DISSIPATIVE AND HAMILTONIAN SYSTEMS WITH CHAOTIC BEHAVIOR: AN ANALYTIC APPROACH

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Some classes of dissipative and Hamiltonian distributed systems are described. The dynamics of these systems is effectively reduced to finite-dimensional dynamics which can be "unboundedly complex" in a sense. Yarying the parameters of these systems, we can obtain an arbitrary (to within the orbital topological equivalence) structurally stable attractor in the dissipative case and an arbitrary polynomial weakly integrable Hamiltonian in the conservative case. As examples, we consider Hopfield neural networks and some reaction–diffusion systems in the dissipative case and a nonlinear string in the Hamiltonian case.

1. Introduction

We consider the problem of chaotic behavior for dissipative and Hamiltonian systems described by systems of coupled oscillators and systems of nonlinear partial differential equations. Our objective is to describe some physical and mechanical models in which the appearance of various types of structurally stable periodic and chaotic behavior and the formation of space–time structures are observed. We consider the dependence of this behavior on the system parameters. The suggested approach is purely analytic, and the related investigation results cannot be obtained using computers, even in principle.

We begin by presenting the physical idea of our approach and then consider examples of fundamental dissipative systems with chaotic behavior constructed in [1], [2]. We note that these results are analytic and do not use computer calculations. The main new results are given in the two concluding sections of the paper. They describe the example of a Hamiltonian system with versatile, complex behavior. We hope that some other simpler and more fundamental examples will also be considered in further publications. Our main hypothesis is that infinite-dimensional Hamiltonian systems that are close to completely integrable ones must, in a sense, manifest "unboundedly complex" finite-dimensional Hamiltonian behavior depending on their parameters (by analogy with the dissipative systems found in [1], [2] that can have "unboundedly complex" dissipative behavior). This behavior results from the interaction between localized modes. We note that the abovementioned notions of "complexity," "unbounded complexity," and chaos, of course, require rigorous mathematical statements, which are given below.

2. An analytic approach to the problem of chaos for distributed dissipative systems

Dissipative dynamic systems such as reaction–diffusion systems and systems of coupled oscillators describe many interesting and important effects in physics, chemistry, and biology (see, e.g., [3]–[5]), but no explicit analytic description, as a rule, is known for the solutions of these systems. There are some results showing that under certain conditions, a so-called global attractor exists and its fractal or Hausdorff dimension is finite.

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For these basic notions, we recall some definitions that are used in what follows [6]–[9]. A dynamic system in a Banach space H is a semigroup S^t of operators acting from H into H. Many equations in mathematical physics define semigroups for an appropriately chosen H [6], [7]. A dynamic system is said to be dissipative if there is a bounded set $B \subset H$ such that any trajectory $S^t u$, $u \in H$, eventually enters B and stays there, i.e., $S^t u \in B$ for $t > T_0(B, u)$. Such a set is said to be absorbing. A set A is said to be attracting with the attraction basin $\mathcal{B}(A)$ if for any neighborhood $V(A)$, the image $S^t B_0$ of any bounded set lying in $\mathcal{B}$, $B_0 \subset \mathcal{B}$, belongs to this neighborhood at sufficiently large times, i.e., $S^t B_0 \subset V(A)$ for $t > T_0(B_0, V(A))$. A set is said to be globally attractive if its attraction basin is so extensive that it coincides with the entire space H. A set I is said to be invariant if the semiflow S^t does not change it, i.e., $S^t I = I$, $t \geq 0$. There are various possible definitions of an attractor. It seems that up to now, there is no fully satisfactory definition from both the physical and the mathematical standpoints [10]. We use the definition of the so-called global attractor that is most widespread in mathematical physics: a set $A \subset H$ is the global attractor of a semigroup S^t in H if it is the smallest closed invariant globally attracting set. In a sense, an attractor thus defines the system behavior at large times t. The existence of an attractor can be proved for a wide class of dissipative systems. Because of the invariance, the semigroup S^t can be restricted to the attractor A to obtain the reduced dynamics $S^t|_A$. Unfortunately, an attractor is usually a nonsmooth set and often not even a finite union of smooth submanifolds. Therefore, generally speaking, the dynamics S^t restricted to the attractor cannot be described by a system of differential equations.

In some cases, it is possible to find a global attracting invariant set $M \subset H$ that is a smooth manifold. It received the name *inertial surface* [9]. This surface is not the minimal attracting set, although both an attractor and even an arbitrary invariant set must lie in M. At the same time, the reduced dynamics $S^t|_M$ restricted to M can be found using a system of differential equations. This reduced dynamics received the name *inertial dynamics*. We note the physical interpretation of this dynamics. Even in an infinite-dimensional system, if it is dissipative, there can exist finitely many "hidden modes" determining the system behavior as $t \to \infty$. In most cases, of course, it is impossible to find these variables explicitly. In this paper, we give some examples where this can be done.

