Mechanics of Solids
Vol. 36, No. 6, pp. 11–24, 2001

Izvestiya AN. MTT
No. 6, pp. 15–30, 2001

EXACT SOLUTION OF A PROBLEM OF ROTATION OF AN AXISYMMETRIC RIGID BODY IN A LINEAR VISCOUS MEDIUM

E. A. Ivanova

(Received 14 January 1999)

An exact closed-form solution is obtained for the problem of free rotation of an axisymmetric rigid body subjected to the viscous friction torque linearly depending on the angular velocity of the body. The solution is represented by exponential series.

1. INTRODUCTION

The problem of rotation of an axisymmetric rigid body in a linear viscous medium has drawn attention of researchers for a long time. For the first time, this problem was considered in [2] with the assumption that the body performing a spherical motion is subjected to the friction torque, $\mathbf{M}_{fr}$, proportional to the angular momentum, $\mathbf{L}$, i.e., $\mathbf{M}_{fr} = -k\mathbf{L}$. Accordingly, we have

$$\dot{\mathbf{L}} = -k\mathbf{L}. \tag{1.1}$$

The exact solution of problem (1.1) can be constructed in quadrature for any tensor of inertia. For a transversely-isotropic tensor of inertia, the solution of this problem can be expressed in terms of elementary functions. R. Grammel [2] has considered the rotatory motion of an axisymmetric body for the case where the viscous friction torque is proportional to the angular velocity, $\mathbf{M}_{fr} = -k\boldsymbol{\omega}$, and, accordingly,

$$(\boldsymbol{\theta} \cdot \boldsymbol{\omega})^{\cdot} = -k\boldsymbol{\omega}, \qquad \boldsymbol{\theta} = \theta_3\mathbf{nn} + \theta_{12}(\mathbf{E} - \mathbf{nn}). \tag{1.2}$$

In [2], the projections of the angular velocity onto the axes rigidly fixed to the body are determined. Thus, problem (1.2) is reduced to the Darboux problem the solution of which has not been discussed in the cited paper.

The torque due to linear viscous friction that depends not only on the angular velocity but also on the quantities defining the orientation of the body was first considered by Magnus [3]. In this case, the motion of the body is governed by the equation

$$(\boldsymbol{\theta} \cdot \boldsymbol{\omega})^{\cdot} = -\mathbf{K}_{fr} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\theta} = \theta_3\mathbf{nn} + \theta_{12}(\mathbf{E} - \mathbf{nn}), \quad \mathbf{K}_{fr} = k_3\mathbf{nn} + k_{12}(\mathbf{E} - \mathbf{nn}). \tag{1.3}$$

As was the case for problem (1.2), problem (1.3) also can be reduced to the Darboux problem. Note that from the viewpoint of mathematics, problem (1.3) is in no way more complicated than problem (1.2). Note that the last observation is valid only for the transversely-isotropic friction tensor $\mathbf{K}_{fr}$. For an arbitrary friction tensor, the problem in question is rather more complicated. The friction torque of the form $\mathbf{M}_{fr} = -\mathbf{K}_{fr} \cdot \boldsymbol{\omega}$, where the tensor of viscous friction, $\mathbf{K}_{fr}$, is aligned with the body tensor of inertia but is not transversely-isotropic, has been considered in [4, 5]. The problem suggested in [4] has the form

$$(\boldsymbol{\theta} \cdot \boldsymbol{\omega})^{\cdot} = -\mathbf{K}_{fr} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\theta} = \theta_3\mathbf{nn} + \theta_{12}(\mathbf{E} - \mathbf{nn}), \quad \mathbf{K}_{fr} = k_1\mathbf{e}_1\mathbf{e}_1 + k_2\mathbf{e}_2\mathbf{e}_2 + k_3\mathbf{nn}. \tag{1.4}$$

Problem (1.4) also can be reduced to the Darboux problem, however in this case, the expressions for the projections of the angular velocity onto the axes rigidly fixed to the body are more complicated and involve Bessel functions [4, 5].

In [4, 5], a problem of rotation in a linearly resisting medium is considered for a rigid body having an arbitrary shape. This problem is formulated as follows:

$$(\boldsymbol{\theta} \cdot \boldsymbol{\omega})^{\cdot} = -\mathbf{K}_{\text{fr}} \cdot \boldsymbol{\omega}, \qquad \boldsymbol{\theta} = \theta_1 \mathbf{e}_1 \mathbf{e}_1 + \theta_2 \mathbf{e}_2 \mathbf{e}_2 + \theta_3 \mathbf{e}_3 \mathbf{e}_3, \quad \mathbf{K}_{\text{fr}} = k_1 \mathbf{e}_1 \mathbf{e}_1 + k_2 \mathbf{e}_2 \mathbf{e}_2 + k_3 \mathbf{e}_3 \mathbf{e}_3. \tag{1.5}$$

In [4, 5], asymptotic relations for the projections of the angular velocity onto the principal axes of inertia have been obtained, with the assumption that the tensor of inertia is nearly transversely-isotropic.

In [2–5], problems of the motion of a rigid body in a viscous medium have also been considered, with the torques modeling the resistance of the medium involving terms quadratic in the angular velocity.

Various models of the interaction of the environment with a moving body are discussed in [6–8]. These books, in particular, prove that the friction torque taken in the form

$$\mathbf{M}_{\text{fr}} = -\mathbf{K}_{\text{fr}} \cdot \boldsymbol{\omega}, \qquad \mathbf{K}_{\text{fr}} = k_3 \mathbf{n}\mathbf{n} + k_{12}(\mathbf{E} - \mathbf{n}\mathbf{n}) \tag{1.6}$$

can be used to describe the action of a rarefied medium (for example, upper atmosphere), provided that (i) the body surface is convex, the axes of dynamic and geometric symmetry coincide, and the center of mass lies on the symmetry axis; (ii) the center of mass is fixed, i.e., the velocity of the incident flow is zero; (iii) the angular velocity of the body is relatively small, i.e., one can neglect the terms involving the square of the angular velocity, as compared with the terms that are linear in the angular velocity. The coefficients of viscous friction, k_{12} and k_3 (constant quantities), depend on the shape of the body surface and the model of the interaction of the body with the environment utilized to obtain integral characteristics [8]. Note also the paper [9] addressing a simple physical model consisting of thin rods with balls attached to these rods. The tensor of inertia of this structure is transversely isotropic. The structure moves in a linear viscous medium, each ball being acted upon by viscous friction force $\mathbf{f}_i = -k\mathbf{v}_i$. It is shown that the total resistance torque calculated with respect to the center of mass exactly corresponds to the relation of (1.6).

In the present paper, we consider problem (1.3). Apart from the cited book [5], in which this problem is reduced to the Darboux problem, one should also mention the paper [9] in which the solution of problem (1.3) has been constructed in quasi-coordinates. This solution is not an exact solution in the classical sense. However, it combines the simplicity of analytical expressions with the clearness of the representation of the results and permits one to form a fairly complete notion about the motion of the body. A method for constructing the exact solution in the form of uniformly converging series has been suggested in [10]. The present paper is a continuation of the investigations of [10].

2. STATEMENT OF THE PROBLEM. FIRST INTEGRALS

Consider free rotation of a rigid body fixed at its center of mass. The tensor of inertia of the body is transversely isotropic. The resistance of the environment is modeled by the torque of linear viscous friction in the form of (1.6), where the viscous friction tensor $\mathbf{K}_{\text{fr}}$ is aligned with the body tensor of inertia. The equation of motion of the body has the form

$$\dot{\mathbf{L}} = \mathbf{M}_{\text{fr}}, \qquad \mathbf{L} = (\theta_3 - \theta_{12})(\mathbf{n} \cdot \boldsymbol{\omega})\mathbf{n} + \theta_{12}\boldsymbol{\omega}, \quad \mathbf{M}_{\text{fr}} = -(k_3 - k_{12})(\mathbf{n} \cdot \boldsymbol{\omega})\mathbf{n} - k_{12}\boldsymbol{\omega}, \tag{2.1}$$

where $\mathbf{L}$ is the angular momentum and $\mathbf{n}$ is the unit vector defining the direction of the symmetry axis in the actual configuration of the system. To make system (2.1) complete, one should supplement it with Poisson's kinematical equation

$$\dot{\mathbf{n}} = \boldsymbol{\omega} \times \mathbf{n}. \tag{2.2}$$

The method utilizing the angular velocity $\boldsymbol{\omega}$ and the unit vector $\mathbf{n}$ as the basic variables has been first suggested by Zhuravlev [11] who has successfully applied this method to investigate the motion of the Lagrange top. Knowing the vector $\mathbf{n}$, one can define the orientation of the body by the rotation tensor

$$\mathbf{P} = \mathbf{P}(\mathbf{n}, \mathbf{k}) \cdot \mathbf{P}(\varphi \mathbf{k}), \qquad \mathbf{n} = \mathbf{P}(\mathbf{n}, \mathbf{k}) \cdot \mathbf{k}, \quad \mathbf{k} = \mathbf{n}(0), \tag{2.3}$$

where $\mathbf{P}(\mathbf{n}, \mathbf{k})$ is the rotation tensor transforming the symmetry axis of the body form the initial position to the actual one. It is convenient to represent this tensor in the form [12]

$$\mathbf{P}(\mathbf{n}, \mathbf{k}) = \mathbf{E} - \frac{1}{1 + \mathbf{k} \cdot \mathbf{n}}(\mathbf{n} + \mathbf{k})(\mathbf{n} + \mathbf{k}) + 2\mathbf{n}\mathbf{k}. \tag{2.4}$$

