WAVELET APPROACH TO ACCELERATOR PROBLEMS, II. METAPLECTIC WAVELETS

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Abstract

This is the second part of a series of talks in which we present applications of wavelet analysis to polynomial approximations for a number of accelerator physics problems. According to the orbit method and by using construction from the geometric quantization theory we construct the symplectic and Poisson structures associated with generalized wavelets by using metaplectic structure and corresponding polarization. The key point is a consideration of semidirect product of Heisenberg group and metaplectic group as subgroup of automorphisms group of dual to symplectic space, which consists of elements acting by affine transformations.

1 INTRODUCTION

In this paper we continue the application of powerful methods of wavelet analysis to polynomial approximations of nonlinear accelerator physics problems. In part 1 we considered our main example and general approach for constructing wavelet representation for orbital motion in storage rings. But now we need take into account the Hamiltonian or symplectic structure related with system (1) from part 1. Therefore, we need to consider instead of compactly supported wavelet representation from part 1 the generalized wavelets, which allow us to consider the corresponding symplectic structures. By using the orbit method and constructions from the geometric quantization theory we consider the symplectic and Poisson structures associated with Weyl– Heisenberg wavelets by using metaplectic structure and the corresponding polarization. In part 3 we consider applications to construction of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems.

In wavelet analysis the following three concepts are used now: 1). a square integrable representation U of a group G, 2). coherent states over G 3). the wavelet transform associated to U.

We have three important particular cases:
a) the affine $(ax+b)$ group, which yields the usual wavelet analysis

$$[\pi(b,a)f](x)=\frac{1}{\sqrt{a}}f\left(\frac{x-b}{a}\right)$$

b). the Weyl-Heisenberg group which leads to the Gabor functions, i.e. coherent states associated with windowed Fourier transform.

$$[\pi(q,p,\varphi)f](x)=\exp(i\mu(\varphi-p(x-q))f(x-q)$$

In both cases time-frequency plane corresponds to the phase space of group representation.
c). also, we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group [7]. But usual representation of it is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in that case) restores square – integrability: $G_{aWH\longrightarrow}$ homogeneous space. Also, we have more general approach which allows to consider wavelets corresponding to more general groups and representations [8], [9]. Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (c) analogously.

2 METAPLECTIC GROUP AND REPRESENTATIONS

Let $Sp(n)$ be symplectic group, $Mp(n)$ be its unique two- fold covering – metaplectic group. Let V be a symplectic vector space with symplectic form (,), then $R\oplus V$ is nilpotent Lie algebra - Heisenberg algebra:

$$[R,V]=0,\quad [v,w]=(v,w)\in R,\quad [V,V]=R.$$

$Sp(V)$ is a group of automorphisms of Heisenberg algebra.

Let N be a group with Lie algebra $R\oplus V$, i.e. Heisenberg group. By Stone– von Neumann theorem Heisenberg group has unique irreducible unitary representation in which $1\mapsto i$. This representation is projective: $U_{g_1}U_{g_2}=c(g_1,g_2)\cdot U_{g_1g_2}$, where c is a map: $Sp(V)\times Sp(V)\rightarrow S^1$, i.e. c is S^1-cocycle.

But this representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F = N' \bowtie Mp(V)$, $\bowtie$ is semidirect product (we consider instead of $N = R \oplus V$ the $N' = S^1 \times V$, $S^1 = (R/2\pi Z)$). Let V^* be dual to V, $G(V^*)$ be automorphism group of V^*.Then F is subgroup of $G(V^*)$, which consists of elements, which acts on V^* by affine transformations.
This is the key point!

Let $q_1, ..., q_n; p_1, ..., p_n$ be symplectic basis in V, $\alpha = pdq = \sum p_i dq_i$ and $d\alpha$ be symplectic form on V^*. Let M be fixed affine polarization, then for $a \in F$ the map $a \mapsto \Theta_a$ gives unitary representation of G: $\Theta_a : H(M) \rightarrow H(M)$