The notion of an invariant surface makes sense for Hamiltonian systems as well. In particular, the n-soliton solutions of the Korteweg–de Vries (KdV) equation is an example of an inertial surface. The differential equations describing the evolution of soliton parameters give an example of reduced dynamics.

The structure of the global attractor or of the inertial dynamics can be investigated only under some essential constraints on the dynamic system in question. The attractor and the trajectory behavior at large times were described for systems with a Lyapunov function (gradient-like systems) and for the so-called strictly monotonic systems [6], [9], [11], [12].

We present two examples important in what follows. The first is the famous basic model by Hopfield [13] in the theory of neural networks,

$$\frac{dx_i}{dt} = \sum_{j=1}^{N} K_{ij} \tanh x_j - x_i + \theta_i, \tag{2.1}$$

where x_i is the state of the ith neuron, N is the number of neurons, K_{ij} is the connection matrix describing the action of the jth neuron on the ith neuron, and θ_i are parameters called "thresholds." The set of the parameters K, N, and θ is denoted by $\mathcal{P}$. If K is a symmetric matrix, then the system is gradient-like, i.e., there is a function $L(x)$ depending on the system state $x = (x_1, \ldots, x_N)$ and not increasing on trajectories (2.1). We have $dL/dt \leq 0$, where the equality occurs if and only if x is an equilibrium point x_{eq} (the right-hand side of (2.1) vanishes for all i at $x = x_{\text{eq}}$). Attractor (2.1) can be described in the case of the "general position." All trajectories $x(t)$ tend to the equilibriums, $x(t) \to x_{\text{eq}}$ as $t \to \infty$. Generally, the number of equilibriums increases as the number N of neurons increases. We note that the function L

has the physical meaning of system energy and that the most significant contribution to L has the form $\sum_{ij} K_{ij} \tanh(x_i) \tanh(x_j)$.

We recall the meaning of the expression "in the case of the general position." For every N, the set of parameters K and θ for which the above description of the behavior of trajectories (2.1) may be incorrect is of measure zero in the parameter space. This can be proved using the Sard theorem.

The second example is the dynamic systems generated by initial boundary problems for second-order parabolic equations, e.g.,

$$u_t = \epsilon^2 \Delta u + f(x, u, \nabla u),$$

$$u(x, 0) = u^0(x), \quad x \in \Omega, \tag{2.2}$$

$$\frac{\partial u}{\partial t}(x, t) = 0, \quad x \in \partial\Omega,$$

where $f \in C^2$. Such a problem generates a dynamic system in the Banach space $L_p(\Omega)$, where the choice of p depends on f and n. If the solutions are a priori bounded in some weak norm, then the classical results show [6] that the system is dissipative and has a finite-dimensional global attractor. If f does not depend on ∇u or if the spatial dimension n of the problem is one, then system (2.2) is gradient-like, and all that has been said about the Hopfield system consequently relates to it (i.e., all trajectories are convergent in the case of the general position [7], [12].) If f depends on ∇u and $n > 1$, then the system belongs to the class of strictly monotonic systems, and it can be asserted that *almost* all trajectories are convergent, i.e., they converge to the equilibriums $u_{\text{eq}}(x)$. The number of equilibriums and the "complexity" (dimension) of the attractor increase as ϵ decreases. In this case, chaotic regimes are possible, which, however, are realizable for equations of a very special form and some special initial data (see [11], [12] for details).

As can be seen from these examples, to obtain "complex" behavior that is stable with respect to small perturbations of the system, it is necessary to consider neural networks (2.1) with nonsymmetric matrices K and at least two-component systems of equations of type (2.2). We note that the matrices K for real biological networks are nonsymmetric and that periodic and chaotic oscillations of $x_i(t)$ can actually be observed in them [14]. A method that permits constructing examples of dissipative systems with complex behavior essentially uses the ideas of structural stability (robustness) and is called the *method of realization of vector fields* [11].

We consider an abstract evolution equation in a Hilbert (Banach) space H,

$$\frac{dw}{dt} = Aw + F(w, \mathcal{P}), \quad w \in H, \tag{2.3}$$

where A is a negative-definite self-adjoint (possibly unbounded) operator, the nonlinear operator F satisfies some conditions ensuring the existence of a semigroup S^t, and $\mathcal{P}$ are the parameters on which F depends. Regarding the existence of parameters, we can say that Eq. (2.3) defines a family of semigroups $S_{\mathcal{P}}^t$. For any of the parameters, we assume that it can be physical, i.e., it can belong to another Banach space B_p as well.

