Motion of soliton perturbations in the elastic bodies interacting with fluid

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1 Introduction. Statement of problem. We consider a simplest model describing a soliton moving in an elastic nonlinear system interacting with a fluid. Such problems are important for ...

Our model, given by the following equations (1.1)-(1.3), describes a simple system, consisting of a elastic nonlinear beam contacting with a fluid. (This beam is a bottom and the fluid occupies a layer under it). The mathematical model is as follows

$$Dw_{xxxx} + Kw - \beta w_{xx} + mw_{tt} - \alpha w^3 = P_+(x,t), \tag{1.1}$$

where $w = w(x,t),\ x \in (-\infty, \infty),\ t \geq 0$ is a deflection of the elastic beam and

$$g\phi_y + \phi_{tt}|_{y=h} = 0, \tag{1.2}$$

$$\phi_y|_{y=0} = w_t, \tag{1.3}$$

$$c^2 \Delta\phi - \phi_{tt} = 0 \tag{1.4}$$

where $\phi(x,y,t)$ is a potential of velocity of the fluid, $y \in [0,h], x \in (-\infty,\infty),\ t \geq 0$. Equations (1.2) and (1.3) are boundary conditions, respectively on the free surface $y = h$ and on the bottom, where one has the contact between the beam and the fluid. Such system can give, for example, a simplest model of an ocean with an elastic nonlinear bottom. The term $P_+ = \epsilon\phi_t$ is the pressure of the fluid on the beam and give a connection between the elastic and fluid components.

The detail mathematical investigation of this model is not quite elementary. To simplify, the statement will be proceed into some steps. Our main hypothesis is that $P_+(x,t)$ is small. Such idea allows us to use the theory of perturbations. Our main aim is to describe a connection between the soliton and surface of fluid, i.e. in other words, we are going to understand how the fluid on surface reacts to soliton moving along bottom.

The first stage is founding of the soliton form distortion under perturbation P_+ and the derivation of an equation for the soliton speed. The second step is a calculation of the fluid state, for given deflection defined by our soliton $w = W = W^{sol}$.

Here we find that, depending of the system parameter $(h, K, \alpha, ...)$, the fluid reaction can be very different. At last, last part of the paper describe complete scheme of iterations allowing at least in principle, to construct complete asymptotic solutions.

2. Preliminary Step: solitons in the beam and its perturbations.

If $P_+ \equiv 0$, the state of the beam does not depend on the fluid state. in this case we can find the following soliton-like solution (found by P. Gordon [1]):

$$W(z) = A\cosh^{-2}(\kappa^{-1}z), \quad z = x - Vt - q \tag{2.1}$$

where $A = (\alpha/15K)^{1/2}$ is the amplitude, $\kappa = 2^{3/2}(K/D)^{-1/4}$ is "width" and $V_0 = m^{-1}(\beta - \frac{5}{2}(KD)^{1/2})$ is the speed of soliton. These three parameters are completely defined by elastic properties of the beam (by numbers K, D, m, β, α. The single "free" parameter (which we can change) is a "shift" (or "coordinate") q which can be chosen arbitrarily.

Now we are going to consider the case when ϵ is small. Then it is natural to seek the solution as a perturbed soliton

$$w = W(z) + \epsilon\tilde{w}, \tag{2.2}$$

where, according to the well known approaches [2][3], in the relation $z = x - Vt - q$, the shift q is an unknown function of time. One can show that $\frac{dq}{dt} \approx \sqrt{\epsilon}$. After substitution, rejecting negligible termes $<< \epsilon$, one finds

$$L\tilde{w} = F(z,t), \tag{2.3}$$

where L is a linear operator defined as $L\tilde{w} = D\tilde{w}_{zzzz} + K\tilde{w} - \beta w_{zz} + V^2 m w_{zz} - 3\alpha W^2$ and

$$F = \phi_t - m[W'_z \frac{d^2q}{d\tau^2} + 2V\ W''_{zz}\frac{dq}{d\tau}], \tag{2.4}$$

and $\tau = \epsilon^{1/2}t$ is a rescaling slow time. We observe that this self -adjoint operator L has the zero eigenfunction ψ such that $L\psi = 0$ where $\psi = \frac{dW}{dz}$. This is so called "zero mode", or "Goldstone mode" playing a fundamental role. In fact, the bounded solution of (2.3) exists if (and only if) $\int_{-\infty}^{\infty} F\psi dz = 0$. By (2.4), this condition gives the dynamics of the perturbed soliton in the newtonian form, namely

$$M\frac{d^2q}{d\tau^2} = \int_{-\infty}^{\infty} \phi_t \frac{dW}{dz}\ dz \tag{2.5}$$

where M, so-called "mass" of soliton, is defined by

$$M = m\int_{-\infty}^{\infty} (\frac{dW}{dz})^2 dz, \tag{2.6}$$

and here $\frac{d^2q}{d\tau^2}$ is the acceleration of the soliton.

