WAVELET APPROACH TO MECHANICAL PROBLEMS. SYMPLECTIC GROUP, SYMPLECTIC TOPOLOGY AND SYMPLECTIC SCALES

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Abstract. We present the applications of methods from wavelet analysis to polynomial approximations for a number of nonlinear problems. According to the orbit method and by using approach from the geometric quantization theory we construct the symplectic and Poisson structures associated with generalized wavelets by using metaplectic structure. We consider wavelet approach to the calculations of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems, for parametrization of Arnold–Weinstein curves in Floer variational approach and characterization of symplectic Hilbert scales of spaces.

1. Introduction. In this paper we consider the application of powerful methods of wavelet analysis to polynomial approximations of nonlinear physical and mechanical problems. In the related paper in this volume we considered the general approach for constructing wavelet representation for solutions of nonlinear polynomial dynamical problems. But now we are interested in underlying Hamiltonian structures. Therefore, we need to consider generalized wavelets, which allow us to take into account the corresponding hidden symplectic, Poissonian, quasicomplex structures, instead of simple compactly supported wavelet representation. By using the orbit method and constructions from the geometric quantization theory we consider the symplectic and Poisson structures associated with Weyl–Heisenberg wavelets by using metaplectic structure and the corresponding polarization in §2–§6. In §7 we consider applications to construction of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems. In §8 we consider the generalization of our variational wavelet approach [1]–[9] to the symplectic invariant calculation of Arnold-Weinstein curves (closed loops) in Hamiltonian systems. In §9 we consider wavelet characterization of symplectic Hilbert scales of spaces, which is important in problems related with KAM perturbations.

2. Metaplectic Group and Representations. In wavelet analysis the following three concepts are used now: 1). a square integrable representation U of a group G, 2). coherent states over G, 3). the wavelet transform associated to U.

We have three important particular cases: a) the affine $(ax+b)$ group, which yields the usual wavelet analysis: $[\pi(b,a)f](x) = 1/\sqrt{a} f\left((x-b)/a\right)$, b). the Weyl-Heisenberg group which leads to the Gabor functions, i.e. coherent states associated with windowed Fourier transform: $[\pi(q,p,\varphi)f](x)$ $= \exp(i\mu(\varphi - p(x-q))f(x-q)$, in both cases time-frequency plane corresponds to the phase space of group representation. c). also, we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group [10]. But usual representation of it is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in that case) restores square – integrability: $G_{aWH\longrightarrow}$ homogeneous space. Also, we have more general approach which allows to consider wavelets corresponding to more general groups and representations [11], [12]. Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (c) analogously.

Let $Sp(n)$ be symplectic group, $Mp(n)$ be its unique two- fold covering – metaplectic group. Let V be a symplectic vector space with symplectic form (,), then $R\oplus V$ is nilpotent Lie algebra - Heisenberg algebra:

$$[R,V]=0, \quad [v,w]=(v,w)\in R, \quad [V,V]=R.$$

$Sp(V)$ is a group of automorphisms of Heisenberg algebra.

Let N be a group with Lie algebra $R\oplus V$, i.e. Heisenberg group. By Stone– von Neumann theorem Heisenberg group has unique irreducible unitary representation in which $1 \mapsto i$. This representation is projective: $U_{g_1}U_{g_2} = c(g_1, g_2) \cdot U_{g_1 g_2}$, where c is a map: $Sp(V) \times Sp(V) \to S^1$, i.e. c is S^1-cocycle.

But this representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F = N' \bowtie Mp(V)$, $\bowtie$ is semidirect product (we consider instead of $N = R \oplus V$ the $N' = S^1 \times V$, $S^1 = (R/2\pi Z)$). Let V^* be dual to V, $G(V^*)$ be automorphism group of V^*.Then F is subgroup of $G(V^*)$, which consists of elements, which acts on V^* by affine transformations.
This is the key point!