Explicitly we have for representation of N on H(M):

$$(\Theta_q f)^*(x) = e^{-iqx} f(x), \quad \Theta_p f(x) = f(x-p)$$

The representation of N on H(M) is irreducible. Let A_q, A_p be infinitesimal operators of this representation

$$A_q = \lim_{t\rightarrow 0} \frac{1}{t}[\Theta_{-tq} - I], \quad A_p = \lim_{t\rightarrow 0} \frac{1}{t}[\Theta_{-tp} - I],$$

then

$$A_q f(x) = i(qx)f(x), \quad A_p f(x) = \sum p_j \frac{\partial f}{\partial x_j}(x)$$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group F is the algebra of all (nonhomogeneous) quadratic polynomials of (p,q) relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, ..., q_n, p_1, ..., p_n, q_i q_j, q_i p_j, p_i p_j, \quad i, j = 1, ..., n, \quad i \leq j,$

$$\begin{aligned}
&PB\ is \quad \{f, g\} = \sum \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \\
&\text{and} \quad \{1, g\} = 0 \quad for \text{ all } g, \\
&\{p_i, q_j\} = \delta_{ij}, \\
&\{p_i q_j, q_k\} = \delta_{ik} q_j, \{p_i q_j, p_k\} = -\delta_{jk} p_i, \\
&\{p_i p_j, q_k\} = \delta_{ik} p_j + \delta_{jk} p_i, \{p_i p_j, p_k\} = 0, \\
&\{q_i q_j, q_k\} = 0, \{q_i q_j, p_k\} = -\delta_{ik} q_j - \delta jk q_i
\end{aligned}$$

so, we have the representation of basic elements
$f \mapsto A_f : 1 \mapsto i, q_k \mapsto ix_k,$

$$\begin{aligned}
p_l &\mapsto \frac{\delta}{\delta x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2}\delta_{ij}, \\
p_k p_l &\mapsto \frac{1}{i}\frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto i x^k x^l
\end{aligned}$$

This gives the structure of the Poisson manifolds to representation of any (nilpotent) algebra or in other words to continuous wavelet transform.

3 THE SEGAL-BARGMAN REPRESENTATION

Let

$$z = \frac{1}{\sqrt{2}}(p - iq), \quad \bar{z} = \frac{1}{\sqrt{2}}(p + iq),$$

$p = (p_1, ..., p_n)$, F_n is the space of holomorphic functions of n complex variables with $(f, f) < \infty$, where

$$(f, g) = (2\pi)^{-n} \int f(z)\overline{g(z)} e^{-|z|^2} dpdq$$

Consider a map $U : H \rightarrow F_n$, where H is with real polarization,F_n is with complex polarization, then we have

$$(U\Psi)(a) = \int A(a, q)\Psi(q)dq,$$

where

$$A(a, q) = \pi^{-n/4} e^{-1/2(a^2+q^2)+\sqrt{2}aq}$$

i.e. the Bargmann formula produce wavelets.We also have the representation of Heisenberg algebra on F_n :

$$\begin{aligned}
U\frac{\partial}{\partial q_j}U^{-1} &= \frac{1}{\sqrt{2}}\left(z_j - \frac{\partial}{\partial z_j}\right), \\
U q_j U^{-1} &= -\frac{i}{\sqrt{2}}\left(z_j + \frac{\partial}{\partial z_j}\right)
\end{aligned}$$

and also : $\omega = d\beta = dp \wedge dq$, where $\beta = i\bar{z}dz$.

4 ORBITAL THEORY FOR WAVELETS

Let coadjoint action be

$$< g \cdot f, Y >=< f, Ad(g)^{-1} Y >,$$

where $<,>$ is pairing $g \in G, \quad f \in g^*, \quad Y \in \mathcal{G}$.

The orbit is $\mathcal{O}_f = G \cdot f \equiv G/G(f)$.

Also,let A=A(M) be algebra of functions, V(M) is A-module of vector fields, A^p is A-module of p-forms. Vector fields on orbit is

$$\sigma(\mathcal{O}, X)_f(\phi) = \frac{d}{dt}(\phi(\exp tXf))\Big|_{t=0}$$

where $\phi \in A(\mathcal{O}), \quad f \in \mathcal{O}$. Then $\mathcal{O}_f$ are homogeneous symplectic manifolds with 2-form