Now it is clear that, to find this acceleration, it is sufficient to resolve the linear boundary value problem with $w_t = W'_t = -V\frac{dW}{dz}$. It is not difficult to see that we can seek a ϕ as a function of y and z.

3 Calculation of fluid state generated by soliton motion.

Under hypothesis $\phi = \phi(y,z)$, we obtain, from (1.2)-(1.4), that depending on case $V < c$ or $V > c$, the problem reduces to either elliptic or hyperbolic ones. We consider here only the elliptic case. (However one notices, in brackets, that case when $V \approx c$ is very interesting and can lead to new effects since there are possible transitions from elliptic situation to hyperbolic one, and inversely). We obtain

$$g\phi_y = V^2\phi_{zz}|_{y=h}, \quad \phi_y|_{y=0} = -VW'_z, \tag{3.1}$$

$$\phi_{yy} + b^2\phi_{zz} = 0, \quad b^2 = c^{-2}(c^2 - V^2). \tag{3.2}$$

To solve this problem, we use the Fourier decomposition. One sets

$$W(z)=\int_0^\infty \hat{W}(k)\cos kz dk,\quad \phi(y,z)=\int_0^\infty \hat{\phi}_k(y)\sin kz dk, \tag{3.3}$$

We find, after some standard calculations, that

$$\phi(y,z)|_{y=h}=Vb^{-1}\int_{-\infty}^\infty S(k)\sin(kz)\hat{W}(k)dk, \tag{3.4}$$

where the kernel S_k has the form

$$S_k=\frac{V^2k}{g}(b\sinh(kbh)-\frac{V^2k}{g}\cosh(kbh))^{-1}=\cosh^{-1}(kbh)(\frac{bg}{V^2k}-\tanh(kbh))^{-1}.\ ??? \tag{3.5}$$

(correct formula see below, appendix A10-A12).

Such complicated kernel means that the response of fluid is given by some pseudo-differential operator.

Let us discuss now the main physical corollaries of these results.

First we notice the following. It is clear now, without any calculations, that the soliton acceleration $\epsilon^2 d^2q/d\tau^2$ is equal to zero, i.e. the soliton, at least with precision $O(\epsilon)$, moves with the constant unperturbed speed V. Indeed, ϕ and dW/dz are odd functions whereas ϕ'_z, naturally, is even. Thus the function in the integral (2.5) also is odd and this integral is zero!

The second consequence is the existence of resonance between the beam and the fluid: the response function of the fluid grows to ∞ as $V\to c$ i.e. when the soliton speed is almost equal to the speed c of the acoustic propagation in the fluid.

To obtain the more simple formula for $\phi(h,z)$ we consider two asymptotic limits: **a** $h\to 0$ and **b**: $h\to\infty$, that corresponds to the cases of the shallow and profound water.

First one notices that $d\hat{W}_k/dk\approx\kappa$ where κ is the soliton width. If $h<<\kappa$ then S_k is a slow function, with comparison to $\hat{W}_k$. Thus we can use the Taylor expansion for S_k that gives $S_k\approx\frac{V^2k}{bg}$. In other words, our pseudo-differential operator becomes the usual differentiation. Thus, we should observe on the surface two solitons: a well and a splash. Moreover, we see that since $\phi(h,z)\approx cdW/dz$, we have $P_+\approx cW_{zz}$. This means that the pressure term P_+ in (1.1) is $\tilde{w}_{zz}$ and that acting of the fluid to the soliton reduces to *the change of mass*, i.e. the fluid acts as *an apparent additional mass*.

If $h\to\infty$ then, on the contrary, $\hat{W}_k$ is slow function of k, with comparison to S_k. The function S_k then will exponentially small, for all k beside values $k\approx 0$ and $k\approx b^2g/V$.

Thus, one can expect that we shall observe harmonic wave corresponding to wave vector $k_c=b^2g/V$.

It is obvious that for h which are not too small and are not too large, we should see the response consisting two oscillating in x solitons.

4 Following approximations

Briefly, let us describe how one can construct the following corrections. From section 3, we can find $P_+(z)$. This allows us to obtain $\tilde{w}(z)$ using (2.3). Thus, we can now substitute $W+\tilde{w}$ instead of W

into equations of section 3 and obtain the correction for the fluid state ϕ and etc. Clearly, this iteration procedure can be continued. Finally, we see, that one can construct all asymptotical solution. However one notices that the basic physical effects follows already from the results of sections 2-3.

Reference

[1] P. Gordon. Thesis. S. Peterburg University.

Appendix: Calculations

Part 1.

