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INFLUENCE OF A LIMITED-POWER DRIVING TORQUE ON THE STABILITY OF STEADY-STATE MOTIONS OF AN ASYMMETRIC TOP

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Steady-state motions of an asymmetric top (unbalanced rotor) with a fixed point under the action of an elastic torque and a driving torque are considered. We investigate the influence of dissipation due to a limited-power motor on the stability of the conservative system which either has a motor of infinite power or does not have a motor.

It is shown that a limited-power motor produces a strong destabilizing effect which begins to manifest itself already at small amplitudes. Both dead and follower driving torque are considered, the destabilizing effect of the dead torque being stronger. Relatively simple relations are obtained for the boundaries of stability regions on the parameter plane. Specific stability criteria for a polynomial of degree 5 are suggested.

It is shown that a nonresonant stable steady-state motion is possible for certain values of the rotor parameters. The case under consideration is compared with that of the plane-parallel motion of an unbalanced rotor.

1. INTRODUCTION

Plane-parallel motion of an unbalanced rigid rotor in nonlinear elastic bearings has been investigated by numerous authors. See the bibliography in [1]. In the present paper we consider a related problem of the spherical motion of an unbalanced rigid rotor (asymmetric top) under the action of an elastic torque and a driving torque (Fig. 1). We will compare in brief these two cases. Both of these problems deal with a system with three degrees of freedom. However, because of its complexity the problem of spherical motion needs more cumbersome calculations. Nonlinear properties of the amplitude-frequency characteristic of the steady-state motion in the case of the plane-parallel motion are accounted for only by nonlinearity of the elastic force. Unlike this, for the spherical motion, these properties are preserved at any characteristics of the elastic torque and to a large extent are determined by the inertial terms of the equations of motion. In the problem of spherical motion, the role of gyroscopic effects is more substantial. This leads to effects which are impossible for plane-parallel motion. For example, the spherical motion can be stable under an overtuning elastic torque and the resonance does not necessarily occur even if the rotor is substantially oblate.

Along with these differences, both problems have much in common. In both systems, similar steady-state motions caused by the rotor unbalance appear. Also, the elastic and dissipative forces acting in these systems are similar in many respects. It was shown in [2, 3] that in the case of plane-parallel motion, a limited-power motor substantially narrows the stability regions of steady-state motions. In the present paper, we investigate this effect for the spherical motion.

The present paper is a continuation of the paper [4], in which the problem of the spherical motion is considered for two conservative cases, specifically, the case of no driving torque (motor torque) and the case of the infinite-power driving torque. In the present paper we investigate a more general case where the power of the driving torque is limited, in this case the system being nonconservative.

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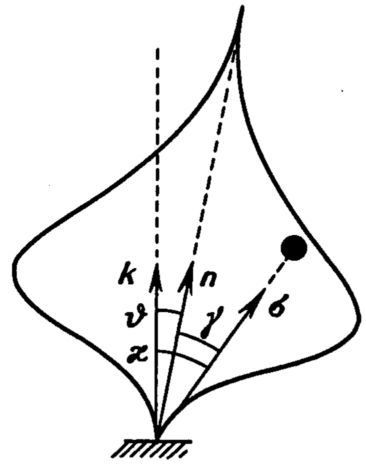

Fig. 1

2. NOTATION

The system under consideration (Fig. 1) consists of an axisymmetric rigid body with a fixed point (symmetric top) and a point mass (unbalance) rigidly attached to the body. The magnitude of this mass is not assumed to be small in the general case. Introduce the basic notation. (For more details, see [4].) The vertical, the axis of the top, and the direction toward the unbalance mass are specified by the unit vectors **k**, **n**, and $\boldsymbol{\sigma}$, respectively. The angles between these vectors are denoted by γ, ϑ and χ (Fig. 1). The decomposition of the vectors **n** and $\boldsymbol{\sigma}$ into the vertical and horizontal components is given by $\mathbf{n} = \boldsymbol{\epsilon} + \eta\mathbf{k}$, $\boldsymbol{\sigma} = \boldsymbol{\tau} + \mu\mathbf{k}$; $\mathbf{k}\cdot\boldsymbol{\epsilon} = \mathbf{k}\cdot\boldsymbol{\tau} = 0$. We denote the sines and cosines of the angles γ, ϑ, and χ as follows:

$$\begin{aligned} &\varepsilon = \sin\vartheta, \qquad \tau = \sin\chi, \qquad \alpha = \sin\gamma, \\ &\eta = \cos\vartheta, \qquad \mu = \cos\chi, \qquad \beta = \cos\gamma. \end{aligned} \tag{2.1}$$

Let φ be the angle of rotation of the top about its axis and Ω the angular velocity of proper rotation. The quantity Ω is defined as $\Omega \stackrel{\text{def}}{=} \mathbf{n}\cdot\boldsymbol{\omega}$. Note, however, that $\Omega \neq \dot{\varphi}$.

The top is acted upon by the external elastic torque $\mathbf{M_e} = C\mathbf{n}\times\mathbf{k}$. The stiffness C can be either positive (restoring torque) or negative (overturning torque). The top is also acted upon by the motor torque (driving torque) $\mathbf{M} = M_1\mathbf{k} + M_2\mathbf{n}$. We will consider two cases, $M_1 = M(\dot{\varphi})$, $M_2 = 0$ and $M_1 = 0$, $M_2 = M(\dot{\varphi})$. The torque in the former and latter cases will be referred to as the dead torque and follower torque, respectively.

