A new model for the study of rain-wind-induced vibrations of a simple oscillator

A.H.P. van der Burgh\textsuperscript{a,}*, Hartono\textsuperscript{a,1}, A.K. Abramian\textsuperscript{b}

\textsuperscript{a}Department of Applied Mathematics, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

\textsuperscript{b}Institute of Problems of Mechanical Engineering, Russian Academy of Sciences, V.O. Bolshoy pr. 61, St. Petersburg, Russia

Accepted 20 April 2005

Abstract

In this paper, a model equation is presented for the study of rain-wind-induced vibrations of a simple oscillator. As will be shown the presence of raindrops in the wind-field may have an essential influence on the dynamic stability of the oscillator. In this model equation the influence of the variation of the mass of the oscillator due to an incoming flow of raindrops hitting the oscillator and a mass flow which is blown and shaken off is investigated. The time-varying mass is modeled by a time harmonic function whereas simultaneously also time-varying lift and drag forces are considered.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Rain-wind-induced vibrations; Variation of the mass of the rain water; Cable stayed bridges; Bifurcation analysis

1. Introduction

Inclined stay cables of bridges are fixed on one end to a pylon and on the other end to the bridge-deck. Usually the stay cables have a polyurethane mantle and a cross-section which is nearly circular. With low structural damping of the bridge, a wind-field containing raindrops may induce vibrations of the cables.

As an example one can refer to Erasmus bridge in Rotterdam of which the stay cables vibrated heavily on November 4, 1996 less than 2 months after its opening. The problem of rain-wind-induced vibrations of stay cables has been reported and studied experimentally for the first time in [1]. Additional experimental studies can be found in [2–4]. In these papers it is remarked that regrettfuly calculation models are not available. A first attempt to model this problem can be found in [5] where in particular time-varying lift and drag forces are modeled. Time-varying lift and drag forces are due to the movement of the water rivulet on the cable. In
recent years some authors [6–9] have studied models for rain-wind-induced vibrations of cables. In [7] the 2D mechanical model was introduced. On the basis of this model, the numerical investigation of the phenomenon was performed. The results obtained shows that rain-wind-induced oscillations occur on vertical cables as well. In [8], an analytical study of wind-rain-induced cable vibrations was presented by considering the influence of a moving upper rivulet on the cable. However, the varying mass of rain water on the cables has not been taken into account by these authors.

In this paper a possible additional effect is taken into account namely the variation of the quantity of rainwater located on the cable. In terms of modeling one can say that the vibrating mass of rainwater on the cable is time-dependent, and is modeled by a time harmonic function, whereas simultaneously also time-varying lift and drag forces are considered. The attention is mainly focused on the interaction of the rain with the oscillator, assuming that this interaction is an instability mechanism. Raindrops hitting the oscillator may form a rivulet or a water ridge on the cable. However in a stationary situation the mass flow of incoming raindrops hitting the oscillator and the mass flow of raindrops shaken off will be equal. If these mass flows are not equal then the mass of raindrops attached to the oscillator varies with time.

One may conclude that the following mechanisms may be relevant for the study of the instability of the oscillator.

• The assumption that the mass of the ridge and hence the mass of oscillator may vary in time, seems realistic.
• Drag and lift forces vary usually in the dynamic situation; however, due to the fact that the position of the ridge on the oscillator is not fixed but varies with time, the aerodynamic coefficients additionally depend on time.

As the second mechanism has been studied in [5] it looks of interest to include the additional effect of time-varying mass. It should be stressed that the dynamics of the mass of the rivulets will not be modeled by a separate equation of motion in this stage: in the modeling we assume that either the position of the rivulets is fixed or varies harmonically in time in the same way as the oscillator.

2. A model equation with time-varying mass and lift and drag forces

In this section, we use the modeling principles as given in [10] or [11]. Consider a horizontal rigid cylinder with uniform circular cross section supported by springs. A rain-wind flow is directed to the axis of the cylinder. The cylinder with springs is constructed in such a way that only vertical oscillations i.e. oscillations in cross-flow are possible. The raindrops that hit the cable may stay on the surface of the cylinder for some time and may form a ridge of water of which the position varies with time. Due to the variation of the acceleration of the cable and the aerodynamic forces, part of the water will be blown and or shaken off and hence the mass of the water ridge varies in time. The system which will be studied is sketched in Fig. 1. \( U \) is the horizontal uniform velocity of the wind containing raindrops. When the cylinder moves in the positive \( y \) direction a virtual wind velocity \( -\dot{y} \) is induced, i.e. a wind flow with velocity \( \dot{y} \) in opposite direction. The drag force \( D \) is indicated in the direction of the resultant wind-velocity \( U_r \), whereas the lift force \( L \) is perpendicular to \( D \) in anti clockwise direction. The water ridge on the cylinder, boldly indicated in Fig. 1, is assumed to carry out harmonic oscillations with a small amplitude on the surface of the cylinder, whereas the mass of the water ridge \( m_r(t) \) is supposed to vary harmonically in time as well. The aerodynamic force \( F_y \) in vertical direction follows from Fig. 1:

\[
F_y = -D \sin \phi - L \cos \phi,  \tag{2.1}
\]

where \( \phi \) is the angle between \( U_r \) and \( U \), positive in clockwise direction: \( |\phi| < \pi/2 \).