We have boundary problem

$$\frac{d^2}{dy^2}\phi_k - k^2 b^2 \phi_k = 0, \tag{A1}$$

$$\frac{d}{dy}\phi_k - V^2 \frac{k^2}{g}\phi_k = 0 \quad (y = h) \tag{A2}$$

$$\frac{d}{dy}\phi_k = kVW_k \quad (y = 0) \tag{A3}$$

General solution of (A1) is:

$$\phi_k(y) = A_k \cosh(kby) + B_k \sinh(kby). \tag{A4}$$

Then

$$\frac{d}{dy}\phi_k(y) = kb(A_k \sinh(kby) + B_k \cosh(kby)). \tag{A5}$$

Thus, (A3) gives

$$kbB_k = kVW_k$$

or

$$B_k = Vb^{-1}W_k. \tag{A6}$$

Let us apply now (A2). We have

$$kb(A_k \sinh(kbh) + Vb^{-1}\cosh(kbh)) = \frac{V^2 k}{gb}(A_k \cosh(kbh) + Vb^{-1}W_k \sinh(kbh))$$

that gives

$$A_k[\sinh(kbh) - \frac{V^2 k}{gb}\cosh(kbh)] == Vb^{-1}[\frac{V^2 k}{gb}\sinh(kbh) - b\cosh(kbh)],$$

and, as a result,

$$A_k = b^{-1}VW_k[\frac{V^2 k}{bg}\sinh(kbh) - \cosh(kbh)][\sinh(kbh) - \frac{V^2 k}{bg}\cosh(kbh)]^{-1},$$

Finally,

$$\phi_k(y) = Vb^{-1}W_k(\frac{\frac{V^2 k}{bg}\sinh(kbh) - \cosh(kbh)}{\sinh(kbh) - \frac{V^2 k}{bg}\cosh(kbh)}\sinh kby + \cosh kby), \tag{A7}$$

where on surface $(y = h)$ we have

$$\phi_k(h) = -Vb^{-1}W_k S_k \qquad (A8)$$

and

$$S_k = -\frac{V^2 k}{gb}\cosh^{-1}(kbh)(\tanh kbh - \frac{V^2 k}{gb})^{-1} \qquad (A9)$$

and thus

$$\phi(h,z) = Vb^{-1}\int \frac{V^2 k}{gb}\frac{W_k \cosh^{-1} kbh \sin kz dk}{\tanh kbh - \frac{V^2 k}{gb}}. \qquad (A10)$$

Part 2.

Theory for $h = \infty$ and comparison of results.

Solution for the case $h = \infty$ has the following form

$$\phi_k(y) = C\exp(-kby), \quad y > 0, \quad k > 0. \qquad (A11)$$

Constant $C = Vb^{-1}W_k$. This expression can be rewritten as

$$\phi_k(y) = Vb^{-1}W_k(\cosh kby - \sinh kby), \quad k > 0. \qquad (A11a)$$

The solution for bounded h is

$$\phi_k(y) = -Vb^{-1}W_k(\frac{\frac{V^2 k}{gb}\sinh kbh - \cosh kbh}{\sinh kbh - \frac{V^2 k}{gb}\cosh kbh}\cosh kby + \sinh kby), \qquad (A12)$$

that coincides with (A11) for $h >> 1$.

Part 3.

3.1 Asymptotics when $h \to 0$.

Substituting into (A11) we find for small h

$$\phi(h,z) \approx -\int_0^\infty W_k \sin kz dk.$$

This is an odd function. For the pressure we have

$$P = const\phi_t(h,z) \approx -constV\phi_z = constV\int_0^\infty kW_k \cos kz dk.$$

This is an even function. The first contribution in acceleration of soliton is $\int dW/dz P(z)dz = 0$ since inside integral we have odd function.

Conclusion: fluid acts on soliton form; fluid uvlekaetsya solitonom tak chto ego skorost kak zelogo ne izmenyetsya. Gidkost acts as additional mass no ne kak dissipation. These conclusions do not depend on glubinu h.

3.2 Asymptotics when $h \to \infty$.

Using (A11) we find

$$\phi(y,z) = Vb^{-1}\int_0^\infty W_k \sin kz \exp(-kby)dk, \quad y>0. \tag{A14}$$

On the surface $y = h$

$$\phi(h,z) = Vb^{-1}\int_0^\infty W_k \sin kz \exp(-kbh)dk. \tag{A15}$$

As $h >> 1$ integral can be calculated asymptotically as

$$\phi(h,z) = zVb^{-1}(bh)^{-2}W_0, \tag{A15}$$

where W_0 est ploshad (area) pod solitonom. $W_0 = \int W(z)dz$.

Pressure is

$$P_+ = constV^2b^{-1}(bh)^{-2}W_0. \tag{A16}$$

I.e. constant.