WAVELET APPROACH TO ACCELERATOR PROBLEMS, III. MELNIKOV FUNCTIONS AND SYMPLECTIC TOPOLOGY

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Abstract

This is the third part of a series of talks in which we present applications of methods of wavelet analysis to polynomial approximations for a number of accelerator physics problems. We consider the generalization of our variational wavelet approach to nonlinear polynomial problems to the case of Hamiltonian systems for which we need to preserve underlying symplectic or Poissonian or quasicomplex structures in any type of calculations. We use our approach for the problem of explicit calculations of Arnold–Weinstein curves via Floer variational approach from symplectic topology. The loop solutions are parametrized by the solutions of reduced algebraical problem – matrix Quadratic Mirror Filters equations. Also we consider wavelet approach to the calculations of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems.

1 INTRODUCTION

In this paper we continue the application of powerful methods of wavelet analysis to polynomial approximations of nolinear accelerator physics problems. In part I we considered our main example and general approach for constructing wavelet representation for orbital motion in storage rings. Now we consider two problems of nontrivial dynamics related with complicated differential geometrical and symplectic topological structures of system (1) from part I. In section 2 we give some points of applications of wavelet methods from parts I, II to Melnikov approach in the theory of homoclinic chaos in perturbed Hamiltonian systems. In section 3 we consider another type of wavelet approach, which gives a possibility to parametrize Arnold–Weinstein curves or closed loops in Hamiltonian systems by generalized refinement equations or Quadratic Mirror Filters equations.

2 ROUTES TO CHAOS

Now we give some points of our program of understanding routes to chaos in some Hamiltonian systems in the wavelet approach [1]-[9]. All points are:

1. A model.
2. A computer zoo. The understanding of the computer zoo.
3. A naive Melnikov function approach.
4. A naive wavelet description of (hetero) homoclinic orbits (separatrix) and quasiperiodic oscillations.
5. Symplectic Melnikov function approach.
6. Splitting of separatrix... $\longrightarrow$stochastic web with magic symmetry, Arnold diffusion and all that.

1. As a model we have two frequencies perturbations of particular case of system (1) from part I:

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= -ax_1 - b[\cos(rx_5)+\cos(sx_6)]x_1 -\\
&\quad dx_1^3 - mdx_1x_3^2 - px_2 - \varphi(x_5)\\
\dot{x}_3 &= x_4\\
\dot{x}_4 &= ex_3 - f[\cos(rx_5)+\cos(sx_6)] - gx_3^3 -\\
&\quad kx_1^2x_3 - gx_4 - \psi(x_5)\\
\dot{x}_5 &= 1\\
\dot{x}_6 &= 1
\end{aligned}$$

or in Hamiltonian form

$$\begin{aligned}
\dot{x} &= J\cdot\nabla H(x) + \varepsilon g(x,\Theta),\\
\dot{\Theta} &= \omega, \quad (x,\Theta)\in R^4\times T^2,\\
T^2 &= S^1\times S^1,
\end{aligned}$$

for $\varepsilon = 0$ we have:

$$\dot{x} = J\cdot\nabla H(x), \quad \dot{\Theta}=\omega \tag{1}$$

2. For pictures and details one can see [3], [8]. The key point is the splitting of separatrix (homoclinic orbit) and transition to fractal sets on the Poincare sections.
3. For $\varepsilon = 0$ we have homoclinic orbit $\bar{x}_0(t)$ to the hyperbolic fixed point x_0. For $\varepsilon \neq 0$ we have normally hyperbolic invariant torus T_ε and condition on transversally intersection of stable and unstable manifolds $W^s(T_\varepsilon)$ and $W^u(T_\varepsilon)$ in terms of Melnikov functions $M(\Theta)$ for $\bar{x}_0(t)$.

$$M(\Theta) = \int_{-\infty}^{\infty} \nabla H(\bar{x}_0(t))\wedge g(\bar{x}_0(t),\omega t+\Theta)dt$$

This condition has the next form:

$$M(\Theta_0) = 0$$

$$\sum_{j=1}^{2} \omega_j \frac{\partial}{\partial \Theta_j} M(\Theta_0) \neq 0$$

According to the approach of Birkhoff-Smale-Wiggins we determined the region in parameter space in which we observe the chaotic behaviour [3], [8].