Let $q_1, ..., q_n; p_1, ..., p_n$ be symplectic basis in V, $\alpha = pdq = \sum p_i dq_i$ and $d\alpha$ be symplectic form on V^*. Let M be fixed affine polarization, then for $a \in F$ the map $a \mapsto \Theta_a$ gives unitary representation of G: $\Theta_a : H(M) \to H(M)$

Explicitly we have for representation of N on H(M):

$$(\Theta_q f)^*(x) = e^{-iqx} f(x), \quad \Theta_p f(x) = f(x - p)$$

The representation of N on H(M) is irreducible. Let A_q, A_p be infinitesimal operators of this representation

$$A_q = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tq} - I], \quad A_p = \lim_{t \to 0} \frac{1}{t}[\Theta_{-tp} - I],$$

$$\text{then} \qquad A_q f(x) = i(qx)f(x), \quad A_p f(x) = \sum p_j \frac{\partial f}{\partial x_j}(x)$$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group F is the algebra of all (nonhomogeneous) quadratic polynomials of (p,q) relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, ..., q_n,\ p_1, ..., p_n,\ q_i q_j,\ q_i p_j,\ p_i p_j, \quad i, j = 1, ..., n, \quad i \le j$,

$$PB\ is \quad \{f, g\} = \sum \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \quad \text{and} \quad \{1, g\} = 0 \quad for \text{ all } g,$$
$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i q_j, q_k\} = \delta_{ik} q_j, \quad \{p_i q_j, p_k\} = -\delta_{jk} p_i,$$
$$\{p_i p_j, p_k\} = 0, \quad \{p_i p_j, q_k\} = \delta_{ik} p_j + \delta_{jk} p_i, \quad \{q_i q_j, q_k\} = 0,$$
$$\{q_i q_j, p_k\} = -\delta_{ik} q_j - \delta_{jk} q_i$$

so, we have the representation of basic elements $f \mapsto A_f : 1 \mapsto i, q_k \mapsto ix_k$,

$$p_l \mapsto \frac{\delta}{\delta x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2}\delta_{ij}, \qquad p_k p_l \mapsto \frac{1}{i}\frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto i x^k x^l$$

This gives the structure of the Poisson manifolds to representation of any (nilpotent) algebra or in other words to continuous wavelet transform.

3. The Segal-Bargman Representation. Let $z = 1/\sqrt{2}\cdot(p-iq)$, $\bar{z} = 1/\sqrt{2}\cdot(p+iq)$, $p=(p_1,...,p_n)$, F_n is the space of holomorphic functions of n complex variables with $(f,f)<\infty$, where

$$(f,g) = (2\pi)^{-n}\int f(z)\overline{g(z)}e^{-|z|^2}dpdq$$

Consider a map $U: H \to F_n$, where H is with real polarization, F_n is with complex polarization, then we have

$$(U\Psi)(a) = \int A(a,q)\Psi(q)dq, \qquad \text{where} \quad A(a,q) = \pi^{-n/4}e^{-1/2(a^2+q^2)+\sqrt{2}aq}$$

i.e. the Bargmann formula produce wavelets.We also have the representation of Heisenberg algebra on F_n :

$$U\frac{\partial}{\partial q_j}U^{-1} = \frac{1}{\sqrt{2}}\left(z_j - \frac{\partial}{\partial z_j}\right), \qquad Uq_jU^{-1} = -\frac{i}{\sqrt{2}}\left(z_j + \frac{\partial}{\partial z_j}\right)$$

and also : $\omega = d\beta = dp\wedge dq$, where $\beta = i\bar{z}dz$.

