WAVELET APPROACH TO POLYNOMIAL MECHANICAL PROBLEMS

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Abstract. We give wavelet description for nonlinear optimal dynamics (energy minimization in high power electromechanical system). We consider two cases of our general construction. In a particular case we have the solution as a series on shifted Legendre polynomials, which is parametrized by the solution of reduced algebraical system of equations. In the general case we have the solution as a multiresolution expansion from wavelet analysis. In this case the solution is parametrized by solutions of two reduced algebraic problems, one as in the first case and the second is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform, Stationary Subdivision Schemes, the method of Connection Coefficients.

1. Introduction. Many important physical and mechanical problems are reduced to the solving of the systems of nonlinear differential equations with polynomial type of nonlinearities. In this paper and related paper in this volume we consider applications of methods of wavelet analysis to such problems. Wavelet analysis is a relatively novel set of mathematical methods, which gives us possibility to work with well-localized bases in functional spaces and with general type of operators (including pseudodifferential) in such bases. Many examples may be found in papers [9]–[18]. Now we apply our approach to a case of constrained variational problem: the problem of energy minimization in electromechanical systems. We consider a synchronous electrical machine and a mill as load (in this approach we can consider instead of a mill any mechanical load with polynomial approximation for the mechanical moment).

We consider the problem of "electrical economizer" as an optimal control problem. As result of the first stage we give the explicit time description of the optimal dynamics for that electromechanical system. As result of the second stage we give the time dynamics of our system via construction based on the set of switched type functions (Walsh functions), which can be realized on the modern thyristor technique. In this paper using the method of analysis of dynamical process in the Park system [1], which we developed in ref. [9], [10], we consider the optimal control problem in that system. As in [9], [10], our goal is to construct explicit time solutions, which can be used directly in microprocessor control systems. Our consideration is based on the Integral Variational Method, which was developed in [21]. As we shall see later, we can obtain explicit time dependence for all dynamical variables in our optimal control problem. It is based on the fact that optimal control dynamic in our case is given by some nonlinear system of equations which is the extension of initial Park system. Moreover, the equations of optimal dynamics also is the system of Riccati type (we use the quadratic dependence of the mechanical moment). It should be noted that this system of equations is not the pure differential system but it is the mixed differential-algebraic or functional system of equations [19].

In §2 we consider the description of our variational approach, which can be generalized in such a way that allows us to consider it in Hamiltonian (symplectic) approach [15].

In §3 we consider the form of explicit time solutions.

Our initial dynamical problem (without control) is described by the system of 6 nonlinear differential equations, which has the next Cauchy form (for definitions see [9], [10])

$$\frac{di_k}{dt} = \sum_\ell A_\ell i_\ell + \sum_{r,s} A_{rs} i_r i_s + A_k(t)$$

where $A_\ell, A_{rs}(\ell, r, s = \overline{1,6})$ are constants, $A_k(t), (k = \overline{1,5})$ are explicit functions of time, $A_6(i_6, t) = a + di_6 + bi_6^2$ is analytical approximation for the mechanical moment of the mill. At initial stage of the solution of optimal control problem in both methods we need to select from initial set of dynamical variables $i_1, \dots, i_6$ the controlling and the controllable variables. In our case we consider i_1, i_2 as the controlling variables. Because we consider the energy optimization, we use the next general form of energy functional in our electromechanical system $Q = \int_{t_0}^{t}[K_1(i_1, i_2) + K_2(\dot{i}_1, \dot{i}_2)]dt$, where K_1, K_2 are quadratic forms. Thus, our functional is the quadratic functional on the variables i_1, i_2 and its derivatives. Moreover, we may consider the optimization problem with some constraints which are motivated by technical reasons [9], [13]. Then after standard manipulations from the theory of optimal control, we reduce the problem of energy minimization to some extended nonlinear system of equations. As a result of the solution of equations of optimal dynamics we have: 1). the explicit time dependence of the controlling variables $u(t) = \{i_1(t), i_2(t)\}$ which give 2). the optimum of corresponding functional of the energy and 3). explicit time dynamics of the controllable variables $\{i_3, i_4, i_5, i_6\}(t)$. The obtained solutions are given in the next form: $i_k(t) = i_k(0) + \sum_{i=1}^{N} \lambda_k^i X_i(t)$, where in our first case (§3) we have $X_i(t) = Q_i(t)$, where $Q_i(t)$ are shifted Legendre polynomials [21] and λ_k^i are the roots of reduced algebraic system of equations. In our second case (§4)–the general wavelet case $X_i(t)$ correspond to multiresolution expansion in the basis of compactly supported wavelets and λ_k^i are the roots of corresponding algebraic Riccati systems with coefficients, which are given by Fast Wavelet Transform (FWT) [2], Stationary Subdivision Schemes(SSS) [6], the method of Connection Coefficients (CC) [23].