The tensor $\mathbf{P}(\varphi\mathbf{k})$ defines the rotation of the body about the symmetry axis. The angular velocity corresponding to the rotation tensor of (2.3) and (2.4) is expressed as

$$\omega = \frac{1}{1 + \mathbf{k}\cdot\mathbf{n}}(\mathbf{n} + \mathbf{k}) \times \dot{\mathbf{n}} + \dot{\varphi}\mathbf{n}. \tag{2.5}$$

Multiplying (2.5) scalarly by $\mathbf{n}$ we obtain, with reference to the kinematic relation (2.2), the expression for $\dot{\varphi}$:

$$\dot{\varphi} = \frac{\mathbf{n}\cdot\omega + \mathbf{k}\cdot\omega}{1 + \mathbf{k}\cdot\mathbf{n}}. \tag{2.6}$$

Thus, having known the angular velocity and the direction of the symmetry axis, one can determine the rotation tensor by calculating a single quadrature.

The system of differential equations (2.1) and (2.2) has two independent first integrals defined by

$$\omega\cos\gamma = \omega_0 e^{-\lambda t}\cos\gamma_0, \qquad \lambda = \frac{k_3}{\theta_3}, \quad \omega = |\omega|, \quad \cos\gamma = \frac{\mathbf{n}\cdot\omega}{\omega}, \tag{2.7}$$

$$\omega^2 = \omega_0^2\left(e^{-2\lambda t}\cos^2\gamma_0 + e^{-2\mu t}\sin^2\gamma_0\right), \qquad \mu = \frac{k_{12}}{\theta_{12}}, \quad \omega_0 = \omega(0), \quad \gamma_0 = \gamma(0). \tag{2.8}$$

The first integrals of (2.7) and (2.8) imply the expression for the absolute value of the angular momentum (which will be utilized in what follows),

$$L^2 = \omega_0^2\left(\theta_3^2 e^{-2\lambda t}\cos^2\gamma_0 + \theta_{12}^2 e^{-2\mu t}\sin^2\gamma_0\right), \tag{2.9}$$

and the expression for the tangent of the angle between the angular velocity and the symmetry axis of the body,

$$\tan\gamma = e^{(\lambda-\mu)t}\tan\gamma_0. \tag{2.10}$$

The analysis of relation (2.10) gives information about the motion of the body at large times. For example, if $\lambda < \mu$, we have $\tan\gamma \to 0$ as $t \to \infty$, which means the coincidence of the angular velocity with the direction of the symmetry axis. For $\lambda > \mu$, we have $\tan\gamma \to \infty$ as $t \to \infty$ and, hence, the angular velocity is perpendicular to the symmetry axis.

3. PARTICULAR SOLUTIONS

The problem of (2.1) and (2.2) has three particular solutions that can be expressed in terms of elementary functions.

1. The first solution corresponds to the case of $k_3/\theta_3 = k_{12}/\theta_{12}$ ($\lambda = \mu$). In this case, the viscous friction tensor is proportional to the body tensor of inertia $\mathbf{K}_{fr} = \lambda\theta$ and, accordingly, the torque due to friction is proportional to the angular momentum, $\mathbf{M}_{fr} = -\lambda\mathbf{L}$. Then the equation of motion has the form of (1.1). Omitting the details of the solution of this problem, we present the final result:

$$\mathbf{L} = \mathbf{L}_0 e^{-\lambda t}, \quad \mathbf{n} = \mathbf{P}(\psi\mathbf{l})\cdot\mathbf{k}, \quad \omega = e^{-\lambda t}\mathbf{P}(\psi\mathbf{l})\cdot\omega_0, \quad \dot{\psi} = \frac{L_0}{\theta_{12}}e^{-\lambda t}, \qquad \mathbf{k} = \mathbf{n}(0), \quad L_0 = |\mathbf{L}_0|, \quad \mathbf{l} = \frac{\mathbf{L}_0}{|\mathbf{L}_0|}. \tag{3.1}$$

The body moves "almost as in Euler's case": the symmetry axis rotates about the vector of angular momentum (which is constant in direction), with the angle between the symmetry axis and the angular momentum remaining unchanged. Apart from this, the body rotates about the symmetry axis in accordance with relation (2.6). This motion differs from that of Euler's case in that the angular velocities of precession and proper rotation exponentially decrease, rather than are constant.

2. For the second solution, the direction of the angular velocity at the initial time instant coincides with the direction of the symmetry axis, i.e., $\gamma_0 = 0$. In this case, in accordance with relation (2.10), we have $\sin\gamma = 0$ and the equation of motion (2.1) becomes

$$\theta_3\dot{\omega} = -k_3\omega, \quad \omega = \omega\mathbf{n}. \tag{3.2}$$

The solution of Eq. (3.2) is given by

$$\omega = \omega_0\mathbf{n}e^{-\lambda t}, \quad \mathbf{n} = \text{const}. \tag{3.3}$$

Accordingly, the symmetry axis of the body remains fixed, and the body rotates about the symmetry axis, with the rotation speed exponentially decreasing.

3. For the third solution, the angular velocity at the initial time instant is perpendicular to the symmetry axis of the body, i.e., $\gamma_0 = \pi/2$. In this case, in accordance with (2.7), we have $\cos\gamma = 0$ and the problem of (2.1) and (2.2) takes the form

$$\theta_{12}\dot{\omega} = -k_{12}\omega, \quad \dot{\mathbf{n}} = \omega \times \mathbf{n}. \tag{3.4}$$

The solution of Eq. (3.4) is given by

$$\omega = \omega_0 e^{-\mu t}, \quad \mathbf{n} = \mathbf{P}(\psi\mathbf{m}) \cdot \mathbf{k}, \quad \dot{\psi} = \omega_0 e^{-\mu t}, \qquad \mathbf{k} = \mathbf{n}(0), \quad \omega_0 = |\omega_0|, \quad \mathbf{m} = \frac{\omega_0}{|\omega_0|}. \tag{3.5}$$

The motion of the body is a rotation with exponentially decreasing speed about a fixed axis. The direction of this axis is determined by the initial direction of the angular velocity. The axis of symmetry of the body is perpendicular to the axis of rotation during the entire motion.

4. THE CHOICE OF THE BASIC VARIABLES. THE EQUATION FOR THE ANGULAR MOMENTUM.

In this section, we discuss the solution of the system of Eqs. (2.1) and (2.2) for arbitrary initial data and any parameters of the problem. The particular cases considered in Section 3 ($\lambda = \mu$, $\gamma_0 = 0$, and $\gamma_0 = \pi/2$) are excluded. We choose the vector of angular momentum, $\mathbf{L}$, to be the basic variable. Such a choice can be justified by the following considerations. The equation of motion (2.1), after eliminating $\mathbf{M}$, with reference to the first integral (2.7) takes the form

$$\dot{\mathbf{L}} = -(k_3 - k_{12})\omega_0 e^{-\lambda t} \cos\gamma_0 \, \mathbf{n} - k_{12}\omega, \quad \mathbf{L} = (\theta_3 - \theta_{12})\omega_0 e^{-\lambda t} \cos\gamma_0 \, \mathbf{n} + \theta_{12}\omega. \tag{4.1}$$

By solving system (4.1) for the vectors ω and $\mathbf{n}$, we express the angular velocity ω and the unit vector $\mathbf{n}$, characterizing the motion of the symmetry axis of the body, in terms of the angular momentum. These expressions have the form

$$\omega = \frac{1}{(\lambda - \mu)\theta_3\theta_{12}} \left[(\theta_3 - \theta_{12})\dot{\mathbf{L}} + (k_3 - k_{12})\mathbf{L}\right], \tag{4.2}$$

$$\mathbf{n} = -\frac{e^{\lambda t}}{(\lambda - \mu)\theta_3\omega_0 \cos\gamma_0}(\dot{\mathbf{L}} + \mu\mathbf{L}). \tag{4.3}$$

Thus, if the vector of angular momentum is known, all quantities characterizing the motion of the body can be determined without additional integration by relations (4.2), (4.3), and (2.6).