Denote by θ_{12}, θ_3 ($\theta_3 < 2\theta_{12}$), and θ the equatorial, axial, and gyroscopic moments of inertia of the top, respectively. The quantity θ is defined as $\theta \stackrel{\text{def}}{=} \theta_{12} - \theta_3$. Let θ_* be the moment of inertia of the unbalance mass with respect to the fixed point. We assume the unbalance to be small, i.e., $\theta_* \ll \theta$. (For brevity, we will write $x \ll y$ for $|x| \ll |y|$.) We consider the quantities γ, $\pi/2-\gamma$, θ_3/θ, and θ_{12}/θ to be nonsmall.

We will distinguish between the general nonlinear case and that of small (not necessarily linear) oscillations. The latter case is characterized by two independent small parameters, the relative unbalance, θ_*/θ, and the amplitude of oscillations, ε. The small oscillations can have very small amplitude, $\varepsilon \ll \theta_*/\theta$, small amplitude, $\varepsilon \sim \theta_*/\theta$, and large amplitude, $\varepsilon \gg \theta_*/\theta$.

3. STEADY-STATE MOTION

The equations for steady-state motions representing permanent rotations about the vertical axis were obtained in [4]. The basic steady-state motion has the form of forced oscillations. For this motion, the vectors **k**, **n**, and $\boldsymbol{\sigma}$ lie in one plane and the angles are related by

$$\chi = \gamma + \vartheta, \tag{3.1}$$

where $|\vartheta| < \gamma < \pi/2$ and $0 < \chi < \pi/2$. The angle ϑ is positive ($\vartheta > 0$) if the vectors **n** and $\boldsymbol{\sigma}$ lie on one side of **k**, otherwise $\vartheta < 0$.

The amplitude-frequency characteristic of the basic steady-state motion is represented by the algebraic equation

$$\theta\varepsilon\eta + \theta_*\tau\mu = \frac{C}{\omega^2}\varepsilon, \tag{3.2}$$

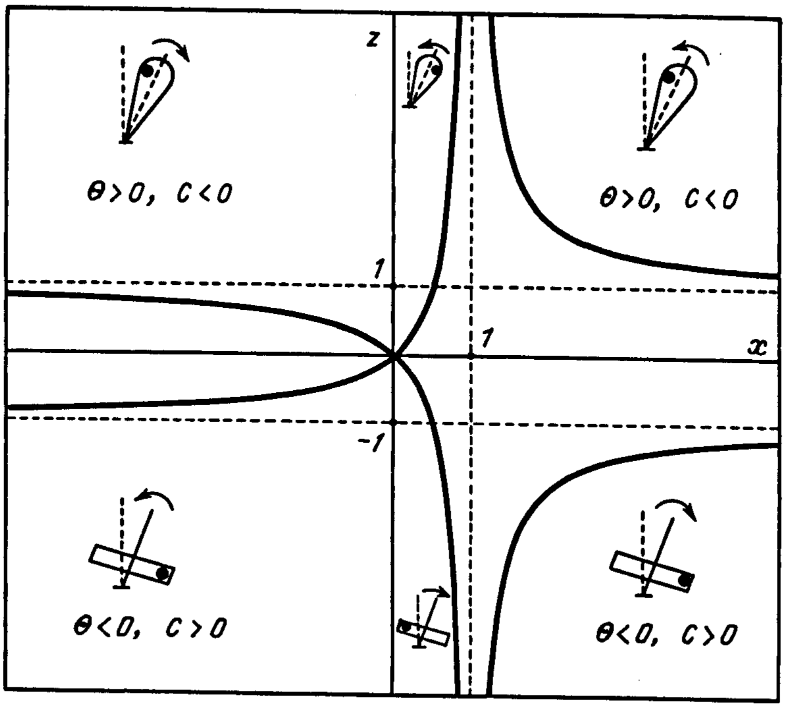

Fig. 2

where ω is the angular velocity (frequency) of precession. Also, this steady-state motion satisfies the relations

$$\dot{\mathbf{n}} = \omega\mathbf{k}\times\mathbf{n}, \quad \dot{\sigma} = \omega\mathbf{k}\times\sigma, \quad \dot{\eta} = 0, \quad \dot{\varphi} = \omega, \quad \Omega = \omega\eta, \quad \dot{\omega} = 0.$$

In what follows, we will be interested mostly in the case of small amplitudes, $\varepsilon \sim \theta_*/\theta \ll 1$ ($\alpha, \beta \sim 1$). For this reason, we will consider the case of $\varepsilon \ll 1$ in more detail. Equation (3.2) in the first approximation becomes

$$\theta\varepsilon + \theta_*\alpha\beta = \frac{C}{\omega^2}\varepsilon \iff \varepsilon = \frac{\theta_*\alpha\beta}{\theta}\frac{\omega^2}{C/\theta - \omega^2}. \tag{3.3}$$

If $C\theta > 0$, then the expression of (3.3) for ε has the form of the amplitude-frequency characteristic of forced oscillations in a single-degree-of-freedom linear system. In this case the system has a resonance at the frequency $\omega_r \overset{\text{def}}{=} \sqrt{C/\theta}$. If $C\theta < 0$, then resonance does not occur. It follows from (3.3) that at large frequencies (as $\omega \to \infty$) we have $\varepsilon = -p$, where

$$p \overset{\text{def}}{=} \frac{\theta_*\alpha\beta}{\theta}. \tag{3.4}$$

Hence, the small oscillations have a residual amplitude as $\omega \to \infty$.