Fig. 1. Cross-section of the cylinder-spring system, fluid flow with respect to the cylinder and wind forces on the cylinder.
The drag and lift forces are given by the empirical relations:

\[ D = \frac{1}{2} \rho d l U^2 C_D(x), \]
\[ L = \frac{1}{2} \rho d l U^2 C_L(x), \]
(2.2)

where \( \rho \) is the density of the flowing medium (air with raindrops), \( d \) the diameter of the cylinder, \( l \) the length of the cylinder, \( C_D(x) \) and \( C_L(x) \) are the drag and lift coefficient curves, respectively, determined by measurements in a wind-tunnel.

From Fig. 1 it follows:

\[ \sin \phi = \dot{y}/U, \]
\[ \cos \phi = U/Ur, \]
\[ z = x_0 + \arctan(\dot{y}/U). \]
(2.3)

The equation of motion of the oscillator readily becomes:

\[ \frac{d}{dt} \left[ (M + m_r(t)) \frac{dy}{dt} \right] + c_y \frac{dy}{dt} + k_y y = F_y, \]
(2.4)

where \( M \) is the mass of the cylinder, \( m_r(t) \) the time-varying mass of the raindrops on the cylinder, \( c_y > 0 \) the structural damping coefficient of the oscillator, and \( k_y > 0 \) the spring constant. By using (2.1)–(2.3) we obtain for \( F_y \):

\[ F_y = -\frac{1}{2} \rho d l U^2 \gamma^2 \left( C_D(x) \frac{dy}{dt} + C_L(x) U \right). \]
(2.5)

Let \( m_r(t)/M = \varepsilon m(t) \), where \( \varepsilon > 0 \) is a small parameter, (2.4) becomes:

\[ (1 + \varepsilon m(t)) \frac{d^2 y}{dt^2} + \left( \frac{c_y}{M} + \varepsilon \frac{dm}{dt} \right) \frac{dy}{dt} + \omega_y^2 y = F_y/M, \]
(2.6)

where \( \omega_y^2 = k_y/M \). The case is studied where the drag and lift coefficient curves can be approximated by

\[ C_D(x) = C_{D0}, \]
\[ C_L(x) = C_{L1}(x - x_0) + C_{L3}(x - x_0)^3, \]
(2.7)

where \( C_{D0} > 0, C_{L1} < 0, C_{L3} > 0 \) and \( x_0 > 0 \). \( x_0 \) is any angle in the domain of the \( x \)-axis of the \( C_L(x) \) curve where the slope is negative i.e. \( C_{L1} < 0 \). The approximations (2.7) fit as a first step with typical curves obtained from wind tunnel experiments as indicated in Fig. 2 (see also [12]). By using (2.3), (2.7) can be written as

\[ C_L(x) = C_{L1}(x - x_0 + \arctan(\dot{y}/U)) \]
\[ + C_{L3}(x - x_0 + \arctan(\dot{y}/U))^3. \]
(2.8)

Now the variation of the position of the water ridge can be modeled by \( x_0 - x_0 = f(t) \).

Substituting of this variation in (2.8) using (2.5) and (2.7), one obtains for (2.6) the following linearized mono equation (around \( \dot{y} = 0 \)):

\[ (1 + \varepsilon m(t)) \frac{d^2 y}{dt^2} + \left( \tilde{K} (C_{D0} + C_{L1}) + \frac{c_y}{M} \right) \frac{dy}{dt} + \omega_y^2 y \]
\[ = -\tilde{K} \left( C_{L1} f(t) + C_{L3} f^3(t) \right) U, \]
(2.9)

where \( \tilde{K} = \rho d l U^2/2M \). Let \( c_y/M = 2\beta \) and \( \tilde{K} = \varepsilon K \omega \), then by dividing \((1 + \varepsilon m(t))\) one obtains

\[ \frac{d^2 y}{dt^2} + (\varepsilon \omega q(t)) \frac{dy}{dt} + \omega_y^2 (1 - \varepsilon m(t)) y \]
\[ = \varepsilon K \omega \left( C_{L1} f(t) + C_{L3} f^3(t) \right) U + O \left( \varepsilon^2 \right), \]
(2.10)