The differential equation for the angular momentum is derived on the basis of the kinematic relation (2.2). By substituting expressions (4.2) and (4.3) into Eq. (2.2) and performing simple transformations we obtain

$$\ddot{\mathbf{L}} + (\lambda + \mu)\dot{\mathbf{L}} + \lambda\mu\mathbf{L} = \frac{\mathbf{L} \times \dot{\mathbf{L}}}{\theta_{12}}. \tag{4.4}$$

The initial conditions for Eq. (4.4) have the form

$$\mathbf{L}(0) = \mathbf{L}_0, \quad \dot{\mathbf{L}}(0) = \dot{\mathbf{L}}_0. \tag{4.5}$$

By differentiating Eq. (4.4) with respect to time, multiplying Eq. (4.4) by $\lambda + \mu$, premultiplying Eq. (4.4) vectorially by $\theta_{12}^{-1}\mathbf{L}$, and then adding the resulting relations, we obtain the third-order equation

$$\dddot{\mathbf{L}} + 2(\lambda + \mu)\ddot{\mathbf{L}} + \left[(\lambda + \mu)^2 + \lambda\mu + \frac{L^2}{\theta_{12}}\right]\dot{\mathbf{L}} + \left[\lambda\mu(\lambda + \mu) - \frac{(L^2)^{\cdot}}{2\theta_{12}}\right]\mathbf{L} = 0. \tag{4.6}$$

With reference to expression (2.9) for the angular momentum, one can interpret Eq. (4.6) as a linear equation with varying coefficients. The initial conditions for Eq. (4.6) are the initial conditions of (4.5) added by Eq. (4.4) for $t = 0$, i.e.,

$$\ddot{\mathbf{L}}(0) = -(\lambda + \mu)\dot{\mathbf{L}}_0 - \lambda\mu\mathbf{L}_0 + \frac{\mathbf{L}_0 \times \dot{\mathbf{L}}_0}{\theta_{12}}. \tag{4.7}$$

Let us prove the equivalence of Eq. (4.4) and the problem of (4.6) and (4.7). Since Eq. (4.6) and the initial condition (4.7) are a consequence of Eq. (4.4), any solution of Eq. (4.4) is a solution of the problem of (4.6) and (4.7). Let us show now that any solution of the problem of (4.6) and (4.7) satisfies Eq. (4.4). To that end, we introduce the notation

$$\mathbf{U} = \ddot{\mathbf{L}} + (\lambda + \mu)\dot{\mathbf{L}} + \lambda\mu\mathbf{L} - \frac{\mathbf{L} \times \dot{\mathbf{L}}}{\theta_{12}}, \tag{4.8}$$

and rewrite the problem of (4.6) and (4.7) as follows:

$$\dot{\mathbf{U}} + (\lambda + \mu)\mathbf{U} + \frac{\mathbf{L} \times \mathbf{U}}{\theta_{12}} = 0, \quad \mathbf{U}(0) = \mathbf{0}. \tag{4.9}$$

The problem of (4.9) implies the initial value problem

$$\tfrac{1}{2}(U^2)^{\cdot} + (\lambda + \mu)U^2 = 0, \quad U^2(0) = 0, \tag{4.10}$$

that has the unique solution $U^2 \equiv 0$. Hence, $\mathbf{U} \equiv 0$, which is equivalent to Eq. (4.4). This completes the proof.

5. COMPLETENESS OF THE SYSTEM OF EXPONENTIAL FUNCTIONS

We will seek a solution of Eq. (4.6) for the angular momentum in the form of a series in terms of exponential functions with decreasing powers, i.e.,

$$\mathbf{L}(t) = \sum_{n=0}^{\infty} \mathbf{C}_n e^{-n\nu t}. \tag{5.1}$$

Let us prove the completeness of the system of functions

$$1, e^{-\nu t}, e^{-2\nu t}, \dots \tag{5.2}$$

on the semi-infinite interval $[0, +\infty)$ with respect to the class of functions $f_*(t)$ such that the integrals

$$\int_\alpha^\beta f_*(t) e^{-\nu t}\, dt, \quad \int_\alpha^\beta f_*^2(t) e^{-\nu t}\, dt \tag{5.3}$$

exist for any $\alpha, \beta \in [0, +\infty)$. By applying the standard orthogonalization procedure on the interval $[0, +\infty)$ with weight $e^{-\nu t}$ to system (5.2) we obtain the system of functions represented by Legendre polynomials

$$P_0(x), P_1(x), P_2(x), \dots, \qquad x = 2e^{-\nu t} - 1 \quad \left(x\big|_{t=0} = 1, \quad x\big|_{t=\infty} = -1\right). \tag{5.4}$$

Legendre polynomials are known to form a complete system of functions on the interval $[-1, 1]$ with respect to functions $f(x)$ such that the integrals

$$\int_a^b f(x)\, dx, \quad \int_a^b f^2(x)\, dx$$

exist for any $a, b \in [-1, 1]$. We proceed from the variable x to the variable t to obtain

$$\int_a^b f(x)\, dx = 2\nu \int_\alpha^\beta f_*(t) e^{-\nu t}\, dt, \quad \int_a^b f^2(x)\, dx = 2\nu \int_\alpha^\beta f_*^2(t) e^{-\nu t}\, dt, \tag{5.5}$$
$$f_*(t) = f(2e^{-\nu t} - 1), \quad \alpha = -\frac{1}{\nu} \ln \frac{b+1}{2}, \quad \beta = -\frac{1}{\nu} \ln \frac{a+1}{2}.$$

The system of functions (5.4) is complete on the interval $t \in [0, +\infty)$ with respect to the class of function $f_*(t)$ such that the integrals of (5.3) exist for any $\alpha, \beta \in [0, +\infty)$. Hence, the original system of functions (5.2) is also complete on the interval $[0, +\infty)$ with respect to the cited class of functions $f_*(t)$.

We will show now that the angular momentum $\mathbf{L}(t)$ belongs to the class of functions $f_*(t)$. To that end, we resolve the vector $\mathbf{L}(t)$ into its projections onto the axes of a Cartesian reference frame to obtain

$$\mathbf{L}(t) = L_1(t)\mathbf{i} + L_2(t)\mathbf{j} + L_3(t)\mathbf{k}. \tag{5.6}$$

It is easy to show that relations (5.6) and (2.9) imply the inequalities

$$\left|\int_\alpha^\beta L_i(t) e^{-\nu t}\, dt\right| \le \frac{\omega_0 \sqrt{\theta_3^2 \cos^2\gamma_0 + \theta_{12}^2 \sin^2\gamma_0}}{\min\{\lambda, \mu\} + \nu} \left[\exp\left(-(\min\{\lambda, \mu\} + \nu)\alpha\right) - \exp\left(-(\min\{\lambda, \mu\} + \nu)\beta\right)\right],$$
$$\left|\int_\alpha^\beta L_i^2(t) e^{-\nu t}\, dt\right| \le \frac{\omega_0^2 (\theta_3^2 \cos^2\gamma_0 + \theta_{12}^2 \sin^2\gamma_0)}{2\min\{\lambda, \mu\} + \nu} \left[\exp\left(-(2\min\{\lambda, \mu\} + \nu)\alpha\right) - \exp\left(-(2\min\{\lambda, \mu\} + \nu)\beta\right)\right]. \tag{5.7}$$

According to these inequalities, the integrals

$$\int_{\alpha}^{\beta} L_i(t)e^{-\nu t}\,dt, \qquad \int_{\alpha}^{\beta} L_i^2(t)e^{-\nu t}\,dt$$

are bounded for any $\alpha, \beta \in [0, +\infty)$ and, hence, the vector $\mathbf{L}(t)$ belongs to the class of functions $f_*(t)$. Thus, we have proved that the solution of Eq. (4.6) can be represented by the series of (5.1), provided that this series and its derivatives up to an appropriate order are uniformly convergent.