$$\Omega(\sigma(\mathcal{O}, X)_f, \sigma(\mathcal{O}, Y)_f) =< f, [X, Y] >,$$

and $d\Omega = 0$. PB on $\mathcal{O}$ have the next form

$$\{\Psi_1, \Psi_2\} = p(\Psi_1)\Psi_2$$

where p is $A^1(\mathcal{O}) \rightarrow V(\mathcal{O})$ with definition $\Omega(p(\alpha), X) = i(X)\alpha$. Here $\Psi_1, \Psi_2 \in A(\mathcal{O})$ and $A(\mathcal{O})$ is Lie algebra with bracket $\{,\}$.
Now let N be a Heisenberg group. Consider adjoint and coadjoint representations in some particular case. $N = (z, t) \in C \times R, z = p + iq$; compositions in N are $(z, t) \cdot (z', t') = (z + z', t + t' + B(z, z'))$, where $B(z, z') =$

$pq' - qp'$. Inverse element is $(-t,-z)$. Lie algebra n of N is $(\zeta,\tau) \in C \times R$ with bracket $[(\zeta,\tau),(\zeta',\tau')] = (0, B(\zeta,\zeta'))$. Centre is $\tilde{z} \in n$ and generated by (0,1); Z is a subgroup $\exp \tilde{z}$. Adjoint representation N on n is given by formula

$$Ad(z,t)(\zeta,\tau) = (\zeta, \tau + B(z,\zeta))$$

Coadjoint:
for $f \in n^*$, $\quad g=(z,t)$,

$$(g \cdot f)(\zeta,\zeta) = f(\zeta,\tau) - B(z,\zeta)f(0,1)$$

then orbits for which $f|_{\tilde{z}} \neq 0$ are plane in n^* given by equation $f(0,1) = \mu$. If $X = (\zeta,0)$, $\quad Y = (\zeta',0)$, $\quad X,Y \in n$ then symplectic structure is

$$\Omega(\sigma(\mathcal{O},X)_f, \sigma(\mathcal{O},Y)_f) = < f,[X,Y] >= \\ f(0,B(\zeta,\zeta'))\mu B(\zeta,\zeta')$$

Also we have for orbit $\mathcal{O}_\mu = N/Z$ and $\mathcal{O}_\mu$ is Hamiltonian G-space.

5 KIRILLOV CHARACTER FORMULA OR ANALOGY OF GABOR WAVELETS

Let U denote irreducible unitary representation of N with condition $U(0,t) = \exp(it\ell)\cdot 1$, where $\ell \neq 0$,then U is equivalent to representation T_ℓ which acts in $L^2(R)$ according to

$$T_\ell(z,t)\phi(x) = \exp\left(i\ell(t+px)\right)\phi(x-q)$$

If instead of N we consider E(2)/R we have S^1 case and we have Gabor functions on S^1.

6 OSCILLATOR GROUP

Let O be an oscillator group,i.e. semidirect product of R and Heisenberg group N.

Let H,P,Q,I be standard basis in Lie algebra o of the group O and H^*, P^*, Q^*, I^* be dual basis in o^*. Let functional f=(a,b,c,d) be

$$aI^* + bP^* + cQ^* + dH^*.$$

Let us consider complex polarizations

$$h = (H, I, P+iQ), \quad \bar{h} = (I, H, P-iQ)$$

Induced from h representation, corresponding to functional f (for $a>0$), unitary equivalent to the representation

$$W(t,n)f(y) = \exp(it(h-1/2))\cdot U_a(n)V(t),$$

where

$$\begin{aligned} V(t) &= \exp[-it(P^2+Q^2)/2a], \\ P &= -d/dx, \quad Q = iax, \end{aligned}$$

and $U_a(n)$ is irreducible representation of N, which have the form $U_a(z) = exp(iaz)$ on the center of N. Here we have: U(n=(x,y,z)) is Schrödinger representation, $U_t(n) = U(t(n))$ is the representation,which obtained from previous by automorphism (time translation) $n \longrightarrow t(n)$; $\quad U_t(n) = U(t(n))$ is also unitary irreducible representation of N.

$$V(t) = \exp(it(P^2+Q^2+h-1/2))$$

is an operator, which according to Stone–von Neumann theorem has the property

$$U_t(n) = V(t)U(n)V(t)^{-1}.$$

This is our last private case, but according to our approach we can construct by using methods of geometric quantization theory many "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it. Very useful particular spline–wavelet basis with uniform exponential control on stratified and nilpotent Lie groups was considered in [9].

Extended version and related results may be found in [1]-[6].

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