For graphic representation of the amplitude-frequency characteristic, it is convenient to introduce the dimensionless parameters

$$z \overset{\text{def}}{=} \frac{|\varepsilon|}{p}, \quad x \overset{\text{def}}{=} \frac{\omega^2}{\omega_r^2} = \frac{\theta}{C}\omega^2. \tag{3.5}$$

With these parameters, Eq. (3.3) becomes $z = \pm x/(1-x)$, $\pm 1 = \operatorname{sign}\varepsilon$.

The amplitude-frequency characteristic is plotted in Fig. 2. Different quadrants on the xz plane correspond to different types of the system characterized by the shape of the top (prolate or oblate) and the nature of the elastic torque (restoring or overturning). To make the figure more graphic, we depicted in each quadrant the corresponding outlines of the top characterizing these differences, as well as the position of the unbalance mass with respect to the symmetry axis.

It is usual for real mechanisms (for example, centrifuges) that $\theta > 0$ and $C > 0$. This case corresponds to the first quadrant of Fig. 2; the system has a resonance. It is apparent from the graph of Fig. 2 that there are two possibilities for completely avoiding resonance by adjusting the system parameters. First, by making the rotor reasonably flat, one can ensure that the gyroscopic moment of inertia is negative. This corresponds to the third quadrant ($\theta > 0$, $C < 0$).

The second possibility, although less realistic, can be implemented by the instantaneous (jump-like) change in the sign of C at some reasonably high value of the angular velocity ω. These methods of suppression of resonance are rather interesting but they require thorough investigations, primarily, the stability analysis of the corresponding steady-state motions.

4. GENERAL CRITERIA OF STABILITY

The characteristic determinant $\Delta(\lambda)$ of the equation of the perturbed motion in the neighborhood of the steady-state motion of the system was obtained in [4]. This determinant has the form

$$\Delta(\lambda) = \begin{vmatrix} (\theta+\theta_*+\theta_3)\lambda^2+\theta\varepsilon^2+\theta_*\left(\frac{\alpha\mu}{\varepsilon}+\tau^2\right) & -\left(2\theta\eta+2\theta_*\mu\frac{\tau-\alpha}{\varepsilon}+\theta_3\right)\lambda & (-2\theta_*\alpha\mu+\theta_3\varepsilon)\lambda \\ (2\theta\eta^2+2\theta_*\mu^2+\theta_3)\lambda & \left(\theta\eta+\theta_*\mu\frac{\tau-\alpha}{\varepsilon}+\theta_3\right)\lambda^2+\theta_*\frac{\alpha\mu}{\varepsilon} & (\theta_*\alpha\mu-\theta_3\varepsilon)\lambda^2+\frac{M_2'}{\omega}\varepsilon\lambda-\theta_*\alpha\mu \\ 2(\theta\varepsilon\eta+\theta_*\tau\mu)\lambda & \left(\theta\varepsilon+\theta_*\tau\frac{\tau-\alpha}{\varepsilon}+\theta_3\frac{1-\eta}{\varepsilon}\right)\lambda^2 & (\theta_*\alpha\tau+\theta_3\eta)\lambda^2-\left(\frac{M_1'}{\omega}+\frac{M_2'}{\omega}\eta\right)\lambda \end{vmatrix}.$$

We can factor out $\theta_3\lambda$ to represent this determinant in the form

$$f(\lambda) \overset{\text{def}}{=} \frac{1}{\theta_3\lambda}\Delta(\lambda) = \begin{vmatrix} A_{11}\lambda^2+C_{11} & B_{12}\lambda & B_{13}\lambda \\ B_{21}\lambda & A_{22}\lambda^2+C_{22} & A_{23}\lambda^2+B_{23}\lambda+C_{23} \\ B_{31} & A_{32}\lambda & A_{33}\lambda+B_{33} \end{vmatrix}. \tag{4.1}$$

The possibility of factoring out the multiplier λ indicates that the system is indifferent to the rotation about **k**. As follows from the expression for $\Delta(\lambda)$, the only entries depending on the parameter $L \overset{\text{def}}{=} -\frac{d}{d\omega}M(\omega)$ characterizing the power of the motor are B_{23} and B_{33}. Hence, the characteristic polynomial $f(\lambda)$ can be represented as

$$f(\lambda) = \lambda a(\lambda^2) + \varkappa b(\lambda^2), \tag{4.2}$$

where $\varkappa$ is a coefficient proportional to L and $a(\lambda^2)$ and $b(\lambda^2)$ are biquadratic polynomials of λ independent of L. This implies that one should consider three cases of the problem in accordance with the value of the power parameter, specifically, $L=\infty$, $L=0$, and $0<L<\infty$. The first two cases, which correspond to the infinite-power motor and unpowered system, respectively, lead to conservative problems. In accordance with (4.2), these two cases correspond to the biquadratic characteristic polynomials

$$b(\lambda^2) = b_2\lambda^4 + b_1\lambda^2 + b_0, \qquad a(\lambda^2) = a_2\lambda^4 + a_1\lambda^2 + a_0.$$

In the third (general) case, the system is nonconservative and corresponds to the characteristic polynomial $f(\lambda)$ of degree 5, which is a linear combination of the polynomials $a(\lambda^2)$ and $b(\lambda^2)$. Thus, the general case is, in a sense, a superposition of two conservative problems.

