WAVELETS IN OPTIMIZATION AND APPROXIMATIONS

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Abstract. We give the explicit time description of the following problems: dynamics of storage rings, optimal dynamics for some important electromechanical system, Galerkin approximation for beam oscillations in liquid, computations of Melnikov functions for perturbed Hamiltonian systems. All these problems are reduced to the problem of the solving of the systems of differential equations with polynomial nonlinearities with or without some constraints. The first main part of our construction is some variational approach to this problem, which reduces initial problem to the problem of the solution of functional equations at the first stage and some algebraical problems at the second stage. We consider also two private cases of our general construction. In the first case (particular) we have the solution as a series on shifted Legendre polynomials, which is parameterized by the solution of reduced algebraical system of equations. In the second case (general) we have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, the first one is the same as in the first case and the second one is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform, Stationary Subdivision Schemes, the method of Connection Coefficients.

We give the explicit time description of the following problems: dynamics and optimal dynamics for important nonlinear dynamical systems, Galerkin approximation for some class of partial differential equations, computations of Melnikov function for perturbed Hamiltonian systems. All these problems are reduced to the problem of the solving of the systems of differential equations with polynomial nonlinearities with or without some constraints. The first main part of our construction is some variational approach to this problem, which reduces initial problem to the problem of the solution of functional equations at the first stage and some algebraical problems at the second stage. We consider also two private cases of our general construction.

In the first case (particular) we have for Riccati type equations the solution as a series on shifted Legendre polynomials, which is parametrized by the solution of reduced algebraical (also Riccati) system of equations [5].

In the second case (general polynomial systems) we have the solution in a compactly supported wavelet basis [8], [11]-[13]. Multiresolution expansion is the second main part of our construction. In this case the solution is parametrized by solutions of two reduced algebraic problems, one as in the first case and the second is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform (FWT) [1], Stationary Subdivision Schemes (SSS) [3], the method of Connection Coefficients (CC) [15].

We use our general construction for solution of important technical problems: minimization of energy and detecting signals from oscillations of a submarine and also for calculations of orbital motions in storage rings [6], [7].

Our initial problem comes from very important technical problem – minimization of energy in electromechanical system with enormous expense of energy. That is synchronous drive of the mill–the electrical machine with the mill as load. It is described by Park system of equations [5]:

$$\frac{di_k}{dt} = \sum_\ell A_\ell i_\ell + \sum_{r,s} A_{r,s} i_r i_s + A_k(t),$$

where $A_\ell, (k,\ell,r,s = \overline{1,6})$ are constants, $A_k(t), (k = \overline{1,5})$ are explicit functions of time, $A_6(i_6, t) = a + di_6 + bi_6^2$ is analytical approximation for the mechanical moment of the mill. In our case we consider i_1, i_2 as the controlling variables. Because we consider the energy optimization, we use the next general form of energy functional in our electromechanical system

$$Q = \int_{t_0}^{t} [K_1(i_1, i_2) + K_2(\dot{i}_1, \dot{i}_2)] dt,$$

where K_1, K_2 are quadratic forms. Moreover, we consider the optimization problem with some constraints, which are motivated by technical reasons. After the manipulations from the theory of optimal control, we reduce the problem of energy minimization to the some nonlinear system of equations. Thus for the Lagrangian optimization we have the system of 13 equations (12– differential equations, 1–functional one). For the Hamiltonian optimization we have the system of 12 equations (10–differential equations, 2–algebraical ones). In both cases obtained systems of equations are the systems of Riccati type. As result of solution of equations of optimal dynamics we have: 1. the explicit time dependence of the controlling variables $u(t) = \{i_1(t), i_2(t)\}$, which give 2. the optimum of corresponding functional of energy and 3. explicit time dynamics of the controllable variables $\{i_3, i_4, i_5, i_6\}(t)$.

Next we consider the construction of explicit time solution. The obtained solutions are given in the next form:

$$i_k(t) = i_k(0) + \sum_{i=1}^{N} \lambda_k^i X_i(t),$$

where in our first case we have $X_i(t) = Q_i(t)$, where $Q_i(t)$ are shifted Legendre polynomials [14] and λ_k^i are roots of reduced algebraic system of equations. In wavelet case $X_i(t)$ correspond to multiresolution expansions in the base of compactly supported wavelets and λ_k^i are the roots of corresponding algebraic Riccati systems with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method of [14] to give the reduction from differential to algebraical system of equations we need compute the objects $\gamma_a^j(i_b)$ and μ_{ji}, where in Lagrangian case $a = \overline{1,13}, b = \overline{1,13}$. We compute it by the formulae:

$$\gamma_a^j(i_b) = t_f \int_0^1 \phi_a(i_b, \tau) X_j(\tau) d\tau, \quad \mu_{ji} = \int\limits_0^1 X_i'(\tau) X_j(\tau) d\tau,$$

where ϕ_a is RHS of initial equations. Then the reduced algebraical system has the form:

$$\sum_{i=1}^{N} \mu_{ji} \lambda_a^i - \gamma_a^j(\lambda_b) = 0,$$

where coefficients of algebraical systems are constructed from objects:

$$\begin{aligned} \sigma_i &\equiv \int_0^1 X_i(\tau) d\tau = (-1)^{i+1}, \qquad (1) \\ \nu_{ij} &\equiv \int_0^1 X_i(\tau) X_j(\tau) d\tau = \sigma_i \sigma_j + \frac{\delta_{ij}}{(2j+1)}, \end{aligned}$$

$$
\begin{aligned}
\beta_{klj} &\equiv \int_0^1 X_k(\tau)X_l(\tau)X_j(\tau)d\tau = \sigma_k\sigma_l\sigma_j + \alpha_{klj} + \frac{\sigma_k\delta_{jl}}{2j+1} + \frac{\sigma_l\delta_{kj}}{2k+1} + \frac{\sigma_j\delta_{kl}}{2l+1}, \\
\alpha_{klj} &\equiv \int_0^1 X_k^*X_l^*X_j^*d\tau = \frac{R(1/2(j+k-l))R(1/2(j-k+l))R(1/2(-j+k+l))}{(j+k+l+1)R(1/2(i+j+k))},
\end{aligned}
$$

if $j+k+l=2m, m\in Z$, and $\alpha_{klj}=0$ if $j+k+l=2m+1$; $R(i)=(2i)!/(2^i i!)^2$, $X_i=\sigma_i+X_i^*$, where the second equality in the formulae for σ,ν,β,α hold for the first case.

Now we give construction for the computations of objects (1) in the wavelet case.

We use some constructions from multiresolution analysis: a sequence of successive approximation closed subspaces V_j

$$
...V_2\subset V_1\subset V_0\subset V_{-1}\subset V_{-2}\subset ...
$$

satisfying the following properties:

$$
\bigcap_{j\in\mathbf{Z}} V_j = 0, \quad \overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2(\mathbf{R}),
$$

where from $f(x)\in V_j$ we have $f(2x)\in V_{j+1}$. There is a function $\varphi\in V_0$ such that $\{\varphi_{0,k}(x)=\varphi(x-k)_{k\in\mathbf{Z}}\}$ forms a Riesz basis for V_0.

We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbf{R})$ [2]. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x)=\varphi(x-i)$. Scaling or refinement equations that defines φ,ψ are

$$
\begin{aligned}
\varphi(x) &= \sum_{k=0}^{N-1} a_k\varphi(2x-k) = \sum_{k=0}^{N-1} a_k\varphi_k(2x), \\
\psi(x) &= \sum_{k=-1}^{N-2} (-1)^k a_{k+1}\varphi(2x+k).
\end{aligned}
$$

Let be $f:\mathbf{R}\longrightarrow\mathbf{C}$ and the wavelet expansion is

$$
f(x) = \sum_{\ell\in\mathbf{Z}} c_\ell\varphi_\ell(x) + \sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}} c_{jk}\psi_{jk}(x). \tag{2}
$$

The indices k,ℓ and j represent translation and scaling, respectively

$$
\varphi_{jl}(x) = 2^{j/2}\varphi(2^jx-\ell), \quad \psi_{jk}(x) = 2^{j/2}\psi(2^jx-k).
$$

The set $\{\varphi_{j,k}\}_{k\in\mathbf{Z}}$ forms a Riesz basis for V_j. Let W_j be the orthonormal complement of V_j with respect to V_{j+1}. Just as V_j is spanned by dilation and translations of the scaling function, so W_j are spanned by translations and dilation of the mother wavelet $\psi_{jk}(x)$.

All expansions which we used are based on the following properties:

$$\{\varphi_{jk}\}_{j\geq 0, k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}),$$

$$V_{j+1} = V_j \bigoplus W_j, \qquad L^2(\mathbf{R}) = \overline{V_0 \bigoplus_{j=0}^{\infty} W_j},$$

$$\{\varphi_{0,k}, \psi_{j,k}\}_{j\geq 0, k\in\mathbf{Z}} \quad \text{is an orthonormal basis for} L^2(\mathbf{R}),$$

$$\{\varphi_{j,k}, \psi_{\ell,k}; 0\leq j\leq J\leq \ell, k\in\mathbf{Z}\} \text{is an orthonormal basis for} L^2(\mathbf{R}),$$

$$\int_{-\infty}^{\infty}\varphi(x)dx = 1.$$

If in formulae (2) $c_{jk} = 0$ for $j \geq J$, then $f(x)$ has an alternative expansion in terms of dilated scaling functions only

$$f(x) = \sum_{\ell\in\mathbf{Z}} c_{J\ell}\varphi_{J\ell}(x) \tag{3}$$

This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet $\psi(x)$, which has k vanishing moments

$$\int x^k\psi(x)dx = 0,$$

or, equivalently

$$x^k = \sum c_\ell\varphi_\ell(x) \quad \text{for each } k, \quad 0\leq k\leq K$$