6. SOLUTION OF THE EQUATION FOR THE ANGULAR MOMENTUM IN THE CASE WHERE THE RATIO λ/μ IS A RATIONAL NUMBER.

In the strict sense, the solution of Eq. (4.6) presented below is a particular solution, since it exists only if a certain relation holds between the parameters of the problem, namely, if the ratio λ/μ is a rational number. Nevertheless, this solution can always be utilized in practice, since for any λ and μ, there exist integers p and q and a real number ν satisfying the relations

$$\lambda = p\nu, \qquad \mu = q\nu, \tag{6.1}$$

with an arbitrarily small error.

We will seek the solution of Eq. (4.6) in the form of series (5.1). We substitute (5.1) into (4.6) and match the coefficients of like powers of the exponential function to obtain

$$\mathbf{L}(t) = \mathbf{A}\sum_{n=p}^{\infty} C_n^{(1)}e^{-n\nu t} + \mathbf{B}\sum_{n=q}^{\infty} C_n^{(2)}e^{-n\nu t} + \mathbf{D}\sum_{n=p+q}^{\infty} C_n^{(3)}e^{-n\nu t}, \tag{6.2}$$

where the coefficients $C_n^{(i)}$ are calculated by the recurrence relations

$$C_p^{(1)} = 1, \quad C_q^{(2)} = 1, \quad C_{p+q}^{(3)} = 1, \quad C_n^{(i)} = \frac{\varkappa_1^2(n-3p)C_{n-2p}^{(i)} + \varkappa_2^2(n-3q)C_{n-2q}^{(i)}}{\nu^2(n-p)(n-q)(n-p-q)}, \tag{6.3}$$

$$\varkappa_1 = \frac{\omega_0\theta_3\cos\gamma_0}{\theta_{12}}, \qquad \varkappa_2 = \omega_0\sin\gamma_0,$$

and the coefficients $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ are determined by the initial conditions (4.5) and (4.7).

To prove the uniform convergence of the series of (6.2) and their first three derivatives, we proceed as follows. We consider the series formed by the absolute values of the coefficients $C_n^{(i)}$,

$$R_i = \sum_{n=1}^{\infty} |C_n^{(i)}|, \tag{6.4}$$

and the comparison series

$$R_* = \sum_{n-1}^{\infty} C_n^*, \qquad C_n^* = C_* \prod_{k=0}^{[n/(2l)]} \frac{1}{n-2(k-1)l}, \qquad l = \max\{p,q\}. \tag{6.5}$$

It is apparent that one can always indicate a sufficiently large number n_* such that the inequality

$$\max\left\{\left|\frac{\varkappa_1^2(n-3p)}{\nu^2(n-p)(n-q)(n-p-q)}\right|, \left|\frac{\varkappa_2^2(n-3q)}{\nu^2(n-p)(n-q)(n-p-q)}\right|\right\} \le \frac{1}{2(n+2l)}$$

holds for all $n \ge n_*$, and that for any n_* there exists a sufficiently large number C_* such that for any $n \le n_*$ we have $|C_n^{(i)}| \le |C_n^*|$. Then for $n \ge n_*$ we have the following relations:

$$|C_n^{(i)}| \le \frac{1}{2(n+2l)}\left(|C_{n-2p}^{(i)}| + |C_{n-2p}^{(i)}|\right) \le \frac{C_*}{2(n+2l)}\left(\prod_{k=0}^{[(n-2p)/(2l)]}\frac{1}{n-2p-2(k-1)l} + \prod_{k=0}^{[(n-2q)/(2l)]}\frac{1}{n-2q-2(k-1)l}\right)$$

$$\le \frac{C_*}{2(n+2l)}\left(\prod_{k=0}^{[(n-2p)/(2l)]}\frac{1}{n-2kl} + \prod_{k=0}^{[(n-2q)/(2l)]}\frac{1}{n-2kl}\right) \le \frac{C_*}{n+2l}\prod_{k=0}^{[n/(2l)]-1}\frac{1}{n-2kl}$$

$$= C_*\prod_{k=0}^{[n/(2l)]}\frac{1}{n-2(k-1)l} = C_n^*.$$

Hence, for sufficiently large C_*, series (6.5) majorizes the series of (6.4). Since series (6.5) converges, the series of (6.4) also converge. Therefore, the numerical series

$$\sum_{n=p}^{\infty} C_n^{(1)}, \quad \sum_{n=q}^{\infty} C_n^{(2)}, \quad \sum_{n=p+q}^{\infty} C_n^{(3)},$$

are absolutely convergent and the functional series of (6.2) corresponding to these numerical series are uniformly convergent on the interval $t \in [0, +\infty)$. To prove the uniform convergence of the first three derivatives of the series of (6.2), it suffices to prove the convergence of the numerical series

$$R_i^{(j)} = \sum_{n=1}^{\infty} (\nu n)^j \left|C_n^{(j)}\right| \quad (j = 1, 2, 3). \tag{6.6}$$

It is apparent that the series

$$R_*^{(j)} = \sum_{n=1}^{\infty} (\nu n)^j C_n^*$$

majorizing the series of (6.6) are convergent. This completes the proof of the uniform convergence of the series of (6.2) and their first derivatives.

A nonstrict asymptotic analysis of the solution (6.2), (4.2), and (4.3) for large t permitted us to draw conclusions as regards the motion of the unit vectors **n**, **m**, and **l** that define the direction of the symmetry axis of the body, the angular velocity $\mathbf{m} = \omega/|\omega|$), and the angular momentum ($\mathbf{l} = \mathbf{L}/|\mathbf{L}|$), respectively. Specifically, we have established the following relations:

$$\lambda < \mu : \quad \mathbf{n} \to \frac{\mathbf{A}}{|\mathbf{A}|}, \quad \mathbf{m} \to \frac{\mathbf{A}}{|\mathbf{A}|}, \quad \mathbf{l} \to \frac{\mathbf{A}}{|\mathbf{A}|}, \tag{6.7}$$

$$\lambda > \mu : \quad \mathbf{n} \to \frac{\mathbf{A}}{|\mathbf{A}|}, \quad \mathbf{m} \to \frac{\mathbf{B}}{|\mathbf{B}|}, \quad \mathbf{l} \to \frac{\mathbf{B}}{|\mathbf{B}|}. \tag{6.8}$$

The results of the numerical solution of the problem (to be presented below) confirm the results of the asymptotic analysis.

7. SOLUTION OF THE EQUATION FOR THE ANGULAR MOMENTUM FOR ARBITRARY λ AND μ.

We will seek the solution of Eq. (4.6) for arbitrary values of λ and μ in the form of a double series

$$\mathbf{L}(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \mathbf{C}_{nm} \exp\bigl(-(n\lambda + m\mu)t\bigr). \tag{7.1}$$

By substituting the series of (7.1) into Eq. (4.6) and matching coefficients of like powers of the exponential function we obtain the following relations:

$$\mathbf{L}(t) = \mathbf{A}R_{10}(t) + \mathbf{B}R_{01}(t) + \mathbf{D}R_{11}(t), \qquad R_{pq}(t) = \sum_{n=p}^{\infty}\sum_{m=q}^{\infty} C_{nm} \exp\bigl(-(n\lambda + m\mu)t\bigr), \tag{7.2}$$

where

$(pq) = (10)$ corresponds to odd n and even m;

$(pq) = (01)$ corresponds to even n and odd m;

$(pq) = (11)$ corresponds to odd n and m.