For conservative systems, the first-approximation stability (i.e., the stability judged on the basis of the linearized equations of the perturbed motion) is not sufficient in general for the Lyapunov stability of the original nonlinear system. This issue was discussed in detail in [4] and we choose not to consider it in the present paper. In what follows, we will understand by stability the first-approximation stability.

For the conservative cases, the stability conditions have the form

$$\begin{aligned} L&=\infty, \quad b_2>0: \quad b_1>0, \quad b_0>0, \quad D_b>0, \\ L&=0, \quad a_2>0: \quad a_1>0, \quad a_0>0, \quad D_a>0, \end{aligned} \tag{4.3}$$

where D_b and D_a are the discriminants of the corresponding quadratic polynomials,

$$D_b \overset{\text{def}}{=} b_1^2 - 4b_0b_2, \quad D_a \overset{\text{def}}{=} a_1^2 - 4a_0a_2.$$

For the nonconservative case, one can use the familiar Routh–Hurwitz stability criterion. However, in this case, the Liénard–Chipart criterion [6] is more convenient, since it permits one to replace complicated inequalities for minors of the Hurwitz matrix by inequalities for the coefficients of the characteristic polynomial. Using the Liénard–Chipart criterion, the author of the present paper derived the stability criterion for polynomials of degree 5. Introduce the coefficients

$$D_{kn} \stackrel{\text{def}}{=} b_k a_n - b_n a_k \quad (k, n = 2, 1, 0); \qquad D \stackrel{\text{def}}{=} D_{21} D_{10} - D_{20}^2. \tag{4.4}$$

Then the stability criterion for the polynomial of (4.2), with $\varkappa > 0$ and $a_2 > 0$, can be represented in the form

$$b_2 > 0, \quad b_1 > 0, \quad b_0 > 0; \qquad D_{kn} > 0 \quad (k > n); \qquad D > 0. \tag{4.5}$$

In this criterion, it suffices to use any of the three conditions $D_{kn} > 0$ $(k > n)$. This permits one to choose a simpler condition. For the problem in question, this simpler condition is $D_{20} > 0$.

Also, one can show that the conditions $a_k > 0$, $D_b > 0$, and $D_a > 0$ are necessary for the polynomial of (4.2) to be stable. This statement leads to the important conclusion that the conditions of (4.5) imply those of (4.3) and, hence, the stability regions for the conservative cases are not narrower than those for the nonconservative case. Thus, a limited-power motor either does not change the stability region of the conservative system or narrows it, thereby producing a destabilizing effect.

If the system contains small parameters, the values of the coefficients a_k and b_k are frequently close to each other. In this case, it is more convenient to express the coefficients D_{kn} as follows:

$$D_{kn} = b_k \alpha_n - b_n \alpha_k, \quad \alpha_k \stackrel{\text{def}}{=} a_k - b_k. \tag{4.6}$$

The closeness of the quantities a_k and b_k leads to similarity of many of the results for conservative systems with $L = \infty$ and $L = 0$.

5. DERIVATION OF SPECIFIC STABILITY CONDITIONS

The task of the investigation of the characteristic determinant in the general case is much more complicated than that of obtaining this determinant. Indeed, the representation of the coefficient D in terms of the coefficients A_{kn}, B_{kn}, and C_{kn} leads to an expression containing more than 60 ninth-order terms. The major problem which hardly could be solved in the general case is that of graphic representation of the results. For this reason, in what follows, we will consider the particular case of small oscillations where $\varepsilon \ll 1$ and $\theta_*/\theta \ll 1$. This case is important for applications.

Introduce the dimensionless parameters

$$q = \frac{\theta}{\theta_*}, \quad \delta = \frac{\theta_* \alpha\beta}{\theta}, \quad g = \frac{C}{\theta_3 \omega^2}, \quad l_1 = -\frac{M_1'}{\theta_3 \omega}, \quad l_2 = -\frac{M_2'}{\theta_3 \omega}. \tag{5.1}$$

Then the first approximation for the determinant $f(\lambda)$ with respect to the independent small parameters ε and δ has the form

$$f_0(\lambda) = \begin{vmatrix} (q+1)\lambda^2 + \delta/\varepsilon + q\varepsilon^2 & -(2q+1)\lambda & (\varepsilon - 2\delta)\lambda \\ (2q+1)\lambda & (q+1)\lambda^2 + \delta/\varepsilon & (-\varepsilon + \delta)\lambda^2 - l_2\varepsilon\lambda - \delta \\ 2(q\varepsilon + \delta) & \left[(q + \frac{1}{2})\varepsilon + \delta\right]\lambda & \lambda + (l_1 + l_2) \end{vmatrix}. \tag{5.2}$$

One can show that to obtain the stability conditions it suffices to use the first approximation for the characteristic determinant. When calculating the coefficients of (4.4) one can control the asymptotic order of the leading terms in order that these terms do not cancel each other. Being substantially simpler than $f(\lambda)$, the determinant $f_0(\lambda)$ is also remarkable for that it contains only three essential parameters, q, δ and ε.

As has been mentioned above, there are three essentially different possibilities: very small amplitudes ($\varepsilon \ll \delta$), small amplitudes ($\varepsilon \sim \delta$), and large amplitudes ($\varepsilon \gg \delta$).