4. Orbital Theory for Wavelets. Let coadjoint action be $<g\cdot f, Y> = <f, Ad(g)^{-1}Y>$, where $<,>$ is pairing $g\in G$, $f\in g^*$, $Y\in\mathcal{G}$. The orbit is $\mathcal{O}_f = G\cdot f \equiv G/G(f)$. Also, let A=A(M) be algebra of functions, V(M) is A-module of vector fields, A^p is A-module of p-forms. Vector fields on orbit is

$$\sigma(\mathcal{O},X)_f(\phi) = \frac{d}{dt}(\phi(\exp tXf))\Big|_{t=0}$$

where $\phi\in A(\mathcal{O})$, $f\in\mathcal{O}$. Then $\mathcal{O}_f$ are homogeneous symplectic manifolds with 2-form $\Omega(\sigma(\mathcal{O},X)_f,\sigma(\mathcal{O},Y)_f) = <f,[X,Y]>$, and $d\Omega=0$. PB on $\mathcal{O}$ have the next form $\{\Psi_1,\Psi_2\} = p(\Psi_1)\Psi_2$ where p is $A^1(\mathcal{O})\to V(\mathcal{O})$ with definition $\Omega(p(\alpha),X) = i(X)\alpha$. Here $\Psi_1,\Psi_2\in A(\mathcal{O})$ and $A(\mathcal{O})$ is Lie algebra with bracket $\{,\}$. Now let N be a Heisenberg group. Consider adjoint and coadjoint representations in some particular case. $N = (z,t)\in C\times R$, $z = p+iq$; compositions in N are $(z,t)\cdot(z',t') = (z+z', t+t'+B(z,z'))$, where $B(z,z') = pq'-qp'$. Inverse element is $(-t,-z)$. Lie algebra n of N is $(\zeta,\tau)\in C\times R$ with bracket $[(\zeta,\tau),(\zeta',\tau')] = (0,B(\zeta,\zeta'))$. Centre is $\tilde{z}\in n$ and generated by (0,1); Z is a subgroup $\exp\tilde{z}$. Adjoint representation N on n is given by formula $Ad(z,t)(\zeta,\tau) = (\zeta,\tau+B(z,\zeta))$ Coadjoint: for $f\in n^*$, $g=(z,t)$, $(g\cdot f)(\zeta,\zeta) = f(\zeta,\tau) - B(z,\zeta)f(0,1)$ then orbits for which $f|_{\bar{z}} \neq 0$ are plane in n^* given by equation $f(0,1) = \mu$. If $X = (\zeta, 0)$, $\quad Y = (\zeta', 0)$, $\quad X, Y \in n$ then symplectic structure is

$$\Omega(\sigma(\mathcal{O}, X)_f, \sigma(\mathcal{O}, Y)_f) =< f, [X, Y] >= f(0, B(\zeta, \zeta'))\mu B(\zeta, \zeta')$$

Also we have for orbit $\mathcal{O}_\mu = N/Z$ and $\mathcal{O}_\mu$ is Hamiltonian G-space.

5. Kirillov Character Formula or Analogy of Gabor Wavelets.

Let U denote irreducible unitary representation of N with condition $U(0,t) = \exp(it\ell) \cdot 1$, where $\ell \neq 0$,then U is equivalent to representation T_ℓ which acts in $L^2(R)$ according to

$$T_\ell(z,t)\phi(x) = \exp\left(i\ell(t + px)\right)\phi(x - q)$$

If instead of N we consider E(2)/R we have S^1 case and we have Gabor functions on S^1.

6. Oscillator Group. Let O be an oscillator group, i.e. semidirect product of R and Heisenberg group N. Let H,P,Q,I be standard basis in Lie algebra o of the group O and H^*, P^*, Q^*, I^* be dual basis in o^*. Let functional f=(a,b,c,d) be $aI^* + bP^* + cQ^* + dH^*$. Let us consider complex polarizations $h = (H, I, P + iQ)$, $\quad \bar{h} = (I, H, P - iQ)$ Induced from h representation, corresponding to functional f (for $a > 0$), unitary equivalent to the representation

$$W(t,n)f(y) = \exp(it(h - 1/2)) \cdot U_a(n)V(t),$$

$$\text{where} \qquad V(t) = \exp[-it(P^2 + Q^2)/2a], \quad P = -d/dx, \quad Q = iax,$$

and $U_a(n)$ is irreducible representation of N, which have the form $U_a(z) = exp(iaz)$ on the center of N. Here we have: U(n=(x,y,z)) is Schrödinger representation, $U_t(n) = U(t(n))$ is the representation obtained from previous by automorphism (time translation) $n \longrightarrow t(n)$; $\quad U_t(n) = U(t(n))$ is also unitary irreducible representation of N. $V(t) = \exp(it(P^2 + Q^2 + h - 1/2))$ is an operator, which according to Stone–von Neumann theorem has the property $U_t(n) = V(t)U(n)V(t)^{-1}$.