Along with (2.3), we consider the system of differential equations

$$\frac{dp}{dt} = Q(p), \quad Q \in C^1(\mathcal{B}^n), \tag{2.4}$$

where $\mathcal{B}^n$ is a unit ball in R^n. If the vector field Q is directed inside the sphere $\partial\mathcal{B}^n$ for $|p| = 1$, then Eq. (2.4) defines a dynamic system in the ball.

Following [2], [11], and [12], we state the following definitions.

Definition 1. We say that prescribed dynamics (2.4) can be *almost realized* by system (2.3) if for any positive number δ, there are parameters $\mathcal{P} = \mathcal{P}(n, Q, \delta)$ such that

1. there is an inertial surface $\mathcal{M}_\mathcal{P} \subset H$ of the form

$$w = W_\mathcal{P}(p),\tag{2.5}$$

where $p \to W_\mathcal{P}(p)$ is a C^1 map from the ball $\mathcal{B}^n$ into H, and
2. the reduced dynamics $S_\mathcal{P}^t|_{\mathcal{M}_\mathcal{P}}$ on $\mathcal{M}_\mathcal{P}$ is determined by the equation

$$\frac{dp}{dt} = \widetilde{Q}(p, \mathcal{P}), \quad \widetilde{Q} \in C^1(\mathcal{B}^n),\tag{2.6}$$

and the inequality

$$|Q - \widetilde{Q}|_{C^1(\mathcal{B}^n)} < \delta\tag{2.7}$$

holds.

Definition 2. We say that a family of semigroups in (2.3) depending on a parameter *realizes everything* and has *maximal dynamic complexity* if every dynamics (2.4) for each dimension n can be almost realized by this family of semigroups. In other words, dynamics (2.6) on $\mathcal{M}_\mathcal{P}$ can be determined with an arbitrary accuracy using the parameters $\mathcal{P}$.

We now mention some consequences. If system (2.3) realizes everything, then it can be shown that its dynamics is, in a sense, unboundedly complex. Namely, to within a homomorphism, all classes of topologically equivalent structurally stable attractors can be realized by these systems. We recall that structural stability of system (2.4) means that (for sufficiently small δ) a δ-small perturbation of the right-hand side of system (2.4) in the C^1 norm does not change the topological structure of the system trajectories [10]. This means that the attractor A_δ of perturbed system (2.4) is topologically equivalent to the original attractor A in system (2.4). Consequently, attractor (2.3) lies on the inertial surface, and the dynamics on (2.3) is topologically equivalent to dynamics (2.4) in A.

Hence, systems that realize everything generate all topological classes of structurally stable types of dynamics. In this sense, we can say that these systems have maximal dynamic complexity. Moreover, this dynamics is controllable using the system parameters. We note that, to some extent, the suggested approach permits avoiding the difficulties related to the existence of very many different definitions of chaos in the literature. Irrespective of the chosen definition, if chaos is structurally stable (and, as is known, this is quite possible [10]), then the type of chaos under consideration arises in the system in question.

3. The Hopfield neural network as a system with dynamic behavior of maximal complexity

It turns out that Hopfield model (2.1) is the simplest example of a system that realizes everything [2]. We first note that system (2.1) is actually dissipative. Indeed, the set $\|x\| < R$, where R depends on the parameters $\mathcal{P} = \{N, K, \theta\}$, can be taken as a dissipating set.

To prove that system (2.1) realizes everything, we fix an arbitrary n in (2.4) and show that (2.1) generates all n-dimensional types of dynamics (2.4). We substitute $K_{ij} = \sum_{s=1}^n A_{is}B_{sj}$, where A and B are new unknown matrices, A being chosen such that $A_{is} = \delta_{is}$ for $i \leq n$. This substitution permits introducing "hidden" variables q_i determining the dynamics at large times. To show this, we change the variables as $x \to (q, z)$, where $q_i = x_i$, $i = 1, 2, \ldots, n$, and $z_i = q_i - \sum_s A_{is}q_s$, $i = n+1, \ldots, N$. After simple calculations, we obtain

$$\frac{dz_i}{dt} = -z_i + \theta_i, \qquad \frac{dq}{dt} = \Phi_\mathcal{P}(q, z), \quad q = (q_1, \ldots, q_n),\tag{3.1}$$

where the subscript $\mathcal{P}$ denotes the dependence of the vector field Φ on the parameters A, B, N, and θ. The first equation in (3.1) shows that the inertial surface is specified by the relations $z_i = \theta_i$. The inertial dynamics is therefore determined by the other equation in (3.1), into which $z = \theta$ must be substituted. It can be shown [2] that the set of the n-dimensional vector fields $\Phi_{\mathcal{P}}(q, \theta)$ on the ball $\mathcal{B}^n$ is everywhere dense in the set of all C^1-smooth vector fields on $\mathcal{B}^n$. This precisely proves that the Hopfield system realizes everything (see [2] for details).