The very small amplitudes appear at low frequencies, at the beginning of the amplitude-frequency characteristic. see Fig. 2 in the vicinity of the point ($x = 0$, $z = 0$). Despite the apparent simplicity of this case, it turns out to be extremely complex, since the leading terms of some coefficients defining the stability are canceled. A detailed analysis involving computer algebra techniques shows that, nevertheless, one can confine oneself to the approximation $f_0(\lambda)$ for the characteristic polynomial. Moreover, the stability conditions in this case are rather simple: the system is stable for $C > 0$ (restoring torque) and unstable for $C < 0$ (overturning torque).

As has already been mentioned, the case of small amplitudes is analogous to the linear treatment of the problem and covers virtually all frequency range, except for very low frequencies and resonance frequencies (if a resonance is observed in the system). The stability conditions in this case are rather sophisticated. They will be the subject matter of the next section. A remarkable feature of the small amplitudes is that the stability conditions in this case depend only on two parameters. This permits a graphic representation of the results on the plane of these parameters.

Large amplitudes occur at near-resonant frequencies. Since the resonance occurs only if $C\theta > 0$, it is possible either for a prolate top with a restoring torque or for an oblate top with an overturning torque. In this case, the stability conditions remain three-parameter. However, they are substantially simpler than the stability conditions for the case of small amplitudes, and a graphic presentation of the results does not lead to difficulties. The case of large amplitudes was thoroughly analyzed in [4] for the conservative cases. If the motor has a limited power, then some of the stability regions disappear.

6. SMALL AMPLITUDES

Since small amplitudes are characterized by the relation $\varepsilon \sim \delta$, we have in fact only one small parameter. Denote $d \overset{\text{def}}{=} \delta/\varepsilon \sim 1$.

For the case of small amplitudes, from (5.2) we obtain $a_k \sim b_k \sim 1$, $\alpha_k = a_k - b_k \sim \varepsilon^2$. Hence, the stability conditions for both conservative cases coincide. In the first approximation, the coefficients defining the stability in the conservative cases have the form

$$\begin{aligned} &b_2 \approx (q+1)^2, \quad b_1 \approx 2(q+1) + (2q+1)^2, \quad b_0 \approx d^2, \\ &D_b \approx (2q+1)^2[4(q+1)d + (2q+1)^2]. \end{aligned} \tag{6.1}$$

These relations can be obtained from (5.2) by relatively simple manipulations. The relations of (6.1) are the same for the dead and follower torques and, hence, the stability conditions do not depend on the direction of the motor torque. It is apparent from (6.1) that the inequality $b_1 > 0$ follows from the inequality $D_b > 0$ and, hence, the stability conditions for the conservative cases are reduced to the inequality $D_b > 0$. Using the original notation, one can represent this condition in the simple form

$$\omega^2 > -\frac{4\theta_{12}C}{\theta_3^2}, \tag{6.2}$$

coinciding with the familiar stability condition for the vertical rotation of a symmetric top.

We will proceed now to the nonconservative case. To this end, we have to calculate the coefficients D_{kn} and D. These calculations are rather cumbersome. We used the computer algebra system Reduce (e.g., see [7]) to carry them out.

Consider first the case of the dead motor torque. For this case we have

$$\begin{aligned} &D_{20} \approx -\tfrac{1}{2}gd(g-d+1)[(6g-1)d - 8g(g+1)]\varepsilon^2, \\ &D \approx \tfrac{1}{4}g^2d(d-2)(g-d+1)(2g-2d+1)^2[(2g-1)^2d + 4g]\varepsilon^4. \end{aligned} \tag{6.3}$$

Here, the coefficient D_{20} was chosen among the coefficients D_{kn} for its simplicity. In the relations of (6.3), the parameter q is replaced by g in accordance with the equation of the steady-state motion (Eq. (3.3)), which in the dimensional variables has the form $q + d = g$. The computer algebra system enabled us to factorize these coefficients into relatively simple multipliers. Without a computer, this would hardly have been possible. Moreover, using a computer allowed us to choose among numerous possibilities the parameters g and d that provide the simplest expressions for the coefficients and, what is the most important, linear inequalities with respect of one of these parameters (d). The last fact makes it possible to solve the inequalities expressing the stability conditions.

We choose to present the final results without dwelling on the solution of these inequalities. To facilitate the interpretation of these results in terms of physics, it is convenient to use the parameters $x = \theta\omega^2/C$ and $y = \theta/\theta_{12}$. The parameter x was introduced above; it is equal to the square of the dimensionless frequency of oscillations. The parameter y reflects the inertial properties of the top; it ranges in the interval $(-1, 1)$.

The stability regions on the xy plane for the dead motor torque are shaded in Fig. 3. As was the case previously, different quadrants of the xy plane correspond to different types of the system. These types differ in the shape of the top and the nature of the elastic torque, which is shown in small pictures of Fig. 3. One of the stability regions consists of two adjacent subregions and is bounded by the line $x = 0$ and the curve $x = y/(2-y)$ labeled by the number *1*. This region corresponds to the restoring elastic torque and relatively small angular velocities. In terms of the original parameters, this region is defined by the simple inequality

$$\omega^2 < \frac{C}{\theta_{12} + \theta_3} \quad \Longleftrightarrow \quad \left(\frac{\omega}{\omega_r}\right)^2 < \frac{\theta_{12} - \theta_3}{\theta_{12} + \theta_3}. \tag{6.4}$$

