficients $r_\ell, \ell \in Z$ satisfy the following system of linear algebraical equations

$$r_\ell = 2\left[r_{2l} + \frac{1}{2}\sum_{k=1}^{L/2} a_{2k-1}(r_{2\ell-2k+1} + r_{2\ell+2k-1})\right]$$

and $\sum_\ell \ell r_\ell = -1$, where $a_{2k-1} = 2\sum_{i=0}^{L-2k} h_i h_{i+2k-1}$, $k = 1,\ldots,L/2$ are the autocorrelation coefficients of the filter H. If a number of vanishing moments $M \geq 2$ then this linear system of equations has a unique solution with finite number of non-zero r_ℓ, $r_\ell \neq 0$ for $-L+2 \leq \ell \leq L-2, r_\ell = -r_{-\ell}$. For the representation of operator d^n/dx^n we have the similar reduced linear system of equations. Then finally we have for action of operator $T_j(T_j : V_j \to V_j)$ on sufficiently smooth function f:

$$(T_j f)(x) = \sum_{k\in Z} \left(2^{-j} \sum_\ell r_\ell f_{j,k-\ell}\right) \varphi_{j,k}(x),$$

where $\varphi_{j,k}(x) = 2^{-j/2}\varphi(2^{-j}x - k)$ is wavelet basis and

$$f_{j,k-1} = 2^{-j/2} \int f(x)\varphi(2^{-j}x - k + \ell)\mathrm{dx}$$

are wavelet coefficients. So, we have simple linear parametrization of matrix representation of our differential operator in wavelet basis and of the action of this operator on arbitrary vector in our functional space. Then we may use such representation in all preceding sections.

IV MAPS AND WAVELET STRUCTURES

A Veselov-Marsden Discretization

Discrete variational principles lead to evolution dynamics analogous to the Euler-Lagrange equations [9]. Let Q be a configuration space, then a discrete Lagrangian is a map $L : Q \times Q \to \mathbf{R}$. usually L is obtained by approximating the given Lagrangian. For $N \in N_+$ the action sum is the map $S : Q^{N+1} \to \mathbf{R}$ defined by

$$S = \sum_{k=0}^{N-1} L(q_{k+1}, q_k), \tag{49}$$

where $q_k \in Q$, $k \geq 0$. The action sum is the discrete analog of the action integral in continuous case. Extremizing S over $q_1, ..., q_{N-1}$ with fixing q_0, q_N we have the discrete Euler-Lagrange equations (DEL):

$$D_2L(q_{k+1}, q_k) + D_1(q_k, q_{q-1}) = 0, \tag{50}$$

for $k = 1, ..., N-1$.

Let

$$\Phi : Q \times Q \to Q \times Q \tag{51}$$

and

$$\Phi(q_k, q_{k-1}) = (q_{k+1}, q_k) \tag{52}$$

is a discrete function (map), then we have for DEL:

$$D_2 L \circ \Phi + D_1 L = 0 \tag{53}$$

or in coordinates q^i on Q we have DEL

$$\frac{\partial L}{\partial q_k^i} \circ \Phi(q_{k+1}, q_k) + \frac{\partial L}{\partial q_{k+1}^i}(q_{k+1}, q_k) = 0. \tag{54}$$

It is very important that the map Φ exactly preserves the symplectic form ω:

$$\omega = \frac{\partial^2 L}{\partial q_k^i \partial q_{k+1}^j}(q_{k+1}, q_k) \mathrm{d}q_k^i \wedge \mathrm{d}q_{k+1}^j \tag{55}$$

B Generalized Wavelet Approach

Our approach to solutions of equations (54) is based on applications of general and very efficient methods developed by A. Harten [20], who produced a "General Framework" for multiresolution representation of discrete data. It is based on consideration of basic operators, decimation and prediction, which connect adjacent resolution levels. These operators are constructed from two basic blocks: the discretization and reconstruction operators. The former obtains discrete information from a given continuous functions (flows), and the latter produces an approximation to those functions, from discrete values, in the same function space to which the original function belongs. A "new scale" is defined as the information on a given resolution level which cannot be predicted from discrete information at lower levels. If the discretization and reconstruction are local operators, the concept of "new scale" is also local. The scale coefficients are directly related to the prediction errors, and thus to the reconstruction procedure. If scale coefficients are small at a certain location on a given scale, it means that the reconstruction procedure on that scale gives a proper approximation of the original function at that particular location. This approach may be considered as some generalization of standard wavelet analysis approach. It allows to consider multiresolution decomposition when usual approach is impossible (δ-functions case).