4. A system of partial differential equations that has maximal dynamic complexity

We consider the following initial-boundary problem in the two-dimensional rectangle $\Omega = [0, 1] \times [0, 1]$:

$$u_t = \Delta u + w(x)u + f_0(x, y) + f_1(x, y)v, \qquad u(x, y, 0) = u^0(x, y), \tag{4.1}$$

$$v_t = \Delta v - a^2 v + g(x, y) \tanh u, \qquad v(x, y, 0) = v^0(x, y), \quad (x, y) \in \Omega, \tag{4.2}$$

$$\frac{\partial u}{\partial t}(x, y, t) = 0, \qquad \frac{\partial v}{\partial t}(x, y, t) = 0, \quad (x, y) \in \partial\Omega, \tag{4.3}$$

where the coefficient functions f_i, g, and w belong to $C^2(\Omega)$ and $a > 0$. These coefficients are regarded as the problem parameters, $\mathcal{P} = \{f_i, g, w, a\}$.

It can be shown that problem (4.1)–(4.3) is well posed and that it defines a dissipative semigroup S^t [2] in Sobolev spaces. Systems of this type can appear in the theory of structure formation and in phase transition theory. It turns out that the dissipative system in question has an inertial surface and inertial dynamics and that it realizes everything [2].

For the further extension of the suggested approach to Hamiltonian systems, it is important to discuss the physical ideas underlying the construction of the above system (see [2] for detailed proofs). We choose the parameters of problem (4.1)–(4.3) such that the inertial dynamics is described explicitly and is reducible to the Hopfield system.

The main idea consists in introducing a small parameter ϵ and choosing a special potential $w = W(x, \epsilon)$ such that the operator $L_\epsilon u = \Delta u + Wu$ has well-localized potential wells and accordingly N almost eigenfunctions $\psi_i(x - x_i, \epsilon)$. These functions are localized in the vicinity of the points x_i in the sense that $\psi_i(x - x_i, \epsilon) < C \exp(-c\epsilon^{-1}|x - x_i|)$, and they satisfy the relation $A_\epsilon \psi_i = h_\epsilon$, where $h_\epsilon < C \exp(-c\epsilon^{-1}|x - x_i|)$. The eigenfunctions A_ϵ orthogonal to all ψ_i have eigenvalues separated from the imaginary axis by an interval of length $O(1)$.

The structure of the solution is as follows. If the coefficients g and f_i are of the order of smallness of $O(\epsilon^\kappa)$ with a finite exponent $\kappa > 0$, then

$$u = \sum_{i=1}^{m} q_i(t)\psi_i(x, \epsilon) + \tilde{u} = U(x, q(t), \epsilon) + \tilde{u} \tag{4.4}$$

for small ϵ, where $q = (q_1, q_2, \ldots, q_n)$, $\tilde{u}$ is a small correction, $\|\tilde{u}\| < c\epsilon^{\kappa_1}$, and $\kappa_1 > 0$. The function v is equal to $V + \tilde{v}$, where $V(x, y, q)$ is the solution of the boundary problem

$$\Delta V - a^2 V + g(x, y, \epsilon)\Phi(U(x, q, \epsilon)) = 0, \qquad \frac{\partial V}{\partial \nu} = 0, \tag{4.5}$$

and $\tilde{v}$ is a small correction, $\|\tilde{v}\| < c\epsilon^{\kappa_2}$, $\kappa_2 > 0$. The functions $q_i(t)$ satisfy the system of equations

$$\frac{dq_i}{dt} = \langle f_0 + f_1 v, \psi_i \rangle, \tag{4.6}$$

where $\langle f, g \rangle$ is the inner product in $L_2(\Omega)$.

It turns out that if the coefficients f_i and g are taken in the form of well-localized Gaussian peaks, then to within small corrections, system (4.6) can be reduced to Hopfield equation (2.1) [2]. This mathematical construction has a simple physical interpretation. The constructed solution describes the slow temporal evolution of the amplitudes $q_i(t)$ of localized modes. The interaction between these modes arises as follows. The term $g\Phi(u)$ specifies small localized inhomogeneities in the nonlinear medium. These inhomogeneities result in the interaction between the localized modes ψ_i via the intermediate field v. This is a strongly nonlocal interaction similar to the neuron interaction in the Hopfield network.

A remarkable property of systems (2.1) and (4.1)–(4.3) is that their parameters can be used to prescribe the topological structure of their dynamics.

5. Complex motion of localized modes for a nonlinear string

5.1. General ideas of the approach. The central question that we answer in what follows is whether there are infinite-dimensional Hamiltonian systems with complex dynamics controllable using the system parameters. We recall that many classical equations of mathematical physics can be regarded as infinite-dimensional Hamiltonian systems (e.g., the KdV equation, the nonlinear Schrödinger equation, etc., are of this type). The theory of nearly integrable systems (in both the finite- and the infinite-dimensional cases) has now been primarily developed.