Giving the controlling variables in the explicit form, we have optimal, according to energy, dynamics in our electromechanical systems. Evidently the technical realization of controlling variables as an arbitrary continuous functions of time is impossible, but we can replace them by their re-expansions on the base of switching type functions, which can be realized now on the modern thyristor technique. We considered this re-expansion in [9], [13], where we used Walsh and Haar functions [3] as a base set of switching type functions. This is a special case of general sequency analysis [20]. About practical realization of expansions from §4 on the base of general wavelet packet basis [4] see [18].

2. Polynomial Dynamics. Our problems may be formulated as the systems of ordinary differential equations $dx_i/dt = f_i(x_j, t)$, $(i, j = 1, ..., n)$ with fixed initial conditions $x_i(0)$, where f_i are not more than polynomial functions of dynamical variables x_j and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \le t \le 1$. Let us consider a set of functions $\Phi_i(t) = x_i dy_i/dt + f_i y_i$ and a set of functionals $F_i(x) = \int_0^1 \Phi_i(t)dt - x_i y_i \mid_0^1$, where $y_i(t)(y_i(0) = 0)$ are dual variables. It is obvious that the initial system and the system $F_i(x) = 0$ are equivalent. In the second paper in this volume we consider the symplectization of this approach. Now we consider formal expansions for x_i, y_i:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t), \tag{1}$$

where because of initial conditions we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients λ_i^k of expansions (1):

$$\sum_k \mu_{kr} \lambda_i^k - \gamma_i^r(\lambda_j) = 0 \tag{2}$$

Its coefficients are $\mu_{kr} = \int_0^1 \varphi_k'(t)\varphi_r(t)dt, \quad \gamma_i^r = \int_0^1 f_i(x_j,t)\varphi_r(t)dt$. Now, when we solve system (2) and determine unknown coefficients from formal expansion (1) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (1) with N terms then we have from (2) the system of $N \times n$ algebraical equations and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (polynomial) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^N \lambda_i^k X_k(t), \tag{3}$$

where coefficients λ_i^k are roots of the corresponding reduced algebraical problem (2). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (2). But in general case, when the problem of computations of coefficients of reduced algebraical system (2) is not solved explicitly as in the quadratic case, which we shall consider below, we have also parametrization of solution (1) by solution of corresponding problems, which appear when we need to calculate coefficients of (2). As we shall see, these problems may be explicitly solved in wavelet approach.