Let F be a linear space of mappings

$$F \subset \{f | f : X \to Y\}, \tag{56}$$

where X, Y are linear spaces. Let also D_k be a linear operator

$$D_k : f \to \{v^k\}, \quad v^k = D_k f, \quad v^k = \{v_i^k\}, \quad v_i^k \in Y. \tag{57}$$

This sequence corresponds to k level discretization of X. Let

$$D_k(F) = V^k = \mathrm{span}\{\eta_i^k\} \tag{58}$$

and the coordinates of $v^k \in V^k$ in this basis are $\hat{v}^k = \{\hat{v}_i^k\}$, $\hat{v}^k \in S^k$:

$$v^k = \sum_i \hat{v}_i^k \eta_i^k, \tag{59}$$

D_k is a discretization operator. Main goal is to design a multiresolution scheme (MR) [20] that applies to all sequences $s \in S^L$, but corresponds for those sequences $\hat{v}^L \in S^L$, which are obtained by the discretization (56).

Since D_k maps F onto V^k then for any $v^k \subset V^k$ there is at least one f in F such that $D_k f = v^k$. Such correspondence from $f \in F$ to $v^k \in V^k$ is reconstruction and the corresponding operator is the reconstruction operator R_k:

$$R_k : V_k \to F, \qquad D_k R_k = I_k, \tag{60}$$

where I_k is the identity operator in V^k (R^k is right inverse of D^k in V^k).

Given a sequence of discretization $\{D_k\}$ and sequence of the corresponding reconstruction operators $\{R_k\}$, we define the operators D_k^{k-1} and P_{k-1}^k

$$\begin{aligned} D_k^{k-1} &= D_{k-1} R_k : V_k \to V_{k-1} \\ P_{k-1}^k &= D_k R_{k-1} : V_{k-1} \to V_k \end{aligned} \tag{61}$$

If the set D_k in nested [20], then

$$D_k^{k-1} P_{k-1}^k = I_{k-1} \tag{62}$$

and we have for any $f \in F$ and any $p \in F$ for which the reconstruction R_{k-1} is exact:

$$\begin{aligned} D_k^{k-1}(D_k f) &= D_{k-1} f \\ P_{k-1}^k(D_{k-1} p) &= D_k p \end{aligned} \tag{63}$$

Let us consider any $v^L \in V^L$, Then there is $f \in F$ such that

$$v^L = D_L f, \tag{64}$$

and it follows from (63) that the process of successive decimation [20]

$$v^{k-1} = D_k^{k-1} v^k, \qquad k = L, ..., 1 \tag{65}$$

yields for all k

$$v^k = D_k f \tag{66}$$

Thus the problem of prediction, which is associated with the corresponding MR scheme, can be stated as a problem of approximation: knowing $D_{k-1} f$, $f \in F$, find a "good approximation" for $D_k f$. It is very important that each space V^L has a multiresolution basis

$$\bar{B}_M = \{\bar{\phi}_i^{0,L}\}_i, \{\{\bar{\psi}_j^{k,L}\}_j\}_{k=1}^L \tag{67}$$

and that any $v^L \in V^L$ can be written as

$$v^L = \sum_i \hat{v}_i^0 \bar{\phi}_i^{0,L} + \sum_{k=1}^{L} \sum_j d_j^k \bar{\psi}_j^{k,L}, \tag{68}$$

where $\{d_j^k\}$ are the k scale coefficients of the associated MR, $\{\hat{v}_i^0\}$ is defined by (59) with $k = 0$. If $\{D_k\}$ is a nested sequence of discretization [20] and $\{R_k\}$ is any corresponding sequence of linear reconstruction operators, then we have from (68) for $v^L = D_L f$ applying R_L:

$$R_L D_L f = \sum_i \hat{f}_i^0 \phi_i^{0,L} + \sum_{k=1}^{L} \sum_j d_j^k \psi_j^{k,L}, \tag{69}$$

where

$$\phi_i^{0,L} = R_L \bar{\phi}_i^{0,L} \in F, \quad \psi_j^{k,L} = R_L \bar{\psi}_j^{k,L} \in F, \quad D_0 f = \sum \hat{f}_i^0 \eta_i^0. \tag{70}$$

When $L \to \infty$ we have sufficient conditions which ensure that the limiting process $L \to \infty$ in (69, 70) yields a multiresolution basis for F. Then, according to (67), (68) we have very useful representation for solutions of equations (54) or for different maps construction in the form which are a counterparts for discrete (difference) cases of constructions from paper [1].

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