Our plan is to consider weakly perturbed integrable infinite-dimensional Hamiltonian systems. In this case, it can be expected that the dynamics of localized modes, e.g., solitons or linear modes, is defined in the first approximation by some reduced Hamiltonians with finitely many degrees of freedom. Below, we present the simplest system of the type of a weakly nonlinear string in which the reduced Hamiltonian has the following form depending on the chosen coefficients and small perturbations:

$$H(p, q) = \sum_{j=1}^{m} p_j^2 + \omega^2 q_j^2 + \epsilon \widetilde{H}(q), \tag{5.1}$$

where p and q are conjugate variables, $q = (q_1, \ldots, q_m)$, and $\widetilde{H}(q)$ is a polynomial in q_i. Hamiltonian (5.1) describes the dynamics of weak interaction between localized modes on a large time interval of the order $O(\epsilon^{-1})$. The small parameter ϵ characterizes the magnitude of the nonlinear perturbation in the string.

Systems of the type (5.1) have been intensively studied in recent decades. The Kolmogorov–Arnold–Moser (KAM) theory permits describing the solution behavior for them. Here, for a finite number $m > 1$ of degrees of freedom, some nontrivial effects such as homoclinic trajectories, separatrix splitting, Arnold's diffusion, and formation of a stochastic web [15] have been studied. It is believed that these effects lead to chaos.

We prove that a wide class of perturbed Hamiltonians of form (5.1) can appear in the dynamics of a nonlinear string. Namely, depending on the string parameters, all possible polynomial perturbations $\widetilde{H}(q)$ of the form

$$\widetilde{H}(q) = \sum_{n=3}^{p} \sum_{\mathbf{i}} C_{i_1, i_2, \ldots, i_n, n} q_{i_1} q_{i_2} \cdots q_{i_n} \tag{5.2}$$

can appear here, where the multi-index $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ in the inner sum is such that $i_1 \leq i_2 \leq \cdots \leq i_n$. This means that the dynamics of Hamiltonian interaction between localized modes can be varied in a rather wide range by changing the system parameters. (The role of the parameters here is played by W_δ and $F(x, u)$; see formula (5.3).) We suggest that this property of the given class of Hamiltonian systems should be regarded as an analogue of the property of maximal complexity for the dynamics of dissipative

systems. We note that some important results on complex solution behavior for the perturbed KdV equation were already obtained in [16], but the perturbations considered there were dissipative and were determined by complicated nonlinear functionals. In contrast to [16], we consider Hamiltonian systems with polynomial nonlinearities.

5.2. The model. We consider the following model of a string with linear and nonlinear defects:

$$u_{tt} - c^2 u_{xx} + W_\delta(x)u = \epsilon F(x, u), \tag{5.3}$$

where $x \in R$ and the potential $W_\delta(x) \in C^\infty$ depends on the second small parameter δ. The construction of W_δ is given below. The terms $W_\delta u$ and F describe linear and nonlinear defects in the string, $F(x, u)$ is a polynomial in u having rapidly decreasing coefficients with respect to x and containing no linear terms, and $F(x, u) = O(u^2)$ as $u \to 0$. Equation (5.3) describes a Hamiltonian system with infinitely many degrees of freedom.

We first describe the asymptotic behavior (as $\epsilon \to 0$ for a fixed small δ) of solutions of Eq. (5.3). We note that nonlinear hyperbolic equations have been thoroughly studied [17].

5.3. Constructing a solution. Let L denote the Schrödinger operator

$$Lu = -c^2 u_{xx} + W_\delta(x)u. \tag{5.4}$$

We assume that $W_\delta - \delta^{-2}$ is a function of the Schwartz class $S(R)$. Moreover, we assume that operator (5.4) has finitely many localized "principal" eigenfunctions $\Psi_i(x) \in L_2(R)$, where $i = 1, 2, \ldots, N$, with eigenvalues $\omega_i^2 = O(1) > 0$, whereas the entire (discrete and continuous) remainder of the spectrum lies in the interval $(c\delta^{-2}, \infty)$, where $c > 0$, and c does not depend on δ for small δ. Without loss of generality, it can also be assumed that the Ψ_j are orthonormal. The functions Ψ_i decrease exponentially with respect to x as $\exp(-\kappa_i|x|)$ as $x \to \infty$, where $c^2 \kappa_i^2 = \delta^{-2} - \omega_i^2$. Such a potential W_δ with given $\omega_1, \ldots, \omega_N$ can be easily constructed as a sum of well-separated potential wells. In essence, the same idea was used in [2]. A simple example of a singular potential with similar properties arises in mechanics [18].