Also we have the shortest possible support: scaling function DN (where N is even integer) will have support $[0, N-1]$ and $N/2$ vanishing moments.

There exists $\lambda > 0$ such that DN has λN continuous derivatives; for small $N, \lambda \geq 0.55$. To solve our second associated linear problem we need to evaluate derivatives of $f(x)$ in terms of $\varphi(x)$.

Let be $\varphi_\ell^n = d^n\varphi_\ell(x)/dx^n$. We derive the wavelet - Galerkin approximation of a differentiated $f(x)$ as $f^d(x) = \sum_\ell c_l\varphi_\ell^d(x)$ and values $\varphi_\ell^d(x)$ can be expanded in terms of $\varphi(x)$

$$\phi_\ell^d(x) = \sum_m \lambda_m\varphi_m(x),$$

where

$$\lambda_m = \int_{-\infty}^{\infty}\varphi_\ell^d(x)\varphi_m(x)dx.$$

The coefficients λ_m are 2-term connection coefficients [15]. In general we need to find

$$\Lambda(\ell_1,\ell_2,...,\ell_n,d_1,d_2,...,d_n)=\Lambda^{d_1 d_2 \dots d_n}_{\ell_1\ell_2\dots\ell_n}=\int\limits_{-\infty}^{\infty}\prod\varphi^{d_i}_{\ell_i}(x)dx$$

For our Riccati case we need to evaluate two and three connection coefficients ($d_i\geq 0$)

$$\Lambda^{d_1 d_2}_\ell=\int_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)dx,\quad \Lambda^{d_1 d_2 d_3}=\int\limits_{-\infty}^{\infty}\varphi^{d_1}(x)\varphi^{d_2}_\ell(x)\varphi^{d_3}_m(x)dx.$$

According to [15] we use the next construction. When N in scaling equation is a finite even positive integer the function $\varphi(x)$ has compact support contained in $[0,N-1]$. For a fixed triple (d_1,d_2,d_3) only some $\Lambda^{d_1d_2d_3}_{\ell m}$ are nonzero : $2-N\leq\ell\leq N-2,\quad 2-N\leq m\leq N-2,\quad |\ell-m|\leq N-2$. There are $M=3N^2-9N+7$ such pairs (ℓ,m). Let $\Lambda^{d_1d_2d_3}$ be an M-vector, whose components are numbers $\Lambda^{d_1d_2d_3}_{\ell m}$. Then we have the first key result: Λ satisfy the system of equations

$$\begin{aligned} A\Lambda^{d_1d_2d_3} &= 2^{1-d}\Lambda^{d_1d_2d_3},\ d=d_1+d_2+d_3, \\ A_{\ell,m;q,r} &= \sum_p a_p a_{q-2\ell+p}a_{r-2m+p} \end{aligned}\tag{4}$$

By moment equations we have created a system of $M+d+1$ equations in M unknowns. It has rank M and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second key result gives us the 2-term connection coefficients ($d=d_1+d_2$):

$$A\Lambda^{d_1d_2}=2^{1-d}\Lambda^{d_1d_2},\quad A_{\ell,q}=\sum_p a_p a_{q-2\ell+p}\tag{5}$$

We use for modelling D6, D8, D10 functions and programs RADAU and DOPRI for testing.

Also, we use FWT and SSS for computing coefficients of reduced algebraic systems.

As a result we obtained the explicit time solution of optimal control problem in form (3). The generalization to arbitrary polynomial systems is evidently. Instead of two reduced linear problems (4), (5) for computational objects (1), we have analogous additional linear problems.

Analogously we consider in wavelet approach related problems: computations in Galerkin approximations and routes to chaos in Melnikov approach [9], [10]. These problems are related to a problem of detecting signals from an oscillating submarine.