The coefficients C_{nm} are calculated by the recurrence relations

$$C_{10} = 1, \quad C_{01} = 1, \quad C_{11} = 1; \qquad C_{n0} = 0 \quad (n > 1); \qquad C_{0m} = 0 \quad (m > 1);$$

$$C_{n1} = -\frac{\varkappa_1^2 C_{n-2,1}[(n-3)\lambda + \mu]}{n(n-1)\lambda^2[(n-1)\lambda + \mu]}, \quad C_{1m} = -\frac{\varkappa_2^2 C_{1,m-2}[\lambda + (m-3)\mu]}{m(m-1)\mu^2[\lambda + (m-1)\mu]}, \tag{7.3}$$

$$C_{nm} = -\frac{\varkappa_1^2 C_{n-2,m}[(n-3)\lambda + m\mu] + \varkappa_2^2 C_{n,m-2}[n\lambda + (m-3)\mu]}{[(n-1)\lambda + (m-1)\mu]\{(n\lambda + m\mu)[(n-1)\lambda + (m-1)\mu] + \lambda\mu\}} \quad (n, m \geq 2).$$

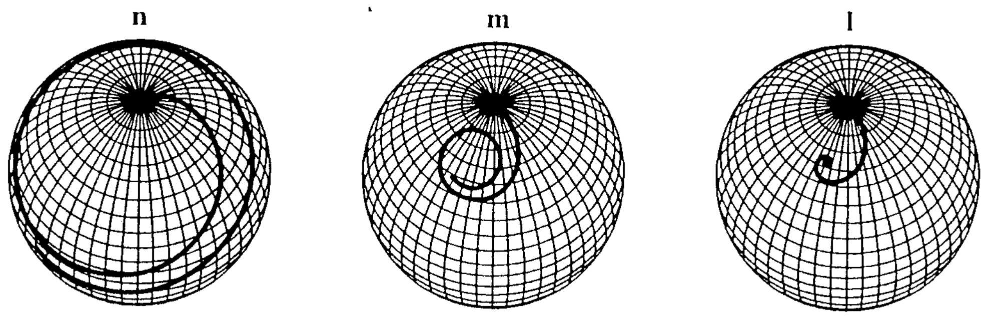

Fig. 1

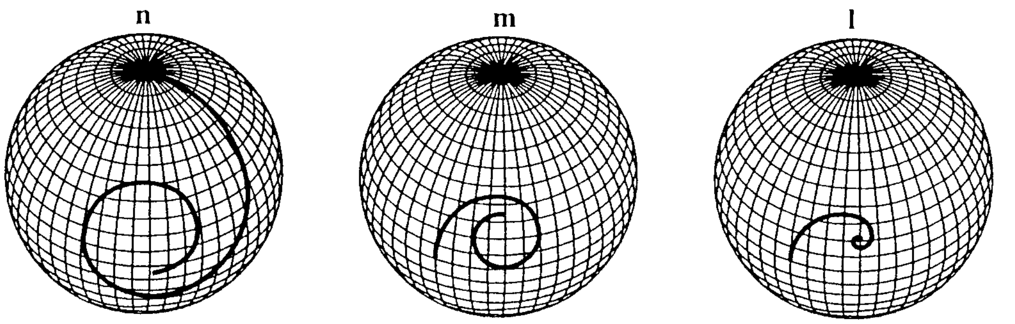

Fig. 2

The unknown constants **A**, **B**, and **D** are determined by the initial conditions (4.5) and (4.7). For the proof of the uniform convergence of the series $R_{10}(t)$, $R_{01}(t)$, $R_{11}(t)$ and their first derivatives, see, for example [10]. The asymptotic behavior of the solution (7.2), (4.2), and (4.3) for large t is determined by the relations of (6.7) and (6.8), where the vectors **A** and **B** are the coefficients of (7.2).

Thus, the general solution of Eq. (4.6) is expressed by relations (7.2) and (7.3). An apparent disadvantage of this solution is that it involves multiple series, which worsens the convergence and complicates computations. For this reason, the general solution of (7.2) and (7.3) has a purely theoretical value. In practice, it is more convenient to use the particular solution of (6.2) and (6.3) approximating the ratio λ/μ by a rational number.

8. NUMERICAL ANALYSIS OF THE SOLUTION FOR VARIOUS PARAMETERS OF THE PROBLEM

In this section, we will investigate the behavior of hodographs of the unit vectors **n**, **m**, and **l** that characterize the direction of the symmetry axis of the body, the angular velocity, and the angular momentum, respectively. The behavior of the trajectories of the ends of the vectors **n**, **m**, and **l** on the unit sphere is affected primarily by the following parameters:

$$\delta = \frac{\lambda}{\mu}, \quad \varepsilon = \frac{\omega_0}{\min\{\lambda, \mu\}}, \quad \gamma_0.$$

1. General behavior. The dependence on the parameter δ. For any values of the parameters of the problem, the hodographs of the vectors **n**, **m**, and **l** are spirals. These spirals may wind or unwind as t increases; as $t \to \infty$, they tend to certain points whose positions are defined by the initial conditions (Figs. 1 and 2). The parameter δ divides the totality of solutions of the problem of (4.2), (4.3), and (7.2) into two classes.

I. For $\delta < 1$, the hodographs of the vectors **n**, **m**, and **l** are winding spirals; as $t \to \infty$, these spirals tend to a single point whose position on the unit sphere is determined by the direction of the vector **A** (Fig. 1).

II. For $\delta > 1$, the hodograph of the vector **n** is an unwinding spiral tending (as $t \to \infty$) to the point whose position is defined by the vector **A**; the hodographs of the vectors **m** and **l** are also unwinding spirals but tending to another point whose position is defined by the direction of the vector **B** (Fig. 2).

For $\delta = 1$, the solution in the form of (4.2), (4.3), and (7.2) does not exist. The variation of the parameter δ in the intervals $(0, 1)$ and $(1, \infty)$ does not substantially influence the character of the motion of the body.

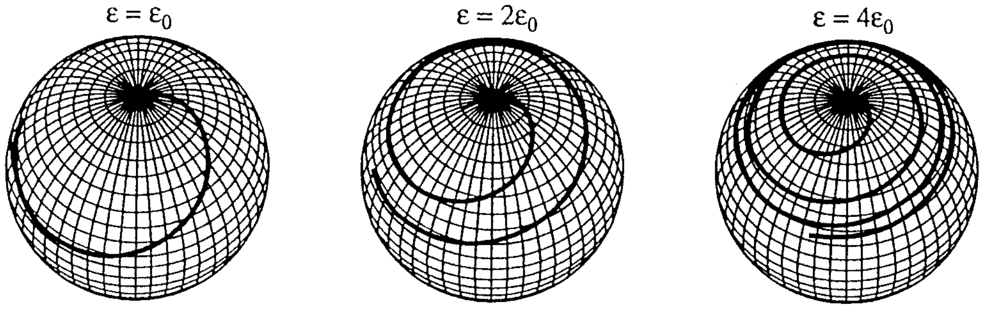

Fig. 3

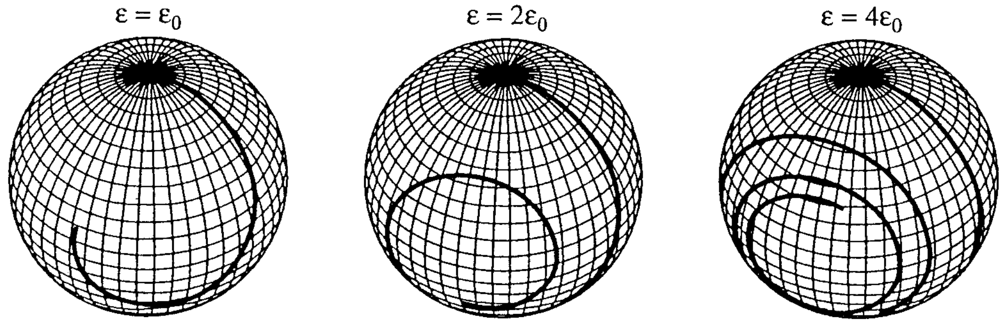

Fig. 4

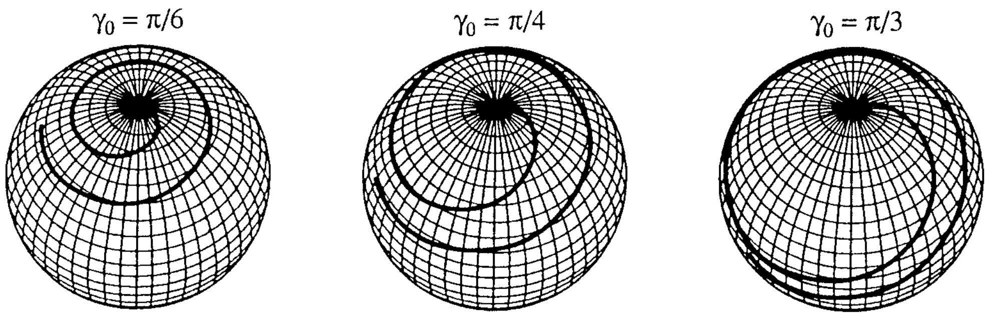

Fig. 5

2. *The dependence on the parameter* $\varepsilon = \omega_0/\min\{\lambda, \mu\}$. Figures 3 ($\delta < 1$) and 4 ($\delta > 1$) show hodographs of the vector **n** for different values of ε. It is apparent from these figures that the increase in ε leads to an increase in the number of turns of the spiral, with the area of the region of the sphere containing the trajectory virtually not changing. If the lesser of the parameters λ and μ decreases, with ε remaining constant, and, accordingly, the initial angular velocity, ω_0, decreases, then the duration of the motion until the direction of the vector **n** "almost" coincides with the direction of the vector **A** increases. This reasoning can be repeated word for word for the vectors **m** and **l**.