This is our last private case, but according to our approach we can construct by using methods of geometric quantization theory many "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it. Very useful particular spline–wavelet basis with uniform exponential control on stratified and nilpotent Lie groups was considered in [12]. In particular case of Heisenberg group $N \ni (p, q, t)$ we may use as bases in our space of representations $L^2(N)$ the following orthonomal bases [12] $\psi_{i,j,k}$

generated from a finite set of 15 regular, localized and oscillating functions $\psi_i : \psi_{i,j,k} = 4^j\psi_i(2^jp - k_1, 2^jq - k_2, 2^jt - k_3 + 2^jk_2p - 2^jk_1q)$.

7. Melnikov Functions Approach. We give now some points of applications of wavelet methods from the preceding consideration to Melnikov approach in the theory of homoclinic chaos in perturbed Hamiltonian systems. We consider some points of our program of understanding routes to chaos in Hamiltonian systems in the wavelet approach [1]-[9]. All points are: 1. A model. 2. A computer zoo. The understanding of the computer zoo. 3. A naive Melnikov function approach. 4. A naive wavelet description of (hetero) homoclinic orbits (separatrix) and quasiperiodic oscillations. 5. Symplectic Melnikov function approach. 6. Splitting of separatrix... $\longrightarrow$stochastic web with magic symmetry, Arnold diffusion and all that.

1. As a model we have two frequencies perturbations of two-mode Galerkin approximations to beam oscillations in liquid [7]:

$$\begin{aligned}
\dot x_1 &= x_2, \qquad \dot x_3 = x_4, \qquad \dot x_5 = 1, \qquad \dot x_6 = 1,\\
\dot x_2 &= -ax_1 - b[\cos(rx_5) + \cos(sx_6)]x_1 - dx_1^3 - mdx_1x_3^2 - px_2 - \varphi(x_5)\\
\dot x_4 &= ex_3 - f[\cos(rx_5) + \cos(sx_6)]x_3 - gx_3^3 - kx_1^2x_3 - gx_4 - \psi(x_5)
\end{aligned}$$

or in Hamiltonian form

$$\dot x = J\cdot\nabla H(x) + \varepsilon g(x,\Theta),\ \dot\Theta \ = \ \omega,\ (x,\Theta)\in R^4\times T^2,\ T^2 = S^1\times S^1,$$

for $\varepsilon = 0$ we have:

$$\dot x = J\cdot\nabla H(x), \quad \dot\Theta = \omega \tag{1}$$

2. For pictures and details one can see [4], [7]. The key point is the splitting of separatrix (homoclinic orbit) and transition to fractal sets on the Poincare sections.

3. For $\varepsilon = 0$ we have homoclinic orbit $\bar x_0(t)$ to the hyperbolic fixed point x_0. For $\varepsilon \neq 0$ we have normally hyperbolic invariant torus T_ε and condition on transversally intersection of stable and unstable manifolds $W^s(T_\varepsilon)$ and $W^u(T_\varepsilon)$ in terms of Melnikov functions $M(\Theta)$ for $\bar x_0(t)$. $M(\Theta) = \int\limits_{-\infty}^{\infty} \nabla H(\bar x_0(t))\wedge g(\bar x_0(t), \omega t + \Theta)dt$. This condition has the next form: $M(\Theta_0) = 0, \quad \sum\limits_{j=1}^{2}\omega_j\frac{\partial}{\partial\Theta_j}M(\Theta_0)\neq 0$. According to the approach of Birkhoff-Smale-Wiggins we determined the region in parameter space in which we observe the chaotic behaviour [7].