To construct W_δ, we first consider a potential $W(x)$ such that $W - 1 \in S(R)$ and W has a minimum at $x = 0$. This potential, which does not depend on δ, can be constructed such that the smallest eigenvalue of the corresponding operator L is zero and the entire remainder of the spectrum lies in the interval $(2\lambda_1, \infty)$, $\lambda_1 > 0$. Furthermore, W can be supplemented with a small order-$O(\delta^2)$ Schwartz-class perturbation such that the smallest eigenvalue of the perturbed operator L becomes equal to $\delta^2 \omega^2$ and the entire remainder of the spectrum belongs to the interval (λ_1, ∞). Let the perturbed potential W thus constructed be denoted by $\widetilde{W}_\omega$. We then take arbitrary points $x_1, x_2, \ldots, x_N$ and construct W_δ in the neighborhood of each of these points in the form $\delta^{-2} \widetilde{W}_{\omega_i}((x - x_i)/\delta)$. These potentials are then glued together to obtain the resulting potential, which is constant to within $O(\exp(-cd\delta^{-1}))$ for the values of x such that $\min_i |x - x_i| > d > 0$ [2]. It can be shown that this leads to an operator L with the abovementioned properties for a small δ.

We seek the solution of (5.3) for small $\epsilon \ll \delta$ in the form

$$u(x, t) = U_0(x, q(t)) + \epsilon \tilde{u}(x, t), \tag{5.5}$$

where

$$U_0(x, q) = \sum_{j=1}^{N} q_j \Psi_j(x) \tag{5.6}$$

is the contribution from the principal localized modes with amplitudes q_j and $\tilde{u}$ is a small correction such that

$$\mathbf{P}\tilde{u} = 0. \tag{5.7}$$

Here, $\mathbf{P} = \sum_{j=1}^{N} \mathbf{P}_j$ is the orthogonal projection onto the "principal" discrete spectrum, and the operators $\mathbf{P}_j$ are defined using the relations $\mathbf{P}_j u = \langle u, \Psi_j \rangle \Psi_j$. We set $\mathbf{Q} = \mathbf{I} - \mathbf{P}$. Then the operator $\mathbf{Q}$ is the projection onto the orthogonal complement of Ψ_j. Substituting (5.5) in (5.3) and applying the projections $\mathbf{P}_j$ and $\mathbf{Q}$ to the two sides of the resulting equation, we obtain the system of equations

$$\frac{d^2 q_j}{dt^2} + \omega_j^2 q_j = \big\langle F\big(x, U_0(x,q) + \tilde{u}\big), \Psi_j \big\rangle, \quad j = 1, 2, \ldots, N, \tag{5.8}$$

$$\tilde{u}_{tt} + L\tilde{u} = \mathbf{Q}F\big(x, U_0\big(x, q(t)\big) + \tilde{u}\big). \tag{5.9}$$

The first step in solving system (5.8), (5.9) is to exclude $\tilde{u}$ from (5.8) on a large time interval with length of the order ϵ^{-1}. As is shown below, this can be done quite rigorously.

To describe the solutions of (5.9), we introduce the Banach space B_s of functions $u(x)$, $x \in R$, with a bounded norm

$$|u|_s = \max\Big\{ \sup_{k \in R} (1 + |k|)^s |\hat{u}_k|, \max_l |\hat{u}_l| \Big\}, \tag{5.10}$$

where $\hat{u}_k$ and $\hat{u}_l$ are the Fourier coefficients in the expansion of $\tilde{u}$ with respect to the eigenfunctions of the operator L. Here, the index l ranges the discrete set corresponding to the discrete eigenfunctions that do not enter the set of the principal modes Ψ_j selected above. The parameter k varies throughout the real axis $(-\infty, \infty)$ and corresponds to the functions in the continuous spectrum. As the phase space for problem (5.8), (5.9), we take the space $H_{s,T}$ containing the pairs $\big(q(t), \tilde{u}(t)\big)$ of continuous functions of t, where $t \in [0, T]$, $\tilde{u}(t) \in B_s$, and $\mathbf{Q}\tilde{u} = 0$ with the norm

$$\big\|(q, u)\big\|_s = \sup_{t \in [0,T]} \big|\tilde{u}(t)\big|_s + \sup_{t \in [0,T]} \big|q(t)\big|. \tag{5.11}$$

Proposition 5.1. *For a sufficiently small T, there is a classical solution of* (5.8), (5.9) *bounded in the $C^{2,2}$ norm.*