where \( \omega q(t) = K \omega (C_{D0} + C_{L1}) + 2\beta + 3C_{L3} K \omega f^2(t) + \frac{dm}{dt} \), and where \( \omega \) is defined in the expressions for \( f(t) \) and \( m(t) \) below. The accepted law of the time variation of water mass is based on the assumption that the water comes off the oscillating cable surface in one of the cables extreme position when the inertia is of maximum value. It can be also assumed that the water mass is being stored during the cable period of oscillation following the one when the water
come-off occurs. Consider the following model variations of the water ridge and \( m(t) \):
\[
\begin{align*}
f(t) &= c_1 \cos(\omega t) + d_1 \sin(\omega t), \\
m(t) &= a_2 \cos(2\omega t) + b_2 \sin(2\omega t), \\
f^2(t) &= c_0 + c_2 \cos(2\omega t) + d_2 \sin(2\omega t),
\end{align*}
\]
where
\[
\begin{align*}
c_0 &= \frac{1}{2} \left( c_1^2 + d_1^2 \right), \\
c_2 &= \frac{1}{2} \left( c_1^2 - d_1^2 \right), \\
d_2 &= c_1 d_1.
\end{align*}
\]
By introducing the new variables \( z = \omega y / U \) and \( \omega t = \tau \) one obtains
\[
\begin{align*}
d^2z \quad (aq(\tau / \omega)) \frac{dz}{d\tau} + \frac{\omega^2}{\omega^2} (1 - \omega m(\tau / \omega)) z \\
= \varepsilon K \left( C_L_1 f(\tau / \omega) + C_L_3 f^3(\tau / \omega) \right) + O \left( \varepsilon^2 \right).
\end{align*}
\]
For the main resonance case as studied in the present paper one should consider
\[
\frac{\omega_0^2}{\omega^2} = 1 - 2a\eta,
\]
where \( \eta \) is a detuning parameter. By using (2.14), (2.13) can be written as
\[
\varepsilon z + z = \varepsilon H(\varepsilon z, \varepsilon \tau),
\]
where \( H(\varepsilon z, \varepsilon \tau) = K(C_L_1 f(\tau / \omega) + C_L_3 f^3(\tau / \omega)) - q(\tau / \omega) \varepsilon z + (2\eta + m(\tau / \omega)) z + O(\varepsilon^2) \). Let
\[
\begin{align*}
z &= y_1 \cos(\tau) + y_2 \sin(\tau), \\
\dot{z} &= - y_1 \sin(\tau) + y_2 \cos(\tau).
\end{align*}
\]
Then (2.15) becomes
\[
\begin{align*}
\dot{y}_1 &= - \varepsilon H \sin(\tau), \\
\dot{y}_2 &= \varepsilon H \cos(\tau).
\end{align*}
\]
By averaging one obtains
\[
\begin{align*}
\left( \begin{array}{c}
\dot{y}_1 \\
\dot{y}_2
\end{array} \right) &= \left( \begin{array}{cc}
s + \frac{1}{4}b_2 + \frac{1}{4}\tilde{c}_2 & \frac{1}{4}d_2 - \frac{1}{4}a_2 - \eta \\
\frac{1}{4}d_2 - \frac{1}{4}a_2 + \eta & s - \frac{1}{4}b_2 - \frac{1}{4}\tilde{c}_2
\end{array} \right) \left( \begin{array}{c}
\bar{y}_1 \\
\bar{y}_2
\end{array} \right) \\
&+ \left( \frac{1}{4} K C_L_1 d_1 + \frac{1}{8} d_1 \tilde{c}_0 \right),
\end{align*}
\]
where
\[
\begin{align*}
s &= - (\beta / \omega + \frac{1}{2} K (C_D_0 + C_L_1) + \frac{1}{2} \tilde{c}_0), \\
\tilde{c}_0 &= 3c_0 K C_L_3, \\
\tilde{c}_2 &= 3c_2 K C_L_3, \\
\bar{d}_2 &= 3d_2 K C_L_3.
\end{align*}
\]
The stability of the critical point of (2.18) depends on the eigenvalues \( \lambda \) of the matrix in Eq. (2.18) and follows from
\[
\begin{align*}
(s - \lambda)^2 &= \left( \frac{1}{4} b_2 + \frac{1}{4} \tilde{c}_2 \right)^2 - \left( \frac{1}{4} d_2 - \frac{1}{4} a_2 \right)^2 + \eta^2 \\
&= 0.
\end{align*}
\]
For \( s = 0 \), implying that the constant damping coefficient vanishes, the transition curve (or manifold in a higher dimensional parameter space) separating stable and unstable regions in the relevant parameter space follows from
\[
\eta^2 = \frac{1}{16} (b_2 + \tilde{c}_2)^2 + \frac{1}{16} (d_2 - a_2)^2.
\]
When one sets \( b_2 + \tilde{c}_2 = X \) and \( d_2 - a_2 = Y \) in (2.20) then the equation represents a cone in the \( (\eta, X, Y) \) space, with the \( \eta \)-axis as central axis (see Fig. 3). The domain inside the cone corresponds with a regime of parameters where all solutions are stable. For \( \eta = 0 \) i.e. \( \omega = \omega_0 \) and \( s = 0 \) only unstable solutions are found (see Fig. 3).
Fig. 3a). Apparently for $\eta = 0$, stable solutions only exist if both eigenvalues which follow from (2.19) are negative. For this case it is clear that $s < 0$ and that the right-hand side of (2.20) should be sufficiently small (see Fig. 3b).

Result (2.20) describes an interesting property. If for instance $\tilde{c}_2 = \tilde{d}_2 = 0$ which implies by using (2.12) that $f(t) \equiv 0$ (the position of the water ridge on the surface is fixed), by varying the amplitude $r_1 = \sqrt{a_1^2 + b_1^2}$ one can control the stability of the trivial solution i.e. pass through the surface of the cone for $\eta \neq 0$ fixed. However, if $\tilde{c}_2 \neq 0$ and $\tilde{d}_2 \neq 0$ the right-hand side of (2.20) can be written in terms of amplitudes and phases as follows:

$$\frac{1}{16} \left[ (b_2 + \tilde{c}_2)^2 + (\tilde{d}_2 - a_2)^2 \right] = \frac{1}{16} \left[ r_1^2 + r_2^2 + 2r_1r_2 \sin(\varphi_1 - \varphi_2) \right],$$ (2.21)

where $r_1^2 = a_1^2 + b_1^2$, $r_2^2 = \tilde{c}_2^2 + \tilde{d}_2^2$, $\varphi_1 = \arctan(b_1/a_1)$ and $\varphi_2 = \arctan(\tilde{d}_2/\tilde{c}_2)$. From (2.21) it follows that by keeping $r_1$ and $r_2$ constant but by varying the phase difference $(\varphi_1 - \varphi_2)$ between $f(t)$ and $m(t)$ one can pass through the surface of the cone separating the stable and unstable regime of parameters.