3. *The dependence on the parameter* γ_0. Figures 5 ($\delta < 1$) and 6 ($\delta > 1$) present hodographs of the vector **n** for different γ_0. As is apparent from these figures, the parameter γ_0 determines the area of the region of the sphere containing the trajectory. As γ_0 increases, this area increases for $\delta < 1$ and decreases for $\delta > 1$. This is also the case for the vectors **m** and **l**, with the only difference that the region occupied by the hodographs of the vectors **m** and **l** is small for all γ_0, apart from γ_0 close to $\pi/2$ for $\delta < 1$ and γ_0 close to 0 for $\delta > 1$.

In conclusion of this section, note that the direct dependence of the character of the motion of the body on the moments of inertia θ_3 and θ_{12} is not observed.

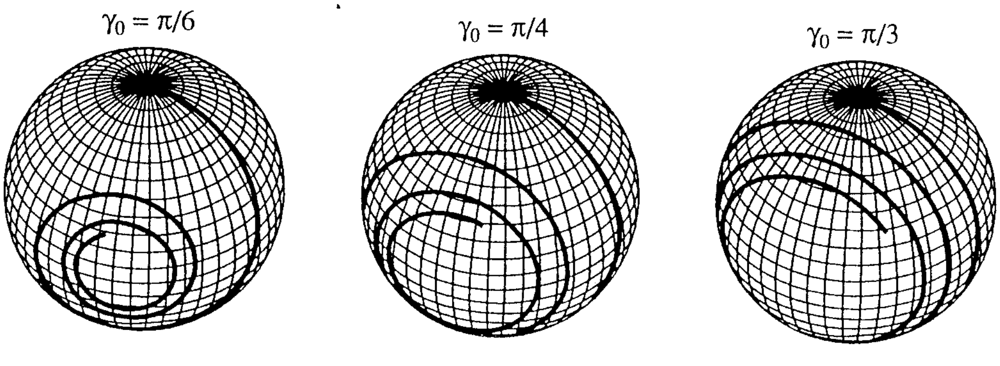

Fig. 6

9. SOLUTION OF THE PROBLEM IN THE CASE OF $k_3 = 0$

For practice, of most interest is the case where the friction torque component along the symmetry axis is zero. In this case, it is natural to anticipate that the motion of the body will tend to the permanent rotation about the symmetry axis. Let us prove this hypothesis. For $k_3 = 0$, one should set $\lambda = 0$ in expressions (2.7) and (2.8) for the first integrals, as well as in expression (2.9) for the angular momentum. Then the absolute values of the angular velocity and the angular momentum will tend to constant values ($\omega^2 \to \omega_0^2 \cos^2\gamma_0$, $L^2 \to \theta_3^2\omega_0^2\cos^2\gamma_0$), as $t \to +\infty$, and the angle between the angular velocity and the symmetry axis of the body will tend to zero. Accordingly, the motion of the body will tend to the permanent rotation about the symmetry axis.

For $\lambda = 0$, the equation for the angular momentum (Eq. (4.6)) becomes

$$\dddot{\mathbf{L}} + 2\mu\ddot{\mathbf{L}} + \left(\mu^2 + \varkappa_1^2 + \varkappa_2^2 e^{-2\mu t}\right)\dot{\mathbf{L}} + \mu\varkappa_2^2 e^{-2\mu t}\mathbf{L} = 0. \qquad (9.1)$$

Since the body does not tend to a state of rest as $t \to +\infty$, it is quite obvious that the solution, apart from exponentially decreasing functions, must also involve harmonic functions. It is clear that these harmonic functions must not depend on the parameter μ characterizing the resistance of the environment. We will seek the solution of Eq. (9.1) in the form

$$\mathbf{L}(t) = \mathbf{A}\sum_{n=1}^{\infty} C_n^{(1)} \exp\bigl(-(n\mu + i\varkappa_1)t\bigr) + \mathbf{B}\sum_{n=1}^{\infty} C_n^{(2)} \exp\bigl(-(n\mu - i\varkappa_1)t\bigr) + \mathbf{D}\sum_{n=0}^{\infty} C_n^{(3)} \exp(-n\mu t). \qquad (9.2)$$

Substitute the relation of (9.2) into Eq. (9.1) and match the coefficients of like powers of the exponential function to obtain

$$\mathbf{L}(t) = (\mathbf{D}_1\cos\varkappa_1 t - \mathbf{D}_2\sin\varkappa_1 t)\sum_{n=1}^{\infty}\operatorname{Re} C_n e^{-n\mu t} + (\mathbf{D}_1\sin\varkappa_1 t + \mathbf{D}_2\cos\varkappa_1 t)\sum_{n=1}^{\infty}\operatorname{Im} C_n e^{-n\mu t} + \mathbf{D}\sum_{n=0}^{\infty} C_n^{(3)} e^{-n\mu t}, \qquad (9.3)$$

where $C_1 = 1$, $C_0^{(3)} = 1$, and the coefficients C_n and $C_n^{(3)}$ can be calculated by the recurrence relations

$$C_n = -\frac{\varkappa_2^2[(n-3)\mu + i\varkappa_1]C_{n-2}}{(n\mu + i\varkappa_1)(n-1)\mu[(n-1)\mu + 2i\varkappa_1]}, \qquad C_n^{(3)} = -\frac{\varkappa_2^2(n-3)C_{n-2}^{(3)}}{n[(n-1)\mu^2 + \varkappa_1^2]}. \qquad (9.4)$$

It is obvious that the series of (9.3) and (9.4), as well as the first three derivatives of these series are uniformly convergent. The unknown constants $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{D}$ are determined by the initial conditions (4.5) and (4.7). The angular velocity, $\boldsymbol{\omega}$, and the unit vector of the symmetry axis of the body, $\mathbf{n}$, can be calculated according to relations (4.2) and (4.3) with $k_3 = 0$ and $\lambda = 0$. The asymptotic behavior of the solution for large t is represented by the relations

$$\mathbf{L} \to \mathbf{D}, \quad \boldsymbol{\omega} \to \frac{\mathbf{D}}{\theta_3}, \quad \mathbf{n} \to \frac{\mathbf{D}}{\theta_3\omega_0\cos\gamma_0}. \qquad (9.5)$$