3. The solutions. Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (3), where in our first case we have $X_k(t) = Q_k(t)$, where $Q_k(t)$ are shifted Legendre polynomials and λ_k^i are roots of reduced quadratic system of equations. In wavelet case $X_k(t)$ correspond to multiresolution expansions in the base of compactly supported wavelets and λ_k^i are the roots of corresponding general polynomial system (2) with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method to give the reduction from differential to algebraical system of equations we need compute the objects γ_a^j and μ_{ji}, which are constructed from objects:

$$\begin{aligned}
\sigma_i &\equiv \int_0^1 X_i(\tau)d\tau = (-1)^{i+1}, \qquad (4)\\
\nu_{ij} &\equiv \int_0^1 X_i(\tau)X_j(\tau)d\tau = \sigma_i\sigma_j + \frac{\delta_{ij}}{(2j+1)},\\
\mu_{ji} &\equiv \int X_i'(\tau)X_j(\tau)d\tau = \sigma_j F_1(i,0) + F_1(i,j),\\
& \quad F_1(r,s) = [1-(-1)^{r+s}]\hat{s}(r-s-1), \quad \hat{s}(p) = \begin{cases} 1, & p \ge 0\\ 0, & p < 0\end{cases}\\
\beta_{klj} &\equiv \int_0^1 X_k(\tau)X_l(\tau)X_j(\tau)d\tau = \sigma_k\sigma_l\sigma_j + \\
& \quad \alpha_{klj} + \frac{\sigma_k\delta_{jl}}{2j+1} + \frac{\sigma_l\delta_{kj}}{2k+1} + \frac{\sigma_j\delta_{kl}}{2l+1},\\
\alpha_{klj} &\equiv \int_0^1 X_k^* X_l^* X_j^* d\tau = \frac{1}{(j+k+l+1)R(1/2(i+j+k))} \times\\
& \quad R(1/2(j+k-l))R(1/2(j-k+l))R(1/2(-j+k+l)),
\end{aligned}$$

if $j+k+l = 2m, m \in Z$, and $\alpha_{klj} = 0$ if $j+k+l = 2m+1$; $R(i) = (2i)!/(2^i i!)^2$, $Q_i = \sigma_i + P_i^*$, where the second equality in the formulae for $\sigma, \nu, \mu, \beta, \alpha$ hold for the first case.

4. Wavelet computations. Now we give construction for computations of objects (4) in the wavelet case. We use some constructions from multiresolution analysis [8]: a sequence of successive approximation closed subspaces V_j: $...V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset ...$ satisfying the following properties: $\bigcap_{j\in\mathbf{Z}} V_j = 0$, $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R})$, $f(x) \in V_j <=> f(2x) \in V_{j+1}$ There is a function $\varphi \in V_0$ such that $\{\varphi_{0,k}(x) = \varphi(x-k)_{k\in\mathbf{Z}}\}$ forms a Riesz basis for V_0. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x) = \varphi(x-i)$. Scaling relation that defines φ, ψ are

$$\begin{aligned}
\varphi(x) &= \sum_{k=0}^{N-1} a_k\varphi(2x-k) = \sum_{k=0}^{N-1} a_k\varphi_k(2x),\\
\psi(x) &= \sum_{k=-1}^{N-2} (-1)^k a_{k+1}\varphi(2x+k)
\end{aligned}$$

Let be $f:\mathbf{R}\longrightarrow\mathbf{C}$ and the wavelet expansion is

$$f(x)=\sum_{\ell\in\mathbf{Z}}c_\ell\varphi_\ell(x)+\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}}c_{jk}\psi_{jk}(x) \tag{5}$$

The indices k,ℓ and j represent translation and scaling, respectively: $\varphi_{jl}(x)=2^{j/2}\varphi(2^jx-\ell)$, $\psi_{jk}(x)=2^{j/2}\psi(2^jx-k)$. The set $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ forms a Riesz basis for V_j. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}. Just as V_j is spanned by dilation and translations of the scaling function, so are W_j spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$. If in formulae (5) $c_{jk}=0$ for $j\geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only $f(x)=\sum\limits_{\ell\in\mathbf{Z}}c_{J\ell}\varphi_{J\ell}(x)$. This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet $\psi(x)$, which has k vanishing moments $\int x^k\psi(x)dx=0$, or equivalently $x^k=\sum c_\ell\varphi_\ell(x)$ for each k, $0\leq k\leq K$. Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0,N-1]$ and $N/2$ vanishing moments. There exists $\lambda>0$ such that DN has λN continuous derivatives; for small $N,\lambda\geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$. Let be $\varphi^n_\ell=d^n\varphi_\ell(x)/dx^n$. We derive the wavelet - Galerkin approximation of a differentiated $f(x)$ as $f^d(x)=\sum_\ell c_l\varphi^d_\ell(x)$ and values $\varphi^d_\ell(x)$ can be expanded in terms of $\varphi(x)$:
$\phi^d_\ell(x)=\sum\limits_m\lambda_m\varphi_m(x),\quad \lambda_m=\int\limits_{-\infty}^{\infty}\varphi^d_\ell(x)\varphi_m(x)dx$ The coefficients λ_m are 2-term connection coefficients. In general we need to find $(d_i\geq 0)$