3. The non-linear model

According to the previous section, by introducing the new variables $z = \omega y/U$ and $\omega t = \tau$ one obtains

$$\ddot{z} + \left( \frac{\text{d}m}{\text{d}t} + \varepsilon \beta \right) \dot{z} + \frac{\omega_0^2}{\omega^2} (1 - \sin(\tau/\omega)) z = -\varepsilon K \sqrt{1 + \dot{z}^2} (C_D(z) \dot{z} + C_L(z)) + O \left( \dot{z}^2 \right),$$ (3.1)

where $K = p d l U / 2 M \omega$, $\varepsilon \beta = c_v / M$, $\omega_0^2 = k_v / M$, $\dot{z} = \text{d}z / \text{d}t$, and

$$C_D(z) = C_{D_0},$$
$$C_L(z) = C_{L_1} (f(\tau/\omega) + \arctan \dot{z}) + C_{L_2} (f(\tau/\omega) + \arctan \dot{z})^3.$$  

Substituting (2.14) into (3.1), expanding the right-hand side of (3.1) with respect to $\dot{z}$ (up to cubical terms), using (2.16) one obtains after first order averaging:

$$\ddot{y}_1 = \varepsilon \left[ \frac{1}{2} KC_{L_1} d_1 + \frac{1}{8} d_1 \tilde{c}_0 + \left( s + \frac{1}{4} b_2 + \frac{1}{4} \tilde{c}_2 \right) \ddot{y}_1 \right. + \left. \left( \frac{1}{4} \tilde{d}_2 - \frac{1}{4} a_2 - \eta \right) \ddot{y}_2 \right. + \left. d_1 \left( \frac{3}{12} \tilde{c}_0 - \frac{1}{48} \tilde{c}_2 \right) \ddot{y}_1 \right]$$
$$+ d_1 \left( q + \frac{1}{24} \tilde{c}_0 + \frac{1}{48} \tilde{c}_2 \right) \ddot{y}_2 - c_1 \left( 2q + \frac{1}{12} \tilde{c}_0 \right. \right.$$  
$$\left. \left. - \frac{1}{24} \tilde{c}_2 \right) \ddot{y}_1 \ddot{y}_2 \right) + \left( p - \frac{1}{16} \tilde{c}_0 + \frac{1}{24} \tilde{c}_2 \right) \ddot{y}_1 \ddot{y}_2 \right. + \left. \frac{1}{48} \ddot{d}_2 \ddot{y}_1 \ddot{y}_2 \right) + \left( p - \frac{1}{16} \tilde{c}_0 \right) \ddot{y}_1 \ddot{y}_2 \right] \right],$$ (3.2)

where $p = -K \left( \frac{1}{16} C_{D_0} + \frac{1}{16} C_{L_1} + \frac{1}{8} C_{L_2} \right)$ and $q = K \left( \frac{1}{16} C_{D_1} + \frac{1}{8} C_{L_2} \right)$. The signs of $p$ and $q$ depend on the values of the aerodynamic coefficients which are for instance given in [13]. System (3.2) is a general cubical system, which includes the linear terms from Eq. (2.18). In the cases where in linear approximation unstable solutions are found the non-linear terms in system (3.2) may provide stable solutions. In what follows, some special cases of the general system (3.2) are studied.

3.1. Fixed position of water ridge and time-varying mass

System (3.2) describes both the effect of the time-varying position of the ridge of water as well as the
time-varying mass of rain water on the oscillator. These effects can be studied separately by first considering the case that the position of the water ridge does not vary in time i.e. $c_2 = d_2 = 0$. In this case (3.2) becomes
\begin{align*}
\ddot{y}_1 &= e \left( \left( \tilde{s} + \frac{1}{4} b_2 \right) \dot{y}_1 - \left( \frac{1}{4} a_2 + \eta \right) \dot{y}_2 ight) \\
&\quad + p \dot{y}_1^3 + p \dot{y}_1 \dot{y}_2^2, \\
\ddot{y}_2 &= e \left[ - \left( \frac{1}{4} a_2 - \eta \right) \dot{y}_1 + \left( \tilde{s} - \frac{1}{4} b_2 \right) \dot{y}_2 + p \dot{y}_2^3 ight] \\
&\quad + p \dot{y}_1^2 \dot{y}_2, \quad (3.3)
\end{align*}
where $\tilde{s} = -(\beta/\omega + \frac{1}{2} K (C_{D_0} + C_{L_1}))$. By using the transformation
\begin{align*}
\tilde{y}_1 &= r \cos \theta, \\
\tilde{y}_2 &= r \sin \theta, \quad (3.4)
\end{align*}
(3.3) becomes
\begin{align*}
\dot{r} &= e r \left[ \frac{1}{4} b_2 \cos(2\theta) - \frac{1}{4} a_2 \sin(2\theta) + \tilde{s} + pr^2 \right], \\
\dot{\theta} &= e \left[ \left( \frac{1}{4} a_2 \cos(2\theta) - \frac{1}{4} b_2 \sin(2\theta) + \eta \right) \right]. \quad (3.5)
\end{align*}