Proof. We write (5.8), (5.9) in the form of integral equations for the Fourier coefficients. For example, $\hat{u}_k$ satisfies the equation

$$\hat{u}_k(t) = b_k \cos\big(\omega(k)t + \beta_k\big) + \epsilon \omega^{-1}(k) \int_0^t \sin\big(\omega(k)(t - t_1)\big) \widehat{\widetilde{F}}_k(t_1)\, dt_1,$$

$$\omega^2(k) = \delta^{-2} + c^2 k^2, \tag{5.12}$$

where b_k are determined by the initial data and $\widetilde{F} = \mathbf{Q}F$. It is easy to show that the space B_s with the norm $|\cdot|_s$ for $s > 5$ forms a Banach algebra with respect to multiplication and that it can be embedded in $C^2(R)$. Because B_s is an algebra, the Fourier coefficients under the above assumptions satisfy the inequalities for F

$$\big|\widehat{\widetilde{F}}_k(t_1)\big| < C_s \big(1 + |k|\big)^{-s} \big(\big|\widehat{F}_{0_k}(t_1)\big| + P\big(|\tilde{u}|_s\big)\big), \tag{5.13}$$

where $F_0 = \mathbf{Q}F\big(x, U_0(q)\big)$ and $P(z)$ is a polynomial such that $P = O(z)$ as $z \to 0$. Similar inequalities hold for $\tilde{u}_l$ (without the factor $\big(1 + |k|\big)^{-s}$). We can now take a sufficiently large R and use (5.13) to show that

for sufficiently small T, the integral operator K defined by the right-hand side in (5.12) is contracting and maps a ball of radius R in $H_{s,T}$ into itself. Consequently, the generalized solution $(q, \tilde{u}) = K(q, \tilde{u})$ exists (it satisfies an integral equation with respect to time instead of a differential equation). In fact, we have $t \to q(t) \in C^2$, and the map $t \to \tilde{u}(t) \in B_s$ is also twice differentiable with respect to time. Embedding theorems can be used to show that these time derivatives belong to C^2 for $s > 5$. The classical solution of the problem under consideration therefore exists on a sufficiently small time interval. Proposition 5.1 is proved.

5.4. Derivation of simplified Hamiltonian equations for q_j. The above assumptions about the decomposition of the spectrum of L into principal modes with the remainder of the spectrum separated from the imaginary axis play the key role in the proof of the following assertion.

Proposition 5.2. *For a sufficiently small ϵ and under the conditions $\left|\tilde{u}(\cdot, 0)\right|_s < c\epsilon$ and $\left|\tilde{u}_t(\cdot, 0)\right|_s < c\epsilon$, the inequalities*

$$\sup_x \left|\tilde{u}(x, t)\right| \leq C_s \left|\tilde{u}(\cdot, t)\right|_s < C(F, s, W_\delta)\epsilon \tag{5.14}$$

hold for $0 < t < c\epsilon^{-1}$.

Proof. The first inequality in (5.14) expresses a simple embedding theorem. We write $X(t) = \left|\tilde{u}(\cdot, t)\right|_s$. Following the procedure in the proof of Proposition (5.1), we can show that there is an open interval $I = (0, T_0)$ such that $\left|X(t)\right| \leq 1$ for $t \in I$. Consequently, the inequality

$$\left|\widehat{\tilde{F}}_k(t) - \widehat{F}_{0_k}(t))\right| < \widetilde{C}_s\left(1 + |k|\right)^{-s} X(t) \tag{5.15}$$

holds on the interval I, where $F_0 = \mathbf{Q}F\big(x, U_0\big(q(t)\big)\big)$. The main problem reduces to estimating the terms

$$S_k = \omega^{-1}(k) \int_0^t \sin\big(\omega(k)(t - t_1)\big)\widehat{F}_{0_k}(t_1)\, dt_1. \tag{5.16}$$

We first pass to the action–angle coordinates J_j and ϕ_j in Eqs. (5.8) for q_j. We set

$$q_j = J^{1/2}(t)\cos(\omega_j t + \phi_j(t)), \qquad p_j = \omega_j J^{1/2}(t)\cos(\omega_j t + \phi_j(t)). \tag{5.17}$$

Then

$$\frac{dJ_j^{1/2}}{dt} = c_j \epsilon F_j \sin(\omega_j t + \phi_j(t)), \qquad \frac{d\phi_j}{dt} = \tilde{c}_j J_j^{-1/2} \epsilon F_j \cos(\omega_j t + \phi_j(t)), \tag{5.18}$$

where $F_j = \langle F, \Psi_j \rangle$, and hence

$$|F_j| < c_2 \sup_x \left|F_0\big(x, U_0(x, q)\big)\right| + c_3|X| < c_4\Big(\sup_x \left|F_0\big(x, U_0(x, q)\big)\right| + 1\Big) \tag{5.19}$$