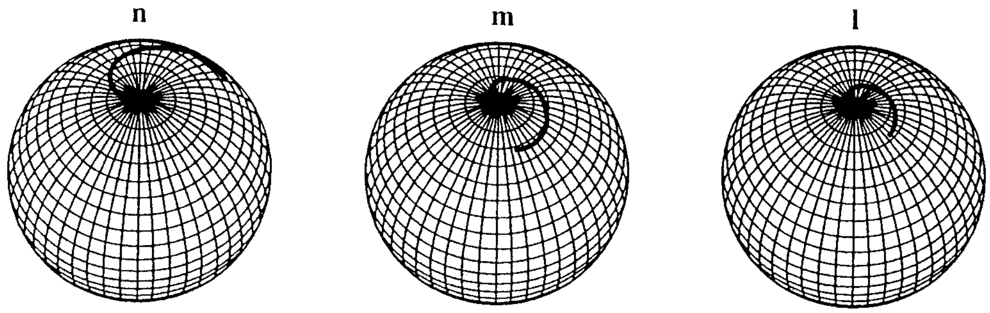

Fig. 7

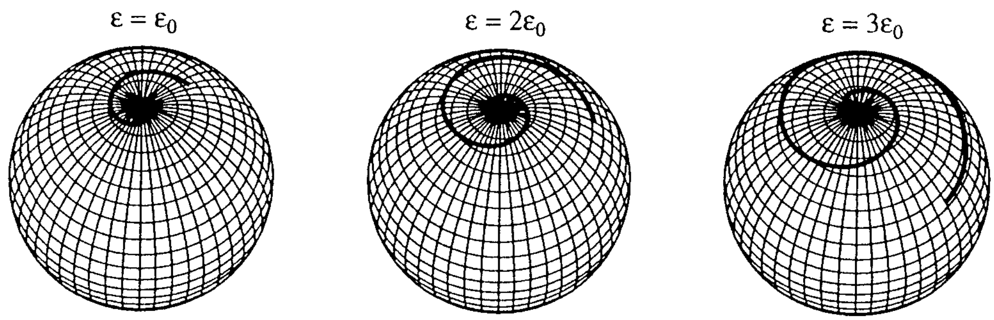

Fig. 8

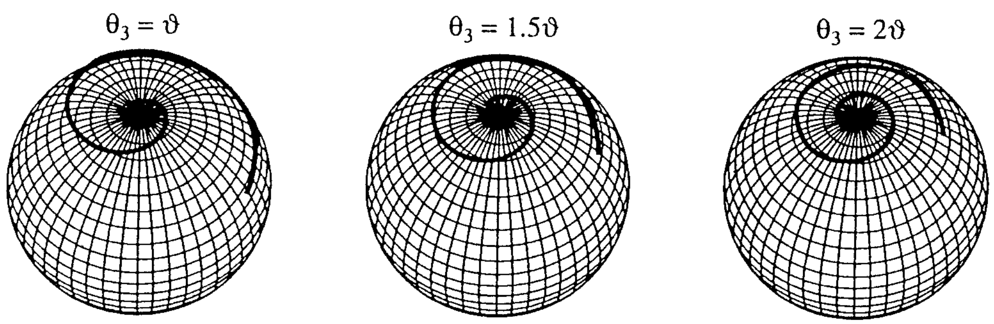

Fig. 9

Figure 7 presents hodographs of **n**, **m**, and **l**. The comparison of Figs. 7 and 1 shows no qualitative difference of the trajectories for $\lambda = 0$ and for $0 < \delta < 1$. The dependence on the parameters γ_0 and ε has the same character for $\lambda = 0$ (in this case, $\varepsilon = \omega_0/\mu$) and $0 < \delta < 1$: the area of the region occupied by the trajectory increases as γ_0 increases, and the number of turns of the trajectory increases as ε increases. Note, however, that for $\lambda = 0$, the increase of ε may lead to an increase in the area occupied by the trajectories (Fig. 8). (This effect has not been observed for $0 < \delta < 1$.) The only essential difference between the cases of $\lambda = 0$ and $0 < \delta < 1$ is that for $\lambda = 0$, the polar moment of inertia, θ_3, plays the role of the third basic parameter of the problem qualitatively affecting the shape of the hodographs of the vectors **n**, **m**, and **l**. Figure 9, depicting hodographs of the vector **n** for different θ_3, shows that the increase of this parameter leads to an increase in the number of turns of the spiral and a decrease in the area occupied by the trajectories.

10. SOLUTION OF THE PROBLEM IN THE CASE OF $k_{12} = 0$

From the viewpoint of mathematics, the case of $k_{12} = 0$ does not differ from the case of $k_3 = 0$. The equation for the angular momentum (Eq. (4.6)) for $\mu = 0$ coincides with Eq. (9.1), obtained for the case of $\lambda = 0$, to within a

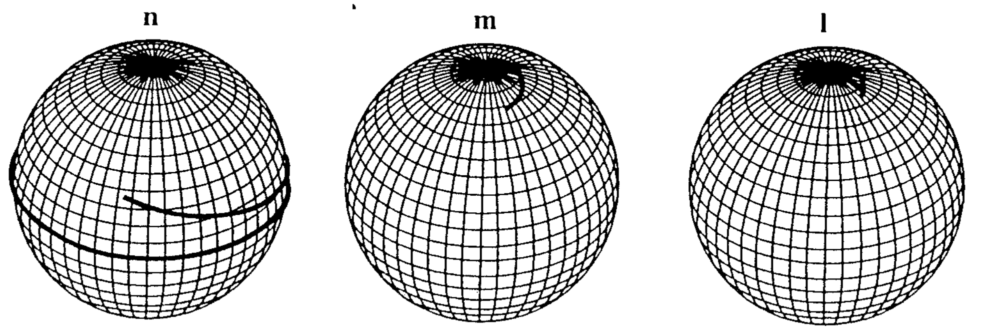

Fig. 10

notation of constant parameters and has the form

$$\dddot{\mathbf{L}} + 2\lambda\ddot{\mathbf{L}} + \left(\lambda^2 + \varkappa_2^2 + \varkappa_1^2 e^{-2\lambda t}\right)\dot{\mathbf{L}} + \lambda\varkappa_1^2 e^{-2\lambda t}\mathbf{L} = 0. \tag{10.1}$$

The solution of Eq. (10.1) can be obtained from relations (9.3) and (9.4) by replacing $\varkappa_1$, $\varkappa_2$, and μ by $\varkappa_2$, $\varkappa_1$, and ε, respectively. Thus we have

$$\mathbf{L}(t) = (\mathbf{D}_1 \cos\varkappa_2 t - \mathbf{D}_2 \sin\varkappa_2 t)\sum_{n=1}^{\infty} \operatorname{Re} C_n e^{-n\lambda t} + (\mathbf{D}_1 \sin\varkappa_2 t + \mathbf{D}_2 \cos\varkappa_2 t)\sum_{n=1}^{\infty} \operatorname{Im} C_n e^{-n\lambda t} + \mathbf{D}\sum_{n=0}^{\infty} C_n^{(3)} e^{-n\lambda t}, \tag{10.2}$$

where $C_1 = 1$, $C_0^{(3)} = 1$, and the coefficients C_n and $C_n^{(3)}$ can be calculated by the recurrence relations

$$C_n = -\frac{\varkappa_1^2[(n-3)\lambda + i\varkappa_2]C_{n-2}}{(n\lambda + i\varkappa_2)(n-1)\lambda[(n-1)\lambda + 2i\varkappa_2]}, \quad C_n^{(3)} = -\frac{\varkappa_1^2(n-3)C_{n-2}^{(3)}}{n[(n-1)^2\lambda + \varkappa_2^2]}. \tag{10.3}$$

The angular velocity, ω and the unit vector $\mathbf{n}$ of the symmetry axis of the body are calculated in accordance with the relations (4.2) and (4.3) with $k_{12} = 0$ and $\mu = 0$. The asymptotic behavior of the solution for large t looks as follows. The absolute values of the angular velocity and the angular momentum tend to constant values ($\omega^2 \to \omega_0^2 \sin^2\gamma_0$, $L^2 \to \theta_{12}^2\omega_0^2 \sin^2\gamma_0$) as $t \to +\infty$; the angle between the angular velocity and the symmetry axis of the body tends to $\pi/2$; the directions of the vectors $\boldsymbol{\omega}$ and $\mathbf{L}$ tend to the constant vector $\mathbf{D}$ determined by the initial conditions; and the motion of the symmetry axis tends to the rotation with constant angular velocity $\varkappa_2 = \omega_o \sin\gamma_0$ in the plane spanned by constant vectors $\mathbf{D}_1$ and $\mathbf{D}_2$ determined by the initial conditions, i.e.,

$$\mathbf{L} \to \mathbf{D}, \quad \boldsymbol{\omega} \to \frac{\mathbf{D}}{\theta_{12}}, \quad \mathbf{n} \to \frac{(\lambda\mathbf{D}_1 + \varkappa_2\mathbf{D}_2)\cos\varkappa_2 t + (\varkappa_2\mathbf{D}_1 - \lambda\mathbf{D}_2)\sin\varkappa_2 t}{\theta_3\omega_0\cos\gamma_0}. \tag{10.4}$$

Hence, for large t, the motion of the body tends to the permanent rotation about the axis the direction of which is defined by the vector $\mathbf{D}$. In this case, the motion of the symmetry axis of the body approaches the uniform rotation in a plane orthogonal to the axis of rotation of the body, $\mathbf{D}$.