The non-trivial critical points of (3.5) are solutions of
\begin{align*}
\frac{1}{4} b_2 \cos(2\theta) - \frac{1}{4} a_2 \sin(2\theta) + \tilde{s} + pr^2 &= 0, \quad (3.6) \\
- \frac{1}{4} a_2 \cos(2\theta) - \frac{1}{4} b_2 \sin(2\theta) + \eta &= 0.
\end{align*}
By elimination of $\theta$ in (3.6) one obtains
\begin{equation*}
\frac{1}{16} a_2^2 + \frac{1}{16} b_2^2 = \eta^2 + (\tilde{s} + pr^2)^2, \quad (3.7)
\end{equation*}
or
\begin{equation*}
r^2 = \frac{1}{p} \left( -\tilde{s} \pm \sqrt{\frac{1}{16} a_2^2 + \frac{1}{16} b_2^2 - \eta^2} \right). \quad (3.8)
\end{equation*}
Thus (3.5) has only one critical point i.e. the origin if $\frac{1}{16} a_2^2 + \frac{1}{16} b_2^2 < \eta^2$, implying that (3.3) has only one critical point. In case $\frac{1}{16} a_2^2 + \frac{1}{16} b_2^2 > \eta^2$ (3.5) may have three or five critical points. To evaluate the stability of these critical points one can linearize the system around each critical point. Jacobi’s matrix of (3.5) evaluated at its critical point $(r_0, \theta_0)$ is
\begin{equation*}
\begin{pmatrix}
2pr_0^2 & -\frac{1}{2} r_0 (b_2 \sin 2\theta_0 + a_2 \cos 2\theta_0) \\
0 & 2\tilde{s} + 2pr_0^2
\end{pmatrix}. \quad (3.9)
\end{equation*}
The eigenvalues of (3.9) are
\begin{align*}
\lambda_1 &= 2pr_0^2 \quad \text{and} \quad \lambda_2 = 2\tilde{s} + 2pr_0^2.
\end{align*}
If for instance $a_2 = 0.6$ or $a_2 = 4$, $b_2 = 0$, $\eta = 0$, $C_{L_3} = 2$, $C_{L_1} = -6$, $C_{D_0} = 1/2$, $\beta/\omega = 2$ and $K = 1$, then five or three critical points are found, respectively, and these phase portraits of system (3.3) are given in Fig. 4a and 4b. In Fig. 4b there are two stable critical points corresponding with two stable periodic solution of (3.1) having equal amplitudes but a phase difference of $\pi$.

In the case the detuning parameter, $\eta$, is not equal to zero, for instance $\eta = 0.9$ and $\eta = 1.2$, these phase portraits are depicted in Fig. 5a and 5b. In these phase portraits the value of $a_2$ is 4. We can observe that the

![Phase Portraits](image.png)

Fig. 4. The phase portrait of (3.3) for $a_2 = 0.6$ and $a_2 = 4$, $b_2 = 0$, $\eta = 0$, $C_{L_3} = 2$, $C_{L_1} = -6$, $C_{D_0} = 1/2$, $\beta/\omega = 2$ and $K = 1$.
Fig. 5. The phase portraits of Eq. (3.3) for several values of $\eta$, the detuning parameter, $a_2 = 4$, $b_2 = 0$, $C_{L_3} = 2$, $C_{L_1} = -6$, $C_{D_0} = 1/2$, $\beta/\omega = 2$ and $K = 1$.

Fig. 6. Relation between $r$ and $\eta$ of formula (3.10).

variation of the detuning parameter $\eta$, leads to a change in the number of the critical points, and when $\eta > 1$ a limit cycle occurs. The change in the number of the critical points can be easily understood by evaluating formula (3.8). In this case the formula (3.8) become

$$r^2 = -\frac{32}{15} \left( -\frac{3}{4} \pm \sqrt{1 - \eta^2} \right) .$$

(3.10)

The graph of $r$ as a function of $\eta$ is depicted in Fig. 6. If $0 < \sqrt{1 - \eta^2} < \frac{3}{4}$, then we obtain four values of $r$. This results implies that Eq. (3.3) has five critical points as the origin is also a critical point. If $\sqrt{1 - \eta^2} = \frac{3}{4}$ then three values of $r$ are obtained (one of them is zero) and if $\frac{3}{4} < \sqrt{1 - \eta^2} < 1$ then we obtain two values for $r$.