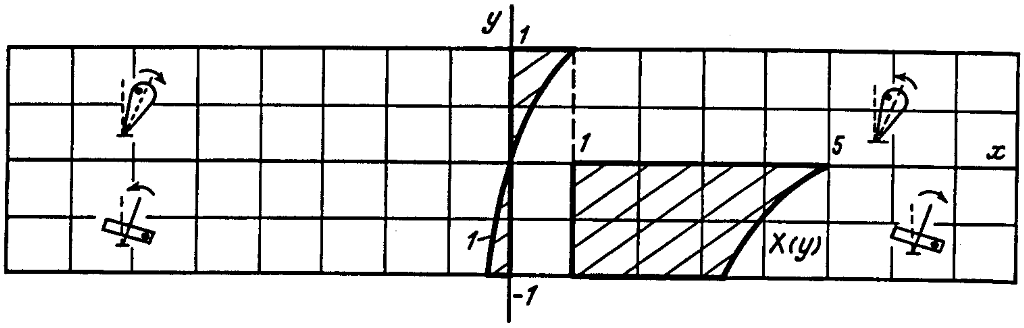

Fig. 3

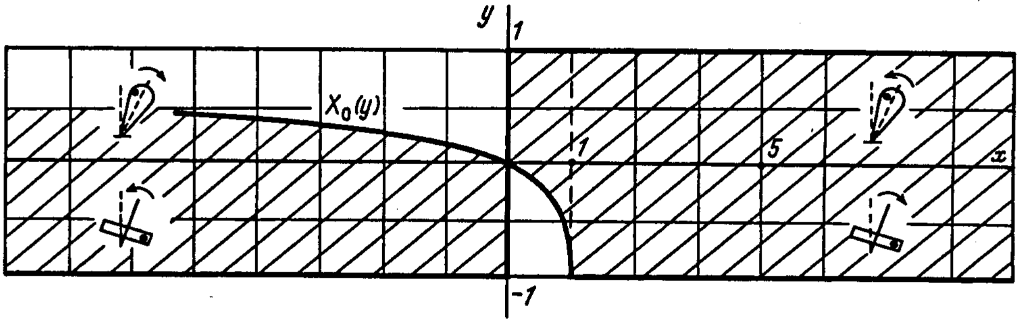

Fig. 4

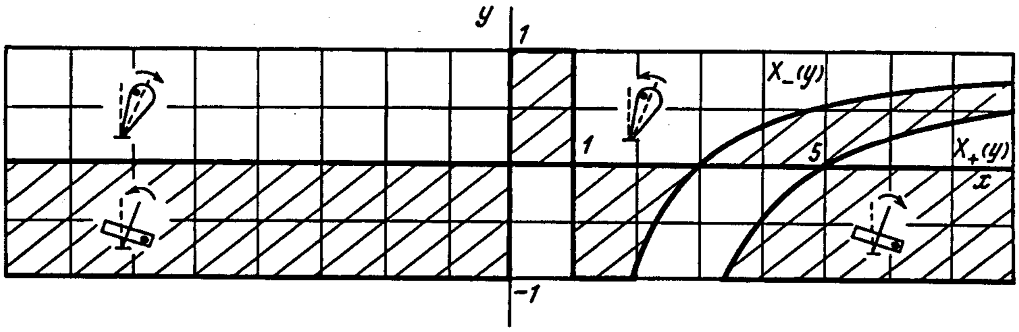

Fig. 5

The other stability region is implemented in the superresonant frequency range for an oblate top with an overturning elastic torque. This region is defined by the inequalities

$$y < 0, \quad 1 < x < X_-(y) \approx 5 + \frac{3y}{1-y} = \frac{3\theta_{12} + 2\theta_3}{\theta_3}. \tag{6.5}$$

The exact definition of the bounding function $X_-(y)$ will be given below. The error of the approximation to this function used in (6.5) does not exceed 3.5%.

Let us compare the stability regions of Fig. 3 with the stability regions for the conservative case. The latter regions on the xy plane are shown (shaded) in Fig. 4. The bounding curve $X_0(y)$ is defined by $X_0(y) = -4y(1-y)^{-2}$. As stated above, a limited-power motor produces a destabilizing effect and, as is apparent from Figs. 3 and 4, destroys the major portion of the stability region for the conservative system.

The situation is somewhat better for the case of the follower motor torque. See Fig. 5. If $\theta < 0$ and $C > 0$, the steady-state motion is stable for all frequencies. If $\theta > 0$ and $C > 0$, the steady-state motion is unstable for all frequencies. For the cases of $\theta > 0$, $C > 0$ and $\theta < 0$, $C < 0$, the steady-state motion is stable in the subresonant and superresonant ranges, respectively. In the last two cases, there exists a relatively narrow strip with the opposite