on the interval I. Formula (5.19) shows that $F_0(x, t)$ has the form

$$F_0(x, t) = \sum_n f_n(x) P_n(t, \tau), \qquad \tau = \epsilon t, \tag{5.20}$$

where each f_n belongs to the Schwartz class and P_n is a polynomial in $\cos \omega_j t$ and $\sin \omega_j t$ with coefficients depending on the slow time τ. We now use the above assumptions about the form of the spectrum of L to estimate the integral of F_0 with respect to t_1 in (5.16). If δ is small and

$$T_0 < c_5 \epsilon^{-1}, \tag{5.21}$$

253

then this integral is of the order $O(1)$ (as $\epsilon \to 0$). This means that

$$X(t) \leq \epsilon C_F + \epsilon C_1 \int_0^t X(t_1)\, dt_1 \tag{5.22}$$

for $t \in \left[0, \min\{T_0, c_5\epsilon^{-1}\}\right]$ and the positive function $X(t)$. Integral inequality (5.22) implies that

$$X(t) \leq \epsilon \widetilde{C}_F < 1 \tag{5.23}$$

for $t \in \left[0, \min\{T_0, c_5\epsilon^{-1}\}\right]$. Inequalities (5.23) show that $T_0 = c_5\epsilon^{-1}$ for small ϵ, which completes the proof of Proposition 5.2.

The above result permits setting $\tilde{u} = 0$ in (5.8) for $t = O(\epsilon^{-1})$. Therefore, introducing the potential $\Phi(x, u)$ by means of the relation $\Phi_u(x, u) = F(x, u)$ and taking

$$\widetilde{H}_F(q) = \int_{-\infty}^{\infty} \Phi\big(x, U_0(x, q)\big)\, dx, \tag{5.24}$$

we conclude that Eqs. (5.8) have a Hamiltonian structure with a Hamiltonian of form (5.1).

5.5. Constructing a nonlinear string with prescribed Hamiltonian dynamics.

Proposition 5.3. *If the frequencies ω_i are "generic," then whatever the polynomial perturbation $\widetilde{H}(q)$ of form (5.2), there is a polynomial perturbation F in (5.3) with respect to u such that $\widetilde{H}_F(q) = \widetilde{H}(q)$.*

Proof. The proof is based on a special choice of $\Phi = \sum_{n=0}^{p} f_n(x)u^{n+2}$. We set

$$f_n(x) = \sum_{k=1}^{M_n} Y_{kn}\delta_1 \rho\left(\frac{x - \bar{x}_{kn}}{\delta_1}\right), \tag{5.25}$$

where $\rho(z)$ is a positive smooth function supported in $(-1, 1)$ such that the integral of $\rho(z)$ is equal to 1, δ_1 is a small parameter, and Y_{kn} are the unknown coefficients to be found. Here, M_n is the number of multi-indices $\mathbf{i}$ of length n (see formula (5.2)). We select the points $\bar{x}_{kn} > 0$ such that the eigenfunction Ψ_j can be replaced with its exponential asymptotic expression. We now note that

$$C_{i_1, i_2, \ldots, i_n} = \int_{-\infty}^{\infty} f_n(x)\Psi_{i_1}(x)\Psi_{i_2}(x)\cdots\Psi_{i_n}(x)\, dx.$$

It follows that the coefficient Y_{kn} can be found for each n using the linear algebraic system of M_n equations with M_n unknowns Y_{kn}. For the chosen points $\bar{x}_{kn}$ and a small δ_1, this system of equations becomes

$$\sum_{k}^{M_n} e^{-\theta_\mathbf{i}} Y_{kn} = B_{\mathbf{i},n}, \tag{5.26}$$

where, as in (5.2), the summation extends over the M_n multi-indices of length n, the right-hand sides $B_{\mathbf{i},n}$ are proportional to the given coefficients $C_{\mathbf{i},n}$, and the exponents $\theta_\mathbf{i}$ are determined by the relations

$$c^2\theta_\mathbf{i}^2 = \sum_{k=1}^{n} \delta^{-2} - \omega_{i_k}^2. \tag{5.27}$$

According to the KAM theory, all exponents $\theta_\mathbf{i}$ are distinct for different multi-indices $\mathbf{i}$ if the frequencies ω_{i_k} are "generic." (We recall that $i_1 \leq i_2 \leq \cdots \leq i_n$.) These frequencies can be selected arbitrarily at the expense of the choice of the potential W_δ. In this case, the determinant of system (5.26) is the Vandermonde determinant and is nonzero. This means that the unknown coefficients Y_{kn} can be found, which completes the proof of Proposition 5.3.

6. Conclusion

The above results show that the interaction between localized modes can be very complex in both dissipative and conservative nonlinear distributed systems. Selecting the inhomogeneity of the medium appropriately, we can obtain any structurally stable dynamics in the case of dissipative systems and an arbitrary polynomial Hamiltonian in the conservative case. Thus, for some classes of nonlinear distributed systems, the dynamics can be unboundedly complex and, moreover, controllable using the system parameters.

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