4. If we cannot solve equations (1) explicitly in time, then we use the wavelet approach from part I for the computations of homoclinic (heteroclinic) loops as the wavelet solutions of system (1). For computations of quasiperiodic Melnikov functions

$$M^{m/n}(t_0) = \int_0^{mT} DH(x_\alpha(t)) \wedge g(x_\alpha(t), t + t_0) dt$$

we used periodization of wavelet solution from part I.

5. We also used symplectic Melnikov function approach

$$\begin{aligned} M_i(z) &= \lim_{j\to\infty} \int_{-T_j^*}^{T_j} \{h_i, \hat{h}\}_{\Psi(t,z)} dt \\ d_i(z,\varepsilon) &= h_i(z_\varepsilon^u) - h_i(z_\varepsilon^s) = \varepsilon M_i(z) + O(\varepsilon^2) \end{aligned}$$

where $\{,\}$ is the Poisson bracket, $d_i(z,\varepsilon)$ is the Melnikov distance. So, we need symplectic invariant wavelet expressions for Poisson brackets. The computations are produced according to part II.

6. Some hypothesis about strange symmetry of stochastic web in multi-degree-of freedom Hamiltonian systems [9].

3 WAVELET PARAMETRIZATION IN FLOER APPROACH.

Now we consider the generalization of our wavelet variational approach to the symplectic invariant calculation of Arnold–Weinstein curves (closed loops) in Hamiltonian systems [10]. We also have the parametrization of our solution by some reduced algebraical problem but in contrast to the general case where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations [11].

The action functional for loops in the phase space is [10]

$$F(\gamma) = \int_\gamma p dq - \int_0^1 H(t, \gamma(t)) dt$$

The critical points of F are those loops γ, which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. By the way, all critical points of F are the saddle points of infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, (M,ω) is symplectic manifolds, $H: M \to R$, H is Hamiltonian, X_H is unique Hamiltonian vector field defined by

$$\omega(X_H(x), v) = -dH(x)(v), \quad v \in T_x M, \quad x \in M,$$

where ω is the symplectic structure. A T-periodic solution $x(t)$ of the Hamiltonian equations

$$\dot{x} = X_H(x) \quad \text{on } M$$

is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1, R^{2n})$, where $S^1 = R/\mathbf{Z}$, of smooth loops in R^{2n}. Let us define a function $\Phi: \Omega \to R$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} < -J\dot{x}, x > dt - \int_0^1 H(x(t)) dt, \quad x \in \Omega$$

The critical points of Φ are the periodic solutions of $\dot{x} = X_H(x)$. Computing the derivative at $x \in \Omega$ in the direction of $y \in \Omega$, we find

$$\begin{aligned} \Phi'(x)(y) &= \frac{d}{d\epsilon} \Phi(x + \epsilon y)|_{\epsilon=0} = \\ & \int_0^1 < -J\dot{x} - \bigtriangledown H(x), y > dt \end{aligned}$$

Consequently, $\Phi'(x)(y) = 0$ for all $y \in \Omega$ iff the loop x satisfies the equation

$$-J\dot{x}(t) - \bigtriangledown H(x(t)) = 0,$$

i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0) = x(1)$, i.e. periodic of period 1. Periodic loops may be represented by their Fourier series:

$$x(t) = \sum_{k\in\mathbf{Z}} e^{k2\pi Jt} x_k, \quad x_k \in R^{2k},$$

where J is quasicomplex structure. We give relations between quasicomplex structure and wavelets in part IV. But now we use the construction [11] for loop parametrization. It is based on the theorem about explicit bijection between the Quadratic Mirror Filters (QMF) and the whole loop group: $LG: S^1 \to G$. In particular case we have relation between QMF-systems and measurable functions $\chi: S^1 \to U(2)$ satisfying

$$\chi(\omega + \pi) = \chi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

in the next explicit form

$$\begin{aligned} \begin{bmatrix} \hat{\Phi}_0(\omega) & \hat{\Phi}_0(\omega+\pi) \\ \hat{\Phi}_1(\omega) & \hat{\Phi}_1(\omega+\pi) \end{bmatrix} &= \chi(\omega) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \chi(\omega+\pi) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where

$$\left|\hat{\Phi}_i(\omega)\right|^2 + \left|\hat{\Phi}_i(\omega+\pi)\right|^2 = 2, \quad i = 0, 1.$$

Also, we have symplectic structure on LG

$$\omega(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} < \xi(\theta), \eta'(\theta) > d\theta$$

So, we have the parametrization of periodic orbits (Arnold–Weinstein curves) by reduced QMF equations.

Extended version and related results may be found in [1]-[9].

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