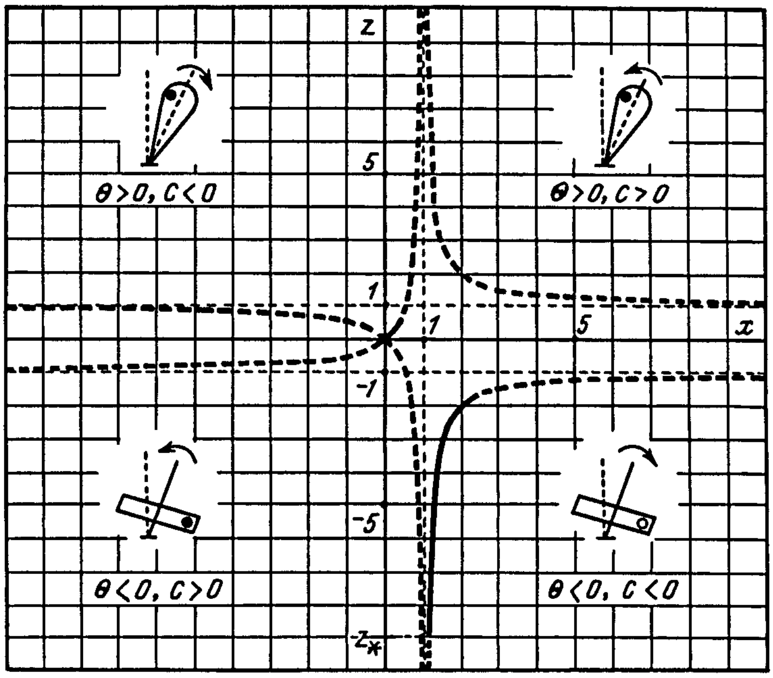

Fig. 6

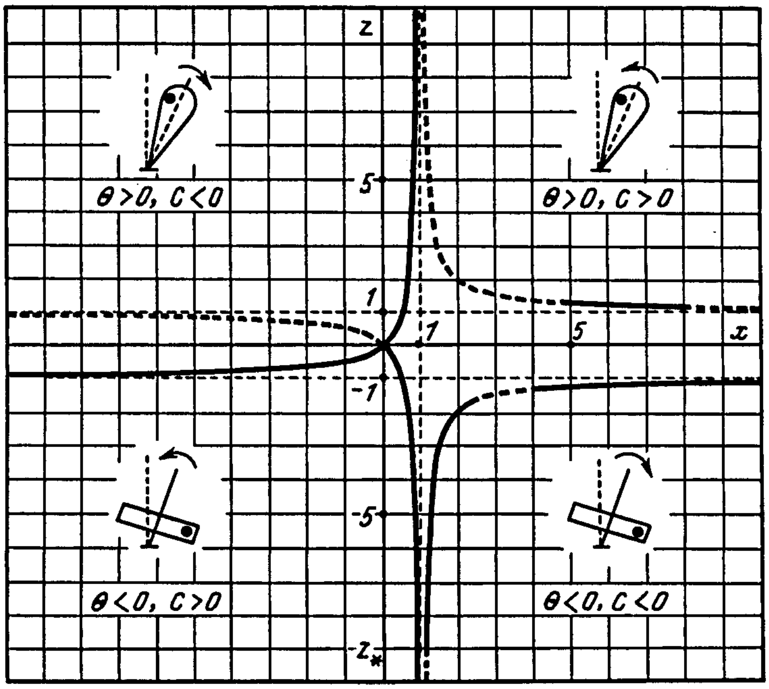

Fig. 7

stability properties. This strip is bounded by the curves $x = X_-(y)$ and $x = X_+(y)$, the curve $X_-(y)$ coinciding with the corresponding curve for the case of the dead torque. While the other stability regions can be interpreted more or less convincingly in terms of physics, this is not the case for the strip indicated. This case is a graphic illustration of how helpless can intuition be in such problems.

Figures 6 and 7 show the stability regions on the amplitude-frequency characteristics for the dead and follower torques, respectively. Dashed lines identify the instability regions. These figures correspond to $|y| = 0.5$. The lower boundary of the stability regions in the variable z corresponds to large amplitudes.

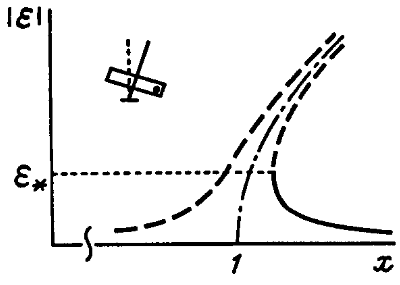

Fig. 8

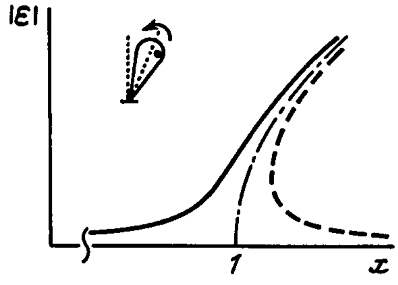

Fig. 9

We will proceed now to a strict definition of the boundary functions. For the follower torque, the coefficients D_{20} and D have the form

$$\begin{aligned} D_{20} &\approx -\tfrac{1}{2}d(g-d+1)[(6g^2-3g-1)d-8g^2(g+1)]\varepsilon^2, \\ D &\approx \tfrac{1}{4}d(g-d+1)(2g-2d+1)^2[(g-1)^2d+2g(g+1)][(2g-1)^2d+4g]\varepsilon^4. \end{aligned} \tag{6.6}$$

The expressions of (6.6) are similar to but somewhat more complex than those of (6.3). Using (6.6) and (6.1), one can obtain implicit expressions for the bounding functions. Introduce the notation

$$F_+(x,y) \overset{\text{def}}{=} (y-1)^2x^3-(y-1)(y-5)x^2+4yx-4y^2, \quad F_-(x,y) \overset{\text{def}}{=} (y-1)^2x^3-(y-1)(y-3)x^2+y^2x-y^2.$$

Then we have

$$\begin{aligned} &F_+(x,y)=0, \quad x>2|y|/(1-y) \quad \Longrightarrow \quad x \overset{\text{def}}{=} X_+(y), \\ &F_-(x,y)=0, \quad \begin{bmatrix} y<0: & x>-2y/(1-y), \\ 0<y<0.957: & x>y/(1-y) \end{bmatrix} \quad \Longrightarrow \quad x \overset{\text{def}}{=} X_-(y). \end{aligned} \tag{6.7}$$

The function $X_-(y)$ is defined for $y < 0.957$, i.e., for tops having a limited prolateness. This function has another shape for $y > 0.957$, but we choose not to consider this case in the present paper, since it lies on the boundary of the applicability area of the model adopted.