3.2. Constant mass and varying position of the water ridge

Next, we consider the case that the position of the ridge of water varies in time and that there is no mass variation due to the rain, i.e. $a_2 = b_2 = 0$, implying that the mass flow of rain water hitting the cylinder is equal to the mass flow of rain water which is blown off the cylinder. As is known $f(t) = f(\tau/\omega) = c_1 \cos(\tau) + d_1 \sin(\tau)$ can be written as $A \cos(\tau + \psi_\tau)$. It is no essential limitation to put $\psi_\tau = 0$, in other words, $c_1 \neq 0$, $d_1 = 0$. In this case (3.2) becomes

$$\begin{align*}
\dot{y}_1 &= e \left[ \left( s + \frac{1}{4} \tilde{c}_2 \right) \tilde{y}_1 - \eta \tilde{y}_2 - c_1 \left( 2q + \frac{1}{12} \tilde{c}_0 \right) \right] y_1^3 - \frac{1}{24} \tilde{c}_2 \tilde{y}_1 \tilde{y}_2 + \left( p - \frac{1}{16} \tilde{c}_0 + \frac{1}{24} \tilde{c}_2 \right) \tilde{y}_1^3 \tilde{y}_2 \\
\dot{y}_2 &= e \left[ -\frac{1}{2} KC_{L_1} c_1 - \frac{1}{8} c_1 \tilde{c}_0 + \eta \tilde{y}_1 + \left( s - \frac{1}{4} \tilde{c}_2 \right) \tilde{y}_2 - c_1 \left( q + \frac{1}{24} \tilde{c}_0 - \frac{1}{48} \tilde{c}_2 \right) \tilde{y}_1^2 - c_1 \left( 3q + \frac{1}{12} \tilde{c}_0 \right) \right] \tilde{y}_2^3 + \left( p - \frac{1}{16} \tilde{c}_0 - \frac{1}{24} \tilde{c}_2 \right) \tilde{y}_2^3 \tilde{y}_1 + \left( p - \frac{1}{16} \tilde{c}_0 \right) \tilde{y}_1^2 \tilde{y}_2, \\
\end{align*}$$

(3.11)

where $\tilde{c}_0 = \tilde{c}_2 = -\frac{3}{4} KC_{L_1}$. One can observe that when $\eta = 0$, $\tilde{y}_1 = 0$ is a solution of the first equation of (3.11). Thus to find the critical points of (3.11) one can substitute $\tilde{y}_1 = 0$ into the second equation of (3.11) and
the result is a cubic equation in $\bar{y}_2$. If, for instance, $C_{L_3} = 2$, $C_{L_1} = -6$, $C_{D_0} = 1/2$, $\beta/\omega = 2$, $K = 1$ and $c_1 = A$ then the relation between $A$ and $\bar{y}_2$ is depicted in Fig. 7a. Clearly when $A$ varies from 0 to 1 the number of real zeros of the cubic equation for $\bar{y}_2$ varies from 3 to 2 and finally to 1, and similar results for the critical points of (3.11). For $A = 0.1$ and the values of the other parameters as given above, the phase portrait of (3.11) is shown in Fig. 7b. The coordinates of the critical points in Fig. 7b correspond with the values $\bar{y}_2$ along line $A = 0.1$ in Fig. 7a. From Fig. 7b one can observe that there are three critical points one stable and two unstable. It follows that there are three periodic solutions in the original system: one asymptotically stable and the others unstable. In the case where $\eta \neq 0$, $\bar{y}_1 = 0$ is not a solution of the first equation of (3.11). When $\eta$ increases from $\eta = 0$ than the three critical points (in Fig. 7b) are moving into the positive direction of the $\bar{y}_1$-axis as is shown in Fig. 9a. Further, following Fig. 8 when the value of $\eta$ is around 0.194 there will be a saddle-node bifurcation and a limit cycle occurs as is depicted in Fig. 9b. The limit cycle corresponds to a modulated oscillation in the original system. For more details one can consult [14].

3.3. Mass and position of water ridge vary both with time

Now we discuss the situation that both effects are included. The effect of the varying position of the ridge of water is indicated by the parameters $c_1$ and $d_1$, and the effect of the varying mass is indicated by $a_2$ and $b_2$. By observing system (3.2) it follows that $a_2$ and $b_2$ affect only the coefficients of the linear terms of the system. Hence by varying $a_2$ and $b_2$ the structure of the solutions near the origin may vary. However $c_1$ and $d_1$ affect both the coefficients of the linear as the non-linear terms and hence this variation has a local and non-local influence on the behavior of the solutions. Clearly the coefficients of the non-linear terms define the structure of the solutions of the critical points away from the origin implying their
non-local relevance. Note that \( s \) depends on \( c_1 \) and \( d_1 \) and are involved in some of the coefficients of the linear terms. We study the situation that both effects are included in two ways, the first one keeps \( a_2 \) and \( b_2 \) fixed and varies \( c_1 \) and \( d_1 \), and the other one keeps \( c_1 \) and \( d_1 \) fixed and varies \( a_2 \) and \( b_2 \). It is known that \( f(t) = f(\tau/\omega) = c_1 \cos \tau + d_1 \sin \tau \) can be written as \( A \cos(\tau + \psi) \). It is not an essential limitation to put \( \psi_0 = 0 \), in other words, \( c_1 \neq 0, d_1 = 0 \), which reduces system (3.2) to:

\[
\dot{y}_1 = \varepsilon \left[ \left( \dot{s} + \frac{1}{4} b_2 - \frac{3}{8} Kc_{L_3} c_1^2 \right) \tilde{y}_1 \right. \\
+ \left( -\frac{1}{4} a_2 - \eta \right) \tilde{y}_2 + c_1 \left( 2q + \frac{1}{16} Kc_{L_3} c_1^2 \right) \tilde{y}_1 \tilde{y}_2 \\
+ \left. \left( p - \frac{1}{32} Kc_{L_3} c_1^2 \right) \tilde{y}_3 + \left( p - \frac{3}{32} Kc_{L_3} c_1^2 \right) \tilde{y}_1 \tilde{y}_2 \right],
\]