In 2-mode Galerkin approximation for beam contacting with ideal compressible liquid in a channel we have the next system of equations:

$$\begin{aligned}
\dot{x}_1 &= x_2, \quad \dot{x}_3 = x_4, \quad \dot{x}_5 = r, \quad \dot{x}_6 = s, \\
\dot{x}_2 &= -ax_1 - b[\cos(x_5) + \cos(x_6)]x_1 - dx_1^3 - mdx_1x_3^2 - px_2 - \varphi(x_5) \\
\dot{x}_4 &= ex_3 - f[\cos(x_5) + \cos(x_6)]x_3 - gx_3^3 - kx_1^2x_3 - gx_4 - \psi(x_5)
\end{aligned}$$

or in Hamiltonian form

$$\dot{x} = J \cdot \nabla H(x) + \varepsilon g(x, \Theta), \quad \dot{\Theta} = \omega, (x, \Omega) \in R^4 \times T^2,$$

for $\varepsilon = 0$ we have:

$$\dot{x} = J \cdot \nabla H(x), \quad \dot{\Theta} = \omega \tag{6}$$

We solve two problems related with these systems of equations. We need to compute explicit time solution for perturbed and unperturbed systems. The solution for unperturbed system (6) we use next for computing Melnikov functions

$$M(\Theta) = \int_{-\infty}^{\infty} \nabla H(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), \omega t + \Theta) dt \tag{7}$$

$$M^{m/n}(t_0) = \int_{0}^{mT} DH(x_\alpha(t)) \wedge (x_\alpha(t), t + t_0) dt \tag{8}$$

which we use for detecting chaotic and quasiperiodic regimes of oscillations [9], [10]. In comparison with wavelet expansion on the real line which we use in optimal control problem and in calculation of Galerkin approximation, in Melnikov function approach for the computation of (quasi)periodic regimes (8) we need to use periodized wavelet expansion, i.e. wavelet expansion on finite interval [2]. Also in the solution of perturbed system we have some problem with variable coefficients. For solving last problem we need to consider one more refinement equation for scaling function $\phi_2(x)$:

$$\phi_2(x) = \sum_{k=0}^{N-1} a_k^2 \phi_2(2x - k)$$

and corresponding wavelet expansion for variable coefficients $b(t)$:

$$\sum_k B_k^j(b)\phi_2(2^j x - k),$$

where $B_k^j(b)$ are functionals supported in a small neighborhood of $2^{-j}k$ [4].

The solution of the first problem consists in periodizing. In this case we expand (quasi)periodic orbit into periodized wavelets defined by [2]:

$$\phi_{-j,k}^{per}(x) = 2^{j/2}\sum_Z \phi(2^j x + 2^j \ell - k).$$

All these modifications lead only to transformations of coefficients of reduced algebraic system, but general scheme remains the same.

All this approaches we also use in very important now nonlinear dynamical problems from accelerator physics. As an example we consider a problem of computations of orbital motions in storage rings. Starting from Hamiltonian, which described classical dynamics in storage rings and using Serret–Frenet parametrization, we have after standard manipulations with truncation of power series expansion of square root the corresponding equations of motion [6]:

$$\begin{aligned}
&\frac{d}{ds}x = \frac{p_x + H\cdot z}{[1+f(p_\sigma)]}; \qquad (9)\\
&\frac{d}{ds}p_x = \frac{[p_z - H\cdot x]}{[1+f(p_\sigma)]}\cdot H - [K_x^2 + g]\cdot x + N\cdot z\\
&+K_x\cdot f(p_\sigma) - \frac{\lambda}{2}\cdot(x^2 - z^2) - \frac{\mu}{6}(x^3 - 3xz^2);\\
&\frac{d}{ds}z = \frac{p_z - H\cdot x}{[1+f(p_\sigma)]};\\
&\frac{d}{ds}p_z = -\frac{[p_x + H\cdot z]}{[1+f(p_\sigma)]}\cdot H - [K_z^2 - g]\cdot z\\
&+N\cdot x + K_z\cdot f(p_\sigma) - \lambda\cdot xz - \frac{\mu}{6}(z^3 - 3x^2 z);\\
&\frac{d}{ds}\sigma = 1 - [1 + K_x\cdot x + K_z\cdot z]\cdot f'(p_\sigma) -\\
&\frac{1}{2}\cdot\frac{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2}{[1+f(p_\sigma)]^2}\cdot f'(p_\sigma);\\
&\frac{d}{ds}p_\sigma = -\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right].
\end{aligned}$$

We use series expansion of function $f(p_\sigma)$ and the corresponding expansion of RHS of equations (9). Then taking into account only an arbitrary polynomial (in terms of dynamical variables) expressions and neglecting all nonpolynomial types of expressions, we consider such approximations of RHS, which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variable s ("time" in our case, if we consider our system of equations as dynamical problem). After applying our approaches we also have description of complicated accelerator dynamics according to formulae (3), (7), (8) [6], [7].

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