The relations of (6.7) are fairly complex. To determine the bounding functions, one should solve cubic equations. The explicit solution of these equations is rather complex and hardly can be constructed in a form convenient for the analysis. The inverse functions $y(x)$ are simpler but they are less convenient for the interpretation in the language of physics. However, a rather accurate approximation to the solution of equations (6.7) can be constructed in the form

$$X_+(y) \approx \frac{1}{2}\left(\frac{5-2y+y^2}{1-y}+\frac{25+4y-y^2}{\sqrt{(1-y)(25-y)}}\right), \qquad X_-(y) \approx \frac{1}{1-y}+\frac{6+y}{\sqrt{3(1-y)(3+y)}}.$$

The error of these relations does not exceed 0.3%. For $y < 0$, simpler approximate relations can be suggested:

$$X_+(y) \approx 5+3.23\frac{y}{1-y} \quad (0.07\%), \qquad X_-(y) \approx 3+2.12\frac{y}{1-y} \quad (0.6\%).$$

The errors of these relations are given in parentheses.

7. LARGE AMPLITUDES

The stability conditions for the conservative cases have the form

$$\theta\varepsilon^3+\theta_*\alpha\beta>0, \quad \theta(4\theta+\theta_3)\varepsilon^3+\theta_3\theta_*\alpha\beta>0. \tag{7.1}$$

The first and second inequalities of (7.1) correspond to $L=\infty$ and $L=0$, respectively. For the nonconservative case, these two inequalities must be satisfied simultaneously and, moreover, the additional condition $\varepsilon > 0$ appears for the follower motor torque and $\theta < 0$ for the dead torque.

The stability regions on the nonlinear amplitude-frequency characteristic for the follower torque are presented in Fig. 8 (for $\theta < 0$) and Fig. 9 (for $\theta > 0$). In these figures, the dashed curves correspond to unstable motions; $\varepsilon_* = |p|^{1/3}$ is the value of $|\varepsilon|$ corresponding to the turning point. For the dead torque, the stability region shown in Fig. 8 disappears. Hence, the destabilizing effect of a limited-power motor manifests itself also for large amplitudes.

Figures 6 and 7 permit one to see simultaneously the stability regions (solid lines) on the amplitude-frequency characteristic of the system for very small, small, and large amplitudes.

8. CONCLUSION

We will enumerate the major changes to which the stability regions of conservative systems (without a motor or with a motor of infinite power) are subjected when passing to the nonconservative case (corresponding to a limited-power motor).

1. New stability regions do not appear. The existing stability regions substantially narrow.
2. Instability manifests itself already at small amplitudes.
3. Instability regions have a complex nature which is not amenable to intuitive predictions.
4. For a motor creating a dead torque, the narrowing effect for stability regions is expressed more clearly than for a motor creating a follower torque.
5. The magnitude of the power of the motor does not influence the size of the stability regions.

Statement 3 deserves particular attention. It turns out that instability regions can occur in nonresonant zones at small amplitudes. Hence, for systems with limited-power motors, the linearized equations of forced oscillations similar to those used in the dynamics of rigid rotors [8] can be unstable. As a consequence, steady-state motions of such systems cannot be implemented in practice. Note that to identify these instability regions we proceeded from the equations of perturbed motions of the full nonlinear system and used the linearization only to simplify the algebraic criteria of stability. If we proceeded from the linearized equations, we would not be able to identify this instability.

Thus, in general, a limited-power motor produces a strong destabilizing effect. This conclusion is in complete agreement with the results of the analysis of plane-parallel motions of an unbalanced rigid rotor in nonlinear elastic bearings [2, 3]. This analysis shows that the stability regions of a system with a limited-power motor are narrower than those of the same system but with an infinite-power motor. However, while in the case of plane-parallel motions [2] the new instability regions can appear only on the right branch of the amplitude-frequency characteristic, for the case considered in the present paper, they can appear also on the left branch. The appearance of large instability regions can call the very implementability of the steady-state motion into question. However, in practice, there are other dissipation factors, primarily, external viscous friction, which can produce a stabilizing effect [1, 2].

Note in conclusion that a limited-power motor leads also to some changes in the behavior of the system in the preserved stability regions. Owing to the dissipative properties of the motor, in these regions the steady-state motions become asymptotically stable rather than stable in the first approximation. In this sense, a limited-power motor can produce a stabilizing effect. From this point of view, the case of an oblate top with a restoring elastic torque is of interest. The steady-state motion of such a top is stable at all frequencies, provided that the motor generates a follower torque. Hence, this top can serve as a model of a centrifuge in which no resonance can occur. (See Section 3.) This result has no analogue in the case of plane-parallel motion.

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