$$\Lambda^{d_1d_2\dots d_n}_{\ell_1\ell_2\dots\ell_n}=\int\limits_{-\infty}^{\infty}\prod\varphi^{d_i}_{\ell_i}(x)dx \tag{6}$$

For Riccati case we need to evaluate two and three connection coefficients $\Lambda^{d_1d_2}_\ell=\int_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)dx,\quad \Lambda^{d_1d_2d_3}=\int\limits_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)\varphi^{d_3}_m(x)dx.$
According to CC method [23] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0,N-1]$. For a fixed triple (d_1,d_2,d_3) only some $\Lambda^{d_1d_2d_3}_{\ell m}$ are nonzero : $2-N\leq\ell\leq N-2,\quad 2-N\leq m\leq N-2,\quad |\ell-m|\leq N-2$. There are $M=3N^2-9N+7$ such pairs (ℓ,m). Let $\Lambda^{d_1d_2d_3}$ be an M-vector, whose components are numbers $\Lambda^{d_1d_2d_3}_{\ell m}$. Then we have the first

key result: Λ satisfy the system of equations ($d = d_1 + d_2 + d_3$): $A\Lambda^{d_1d_2d_3} = 2^{1-d}\Lambda^{d_1d_2d_3}$, $A_{\ell,m;q,r} = \sum\limits_p a_p a_{q-2\ell+p} a_{r-2m+p}$. By moment equations we have created a system of $M + d + 1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second key result gives us the 2-term connection coefficients: $A\Lambda^{d_1d_2} = 2^{1-d}\Lambda^{d_1d_2}$, $d = d_1 + d_2$, $A_{\ell,q} = \sum\limits_p a_p a_{q-2\ell+p}$. For nonquadratic case we have analogously additional linear problems for objects (6). Also, we use FWT [2] and SSS [6] for computing coefficients of reduced algebraic systems. We use for modelling D6, D8, D10 functions and programs RADAU and DOPRI for testing [19].

As a result we obtained the explicit time solution (3) of our problem. In comparison with wavelet expansion on the real line which we use now and in calculation of Galerkin approximation, Melnikov function approach, etc also we need to use periodized wavelet expansion, i.e. wavelet expansion on finite interval. Also in the solution of perturbed system we have some problem with variable coefficients. For solving last problem we need to consider one more refinement equation for scaling function $\phi_2(x)$: $\phi_2(x) = \sum\limits_{k=0}^{N-1} a_k^2 \phi_2(2x - k)$ and corresponding wavelet expansion for variable coefficients $b(t)$: $\sum\limits_k B_k^j(b)\phi_2(2^j x - k)$, where $B_k^j(b)$ are functionals supported in a small neighborhood of $2^{-j}k$ [7].

The solution of the first problem consists in periodizing. In this case we use expansion into periodized wavelets defined by $\phi^{per}_{-j,k}(x) = 2^{j/2}\sum\limits_Z \phi(2^j x + 2^j\ell - k)$ [5]. All these modifications lead only to transformations of coefficients of reduced algebraic system, but general scheme remains the same.

Extended version and related results may be found in [9]–[18].

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