4. If we cannot solve equations (1) explicitly in time, then we use the

wavelet approach from our other paper for the computations of homoclinic (heteroclinic) loops as the wavelet solutions of system (1). For computations of quasiperiodic Melnikov functions: $M^{m/n}(t_0) = \int_0^{mT} DH(x_\alpha(t)) \wedge g(x_\alpha(t), t+t_0)dt$, we used periodization of wavelet solution [3], [4].
5. We also used symplectic Melnikov function approach:
$M_i(z) = \lim_{j\to\infty} \int\limits_{-T_j^*}^{T_j} \{h_i, \hat{h}\}_{\Psi(t,z)}dt$, $d_i(z,\varepsilon) = h_i(z_\varepsilon^u) - h_i(z_\varepsilon^s) = \varepsilon M_i(z) + O(\varepsilon^2)$, where $\{,\}$ is the Poisson bracket, $d_i(z,\varepsilon)$ is the Melnikov distance. So, we need symplectic invariant wavelet expressions for Poisson brackets. The computations are produced according to §2–§6.
6. Some hypothesis about strange symmetry of stochastic web in multi-degree-of freedom Hamiltonian systems [8].

8. Symplectic Topology and Wavelets. Now we consider the generalization of our wavelet variational approach to symplectic invariant calculation of closed loops in Hamiltonian systems [13].

We also have the parametrization of our solution by some reduced algebraical problem but in contrast to the general case where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations [14].

The action functional for loops in the phase space is [13]

$$F(\gamma) = \int_\gamma pdq - \int_0^1 H(t,\gamma(t))dt$$

The critical points of F are those loops γ, which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. By the way, all critical points of F are the saddle points of infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, (M,ω) is symplectic manifolds, $H: M \to R$, H is Hamiltonian, X_H is unique Hamiltonian vector field defined by $\omega(X_H(x), v) = -dH(x)(v), \quad v \in T_xM, \quad x \in M$, where ω is the symplectic structure. A T-periodic solution $x(t)$ of the Hamiltonian equations: $\dot{x} = X_H(x) \quad$ on M is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1, R^{2n})$, where $S^1 = R/\mathbf{Z}$, of smooth loops in R^{2n}. Let us define a function $\Phi: \Omega \to R$ by setting

$$\Phi(x)=\int_0^1\frac{1}{2}< -J\dot{x},x> dt-\int_0^1 H(x(t))dt,\quad x\in\Omega$$

The critical points of Φ are the periodic solutions of $\dot{x}=X_H(x)$. Computing the derivative at $x\in\Omega$ in the direction of $y\in\Omega$, we find

$$\Phi'(x)(y)=\frac{d}{d\epsilon}\Phi(x+\epsilon y)|_{\epsilon=0}=\int_0^1 < -J\dot{x}-\bigtriangledown H(x),y> dt$$

Consequently, $\Phi'(x)(y)=0$ for all $y\in\Omega$ iff the loop x satisfies the equation

$$-J\dot{x}(t)-\bigtriangledown H(x(t))=0,$$

i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0)=x(1)$, i.e. periodic of period 1. Periodic loops may be represented by their Fourier series: $x(t)=\sum_{k\in\mathbf{Z}}e^{k2\pi Jt}x_k,\quad x_k\in R^{2k}$, where J is quasicomplex structure. We give relations between quasicomplex structure and wavelets in §9. But now we use the construction [14] for loop parametrization. It is based on the theorem about explicit bijection between the Quadratic Mirror Filters (QMF) and the whole loop group: $LG: S^1\to G$. In particular case we have relation between **QMF**-systems and measurable functions $\chi: S^1\to U(2)$ satisfying

$$\chi(\omega+\pi)=\chi(\omega)\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

in the next explicit form

$$\begin{bmatrix}\hat{\Phi}_0(\omega)&\hat{\Phi}_0(\omega+\pi)\\\hat{\Phi}_1(\omega)&\hat{\Phi}_1(\omega+\pi)\end{bmatrix}=\chi(\omega)\begin{bmatrix}0&1\\1&0\end{bmatrix}+\chi(\omega+\pi)\begin{bmatrix}0&0\\0&1\end{bmatrix},$$

where $\left|\hat{\Phi}_i(\omega)\right|^2+\left|\hat{\Phi}_i(\omega+\pi)\right|^2=2,\quad i=0,1.$