\[
\dot{y}_2 = \varepsilon \left[ -\frac{1}{2} Kc_{L_1} c_1 - \frac{3}{8} Kc_{L_3} c_1^3 + \left( -\frac{1}{4} a_2 + \eta \right) \tilde{y}_1 \\
+ \left( \dot{s} - \frac{1}{4} b_2 - \frac{9}{8} Kc_{L_3} c_1^2 \right) \tilde{y}_2 \\
- c_1 \left( q + \frac{1}{32} Kc_{L_3} c_1^2 \right) \tilde{y}_1 \tilde{y}_2 \\
- c_1 \left( 3q + \frac{5}{32} Kc_{L_3} c_1^2 \right) \tilde{y}_2 \tilde{y}_2 \\
+ \left( p - \frac{5}{32} Kc_{L_3} c_1^2 \tilde{y}_2 \tilde{y}_2 \right) \\
+ \left( p - \frac{3}{32} Kc_{L_3} c_1^2 \right) \tilde{y}_1 \tilde{y}_2 \right].
\] (3.12)

The critical points of (3.12) for relevant values of the coefficients can be analyzed by using a Gröbner basis algorithm. Keep all parameters fixed except \( c_1 = A \), for instance \( a_2 = 4, b_2 = 0, \eta = 0, C_{L_3} = 2, C_{L_1} = -6, C_{D_0} = 1/2, \beta/\omega = 2 \) and \( K = 1 \). Then one can obtain a relation between \( A \) and \( \tilde{y}_1, \tilde{y}_2 \) as shown in Fig. 10. The number of critical points of (3.12) varies from three to one if \( A \) varies from 0 to 1 and the phase portraits are depicted in Fig. 11a–d for several values of \( A \). In Fig. 9 the vertical axis is \( \tilde{y}_2 \) and the horizontal axis is \( \tilde{y}_1 \). The coordinates of the critical points in Fig. 11 correspond to the value of \( \tilde{y}_1 \) and \( \tilde{y}_2 \) in Fig. 10 for the given value of \( A \). Corresponding (parts of) curves in Fig. 10a and b are indicated with \( S_1, U \), and \( S_2 \) where \( S \) stands for stable and \( U \) stands for unstable. If for example \( A = 0.1 \), the coordinates of the stable critical point \((\tilde{y}_1, \tilde{y}_2)\) follow from Fig. 10a by using the upper \( S_2 \) curve to obtain \( \tilde{y}_1 \) and from Fig. 10b the lower \( S_2 \) curve to obtain \( \tilde{y}_2 \). Further one can observe that a saddle-node bifurcation occurs for \( A \) in the neighborhood of 0.52.

Further, we study the situation \( c_1 = A = 0.1, d_1 = 0 \) and vary \( a_2 \) with \( b_2 = 0 \). By keeping the other parameters fixed as before implying that only \( a_2 \) varies in Eq. (3.12), then by using the Gröbner basis algorithm one can find the number of critical points as shown in Fig. 12a, and 12b. From the diagram in Fig. 12a one can observe that the number of critical points varies as follows:

\( 3 \rightarrow 4 \rightarrow 5 \rightarrow 4 \rightarrow 3 \),

when \( a_2 \) varies from 0 to 5. For example when \( a_2 = 4 \) the number of critical points is three. The phase
Fig. 10. Critical points \((\bar{y}_1, \bar{y}_2)\) of system (3.12) as a function of \(A\) where \(d_1 = 0, a_2 = 4, b_2 = 0\) and \(\eta = 0\).

Fig. 11. The phase portraits of Eq. (3.12) for several values of \(A\), the amplitude of the variation of the position of the water ridge. The variation of the mass of rainwater has in this case a constant amplitude \(a_2 = 4, b_2 = 0\).

Portraits are depicted in Fig. 13 for several values of \(a_2\). In Fig. 13 the vertical axis is \(\bar{y}_2\) and the horizontal axis is \(\bar{y}_1\).

In comparing the phase portraits in Fig. 11 and 13 one can verify that Fig. 11b and 13d are qualitatively equivalent although the values of \(a_2\) differ: 4 in
Fig. 12. Critical points \((\bar{\gamma}_1, \bar{\gamma}_2)\) of system (3.12) as a function of \(a_2\), where \(c_1 = A = 0.1\), \(d_1 = 0\), \(b_2 = 0\) and \(\eta = 0\).

Fig. 13. The phase portraits of Eq. (3.12) for several values of \(a_2\), the amplitude of the variation of the mass of rain water. The variation of the position of the water ridge has constant amplitude \(A = 0.1\).