Figure 10 presents hodographs of the vectors $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{l}$. The comparison of Figs. 10 and 2 shows that the motion of the body in the case of $\mu = 0$ very much resembles the motion of the body in the case of $1 < \delta < +\infty$. The only difference is that all spirals in Fig. 2 have finite lengths, since these spirals, being unwinding, tend to limit points, whereas the hodograph of the vector $\mathbf{n}$ depicted in Fig. 10 is a spiral having infinite length. The spiral in Fig. 10 unwinds making infinitely many turns near a great circle of the unit sphere. For $\mu = 0$ (as was the case for $\lambda = 0$), there exist three basic parameters that qualitatively influence the shape of the hodographs of the vectors $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{l}$. These parameters are γ_0, $\varepsilon = \omega_0/\lambda$, and θ_{12}. The dependence on the parameter γ_0 for $\mu = 0$ has the same character as is the case for $1 < \delta < +\infty$ — the increase of γ_0 leads to a decrease in the area occupied by the trajectories. The dependence on the parameter ε (Fig. 11) manifests itself by the fact that the increase of this parameter increases the number of turns of the portion of the spiral that can be visually distinguished from the limit circle. (This is similar to the case of $1 < \delta < +\infty$ where the increase of ε leads to an increase in the number of turns of the spiral.) In addition, the increase of ε leads to an increase in the area occupied by the trajectory, which was not the case for $1 < \delta < +\infty$. As is apparent from Fig. 12, the increase of θ_{12} leads to a decrease in the number of turns of the spiral visually distinguishable from the limit circle with simultaneous decrease in the area occupied by the trajectories. The dependence of the behavior of the trajectories on the equatorial moment of inertia treated as an independent parameter is characteristic only of the case of $\mu = 0$.

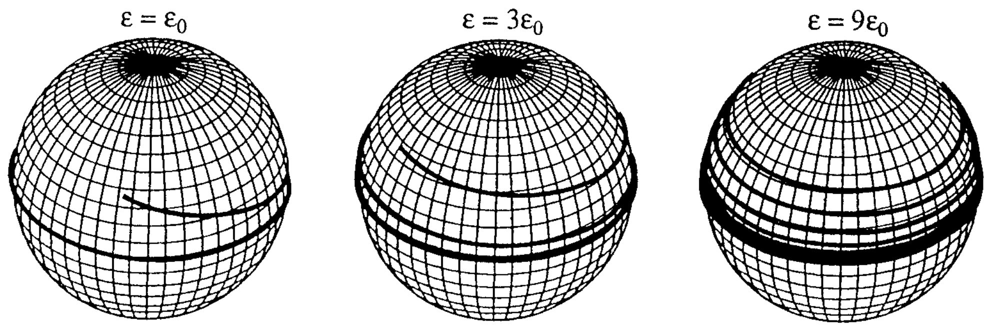

Fig. 11

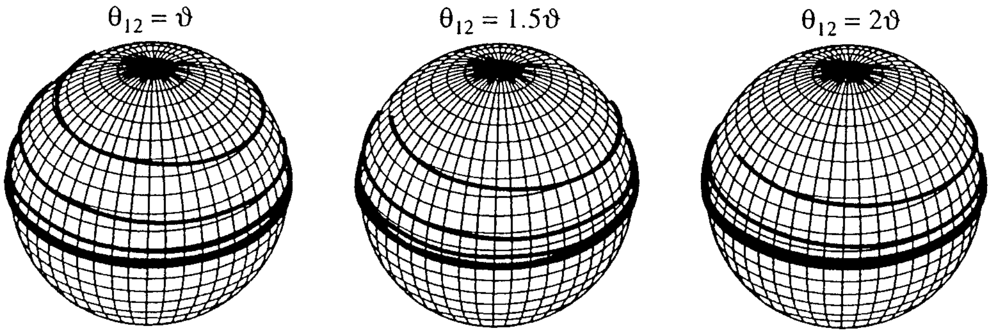

Fig. 12

11. CONCLUSIONS

In dynamics of a rigid body, there are only a small number of problems the exact solution of which can be constructed in closed form. One of such problems is the problem of the rotation of an axisymmetric rigid body in a linear viscous medium. In the present paper, we have constructed the exact solution of this problem in the form of exponential series. In the general case, this solution is represented by the double series of (7.2) and (7.3). Such a representation has obvious disadvantages that are compensated for by the fact that when solving specific problems, one can approximate the ratio λ/μ by a rational number and utilize the particular solution of (6.2) and (6.3) expressed in terms of single series. Particular attention should be given to the cases of $k_3 = 0$ and $k_{12} = 0$. In these cases, the motion of the body tends to a permanent rotation, rather that to a state of rest, as $t \to +\infty$; this rotation is performed about the symmetry axis of the body for $k_3 = 0$ and about an axis perpendicular to the symmetry axis for $k_{12} = 0$. These two special cases may appear to be of interest for practice.

An interest in the problem under study is accounted for not only by the fact that one can construct an exact solution of this problem in closed form but also by the fact that this solution can be represented graphically. The latter possibility in dynamics of a rigid body is as rare case as is the existence of an exact solution. In terms of the ability of the solution to be graphically represented, this problem can be compared with Lagrange's integrable case. Hodographs of the unit vector **n** that defines the direction of the body symmetry axis are divided into three classes. The first class contains spirals winding to a point with diminishing radius of turns ($\lambda < \mu$). The second class consists of spirals which unwind with increasing radius of turns and tend to a great circle of the unit sphere ($\lambda > \mu$). The third class contains spirals degenerated into circles ($\lambda = \mu$). In all these cases, apart from the case of $k_{12} = 0$, the spirals have a finite length [2, 9]. A qualitatively new result related to the analysis of hodographs of the unit vectors **n**, **m**, and **l** characterizing the directions of the symmetry axis of the body, the angular velocity, and the angular momentum, respectively, is the determination of the limit points asymptotically approached by hodographs of these vectors as $t \to +\infty$ by the initial conditions. The shape of these hodographs permits one qualitatively assess the rate of convergence of the series representing the solution of the problem. If the hodograph is a spiral consisting of several turns (which may occur in the case of small friction coefficients or large initial angular velocities), then the series converge slowly and, hence, one has to take into account several dozen terms of these series to obtain an acceptable result. If the hodograph is a spiral containing less than one complete turn, first several terms of the series fairly well approximate the solution. Possibly, the optimal approximation to the solution of the problem is a combination of the asymptotic solution slightly different from the regular precession for $t \in [0, t_*]$ (t_* is the time at which the trajectory goes into last turn) with the approximation of the exact solution by several terms of the series for $t > t_*$. A detailed discussion of the approximate solution is beyond the scope of the present paper aimed at the construction and analysis of an exact solution.

ACKNOWLEDGMENTS

The author thanks P. A. Zhilin for the discussion of this work and useful remarks.

REFERENCES

[1] E. J. Routh, Dynamics of a System of Rigid Bodies [Russian translation], Nauka, Moscow, 1983.
[2] R. Grammel, Der Kreisel Seine Theorie und Seine Anwendungen [Russian translation from German], Izd-vo Inostr. Lit-ry, Moscow, 1952.
[3] K. Magnus, Gyroscope: Theory and Applications [Russian translation], Mir, Moscow, 1974.
[4] V. N. Koshlyakov, "On some special cases of integration of Euler's dynamical equations associated with the motion of a gyro in a resisting medium," PMM [Applied Mathematics and Mechanics], Vol. 17, No. 2, pp. 137–148, 1953.
[5] V. N. Koshlyakov, Problems of Dynamics of a Rigid Body in Applied Theory of Gyroscopes [in Russian], Nauka, Moscow, 1985.
[6] N. F. Krasnov, Aerodynamics of Bodies of Revolution [in Russian], Mashinostroenie, Moscow, 1964.
[7] R. G. Barantsev, Interaction of Rarefied Gas with Streamlined Surfaces [in Russian], Nauka, Moscow, 1975.
[8] V. V. Beletskii and A. M. Yanshin, Influence of Aerodynamic Forces on the Rotation of Satellites [in Russian], Naukova Dumka, Kiev, 1984.
[9] A. M. Krivtsov, "Description of motion of an axisymmetric rigid body in a linearly viscous medium in terms of quasicoordinates," Izv. AN. MTT [Mechanics of Solids], No. 4, pp. 23–29. 2000.
[10] E. A. Ivanova, "Free rotation of an axisymmetric rigid body in a resisting medium. Mechanics and control processes," Trudy SPbGTU, No. 467, pp. 61–69, 1997.
[11] V. Ph. Zhuravlev, Fundamentals of Theoretical Mechanics [in Russian], Nauka, Moscow, 1997.
[12] A. M. Krivtsov, "Steady-state motions of a nonsymmetric top," Izv. AN. MTT [Mechanics of Solids], No. 3, pp. 28–38, 1997.

St. Petersburg