Also, we have symplectic structure on LG

$$\omega(\xi,\eta)=\frac{1}{2\pi}\int_0^{2\pi}<\xi(\theta),\eta'(\theta)> d\theta$$

So, we have the parametrization of periodic orbits (Arnold–Weinstein curves) by reduced QMF equations.

9. Symplectic Hilbert Scales via Wavelets. We can solve many important dynamical problems such that KAM perturbations, spread of energy to higher modes, week turbulence, growths of solutions of Hamiltonian equations only if we consider scales of spaces instead of one functional space. For Hamiltonian system and their perturbations for which we need take into account underlying symplectic structure we need to consider symplectic scales of spaces. So, if $\dot{u}(t) = J\nabla K(u(t))$ is Hamiltonian equation we need wavelet description of symplectic or quasicomplex structure on the level of functional spaces. It is very important that according to [16] Hilbert basis is in the same time a Darboux basis to corresponding symplectic structure. We need to provide Hilbert scale $\{Z_s\}$ with symplectic structure [15], [17]. All what we need is the following. J is a linear operator, $J: Z_\infty \to Z_\infty$, $J(Z_\infty) = Z_\infty$, where $Z_\infty = \cap Z_s$. J determines an isomorphism of scale $\{Z_s\}$ of order $d_J \geq 0$. The operator J with domain of definition Z_∞ is antisymmetric in Z: $< Jz_1, z_2 >_Z = - < z_1, Jz_2 >_Z, z_1, z_2 \in Z_\infty$. Then the triple $\{Z, \{Z_s | s \in R\}, \quad \alpha = < \bar{J}dz, dz >\}$ is symplectic Hilbert scale. So, we may consider any dynamical Hamiltonian problem on functional level. As an example, for KdV equation we have $Z_s = \{u(x) \in H^s(T^1) | \int_0^{2\pi} u(x)dx = 0\}, \ s \in R, \quad J = \partial/\partial x$, J is isomorphism of the scale of order one, $\bar{J} = -(J)^{-1}$ is isomorphism of order -1. According to [18] general functional spaces and scales of spaces such as Holder–Zygmund, Triebel–Lizorkin and Sobolev can be characterized through wavelet coefficients or wavelet transforms. As a rule, the faster the wavelet coefficients decay, the more the analyzed function is regular [18]. Most important for us example is the scale of Sobolev spaces. Let $H_k(R^n)$ is the Hilbert space of all distributions with finite norm $\|s\|^2_{H_k(R^n)} = \int d\xi (1+|\xi|^2)^{k/2} |\hat{s}(\xi)|^2$. Let us consider wavelet transform

$$W_g f(b,a) = \int_{R^n} dx \frac{1}{a^n} \bar{g}(\frac{x-b}{a}) f(x), \quad b \in R^n, \quad a > 0$$

w.r.t. analyzing wavelet g, which is strictly admissible, i.e.

$$C_{g,g} = \int_0^\infty \frac{da}{a} |\hat{g}(\bar{a}k)|^2 < \infty.$$

Then there is a $c \geq 1$ such that

$$c^{-1} \|s\|^2_{H_k(R^n)} \leq \int_{H^n} \frac{dbda}{a} (1 + a^{-2\gamma}) |W_g s(b,a)|^2 \leq c \|s\|^2_{H_k(R^n)}$$

This shows that localization of the wavelet coefficients at small scale is linked to local regularity.

Extended version and related results may be found in [1]-[9].

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