**Fig. 11b and 1.3 in Fig. 13d.** This is in accordance with the qualitative behavior of the critical points in Figs. 12a and b for \(a_2 > 1.3\). Clearly in both phase portraits there are two stable critical points corresponding with stable periodic solutions with approximately the same amplitude but with different phases. Increasing \(A\) in Fig. 11 and decreasing \(a_2\) in Fig. 13 leads to Figs. 11d and 13a with one stable critical point. However, the phase portraits are qualitatively rather different.
Finally the number and the stability of the critical points of system (3.12) for several values of $A$ and $a_2$ are summarized in Table 1. In general, the variation of the position of the water ridge and the variation of the mass of rain water on the oscillator give different effects to the system. The presence of the variation of the position of the water ridge in the system implies that the origin is not a critical point of the system. In other words, the system does not have a trivial solution. Further, if we increase the amplitude of the variation of the position of the water ridge then the system has only one stable critical point. However, if we increase the amplitude of the variation of the mass of rain water on the oscillator then the system can have three critical points of which two are stable and one unstable. Considering the magnitude of the amplitude of the two variations, it seems that in a number of cases the variation of the position of the water ridge leads to bifurcation of critical points. In Fig. 10c for example ($a_2 = 4$ and $A = 0.5$ ) the system has three critical points and in Fig. 11d ($a_2 = 4$ and $A = 0.8$ ) the system has only one stable critical point, thus between $A = 0.5$ and $A = 0.8$ a saddle-node bifurcation occurs. Because a critical point corresponds

<table>
<thead>
<tr>
<th>$a_2$</th>
<th>$A = 0$</th>
<th>$A = 0.1$</th>
<th>$A = 0.14$</th>
<th>$A = 0.5$</th>
<th>$A = 0.8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Origin is an unstable node.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>Origin is an unstable node.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
</tr>
<tr>
<td></td>
<td>(2s, 3u)</td>
<td>(1s, 2u)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>Origin is an unstable node.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
</tr>
<tr>
<td></td>
<td>(2s, 3u)</td>
<td>(2s, 3u)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.3</td>
<td>Origin is an unstable node.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
</tr>
<tr>
<td></td>
<td>(2s, 3u)</td>
<td>(2s, 1u)</td>
<td>(2s, 1u)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Origin is a saddle point.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
<td>o.n.s.</td>
</tr>
<tr>
<td></td>
<td>3 c.p.</td>
<td>3 c.p.</td>
<td>3 c.p.</td>
<td>3 c.p.</td>
<td>1s c.p.</td>
</tr>
<tr>
<td></td>
<td>(2s, 1u)</td>
<td>(2s, 1u)</td>
<td>(2s, 1u)</td>
<td>(2s, 1u)</td>
<td></td>
</tr>
</tbody>
</table>

$A$= the amplitude of the variation of the position of the water ridge, $a_2$= the amplitude of the variation of the mass of rain water, $s$ = stable and $u$= unstable, c.p. = critical point, o.n.s.=origin is not a solution.
with a periodic solution in the original system then according to Fig. 11c the original system has three periodic solutions, two stable and one unstable. The influence of increasing the amplitude of the position of the water ridge of the system is that the two critical points vanish due to saddle-node bifurcation and only one stable critical point remains. The results which are described in Table 1 do not include the influence of the detuning parameter $\eta$. Now, we will see the effect of the detuning parameter $\eta$ by keeping all other parameters fixed. When the value of $\eta$ is “large” then we will have in the phase plane of Eq. (3.12) an unstable critical point and a stable limit cycle around this point. For instance, when we fix the parameters, $a_2 = 4$, $b_2 = 0$, $C_{L3} = 2$, $C_{L1} = -6$, $C_{D0} = 1/2$, $\beta/\omega = 2$, $K = 1$ and $A = 0.1$ and when we vary $\eta$ we will find the following. When $\eta = 0$ the phase portrait is shown in Fig. 14a. By increasing $\eta$ the two critical points come closer to each other and finally collapse when the value of $\eta$ is between 0.8 and 1 (see Fig. 14b and c). This phenomenon is known as a saddle-node bifurcation. When the values of $\eta$ are between 1.24 and 1.25 the Hopf bifurcation occurs, initiating the occurrence of a limit cycle. For $\eta = 2$ the phase portrait is given in Fig. 14d.

### 4. Conclusions

From the model equation describing the interaction of a wind-field containing rain drops and a simple oscillator it follows that both the time-varying mass...
of rain drops attached to the oscillator and the time-varying lift and drag force coefficients are mechanisms leading to an unstable equilibrium position. From the physical point of view it can be understood that regular adding and removing of a marginal quantity of raindrops attached to a mass spring system defined as a Hamiltonian system, may lead to an unstable equilibrium.

On the other hand, the time-varying position of the water ridge leads to time-varying lift and drag forces as an instability mechanism. When the position of the water ridge is fixed \( (A = 0, \text{ Table 1, column 1}) \) an unstable equilibrium position as evolution of three unstable critical points and two stable critical points corresponding with two periodic solutions is found (Fig. 11a). In absence of variation of the mass of rain water on the oscillator \( (a_2 = 0, \text{ Fig. 13a}) \) only one stable critical point i.e. one periodic solution is found. Apparently, the first mechanism displays a certain symmetry with respect to the positions of the critical points which the second mechanism does not show.

In general the variation of the position of the water ridge and the variation of the mass of rain water on the oscillator give different effects to the system. Increasing the amplitude of the variation of the position of the water ridge shows that the system has one stable periodic solution, but when the amplitude of the variation of the mass of rain water increases then the system finally will have three critical points (last row in Table 1) corresponding with three periodic solutions (two stable and one unstable). Variations of the detuning parameter will lead to saddle-node and Hopf bifurcations in the system. From a practical point of view, one may conclude that in order to avoid instabilities one should design the oscillator in such a way that rain water accumulation and variation will not be possible. Since there are no known experimental data on time variation of water ridge mass, it is impossible to compare them with the achieved theoretical results.

References


