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# ELECTROMAGNETIC FIELD IN MOVING SPACE WITH SPHERICAL ENCLOSURE

**V.A. Vestyak[1], L.A. Igumnov[2], D.V. Tarlakovskiy[3*]**

[1]Moscow Aviation Institute (National Research University),

Volokolamskoye shosse 4, A-80, GSP -3, Moscow, 125993, Russia

[2]Research Institute for Mechanics of Lobachevsky State University of Nizhni Novgorod, 23 Prospekt Gagarina (Gagarin Avenue) BLDG 6, Nizhny Novgorod, 603950, Russia

[3]Institute of Mechanics (Lomonosov Moscow State University, Michurinski prospekt 1, Moscow 119192, Russia

*e-mail: tdvhome@mail.ru

**Abstract.** The unsteady axisymmetric problem being considered is related to the identification of the components of electromagnetic field in moving space with spherical enclosure filled with homogeneous isotropic conductor. The model used includes Maxwell's equations and generalized Ohm's law. In order to find a solution, a series expansion in Legendre polynomials and Laplace time transform are applied. Integral representations of the components of the electromagnetic field with Green's function kernels have been generated.

## 1. Introduction

As of today, the problem related to wave propagation in electromagnetoelastic bodies has been mainly considered in regard to stationary problems (see example [1]). When researching non-stationary processes in such bodies (example [2]), the small parameter method is convenient to be used represented by mechanical and electromagnetic field interaction factor. At the same time, a solution is required for the auxiliary problem of identification of the electromagnetic field parameters in accordance with the predetermined motion law. This problem is considered below in respect of a plane with spherical enclosure. It also has its own independent value in terms of, for example, research on the motion of various aircrafts affected by electromagnetic field.

## 2. Setting up the problem

It is assumed that the space with the spherical enclosure of radius R is filled with homogeneous isotropic conductor and moves in accordance with the predetermined law. Axisymmetric change of the electromagnetic field in it is described using the Maxwell's equations and generalized Ohm's law in the spherical coordinate system $r, \theta, \vartheta$ $\left(r \geq 0,\ 0 \leq \theta \leq \pi,\ -\pi < \vartheta \leq \pi\right)$:

$$-r^{-1}\left(rH\right)_{,r} = \eta_e^2\left(\gamma j_\theta + \dot{E}_\theta\right),\ r^{-1}\left(H\sin\theta\right)_{,\theta}\sin^{-1}\theta = \eta_e^2\left(\gamma j_r + \dot{E}_r\right),$$

$$r^{-1}\left[\left(rE_\theta\right)_{,r} - E_{r,\theta}\right] = -\dot{H},\ E_{r,r} + r^{-1}\left(E_{\theta,\theta} + E_\theta \operatorname{ctg}\theta + 2E_r\right) = \rho_e; \tag{1}$$

$$j_r = E_r + \rho_{e0}\dot{u}/\gamma,\ j_\theta = E_\theta + \rho_{e0}\dot{v}/\gamma. \tag{2}$$

From now on, dots stand for time derivatives while the variable after the coma in the lower index indicates its derivative. The following dimensionless quantities are also used (if the tracing is the same, the prime corresponds to dimensional analogues):

$$\tau = \frac{c_1 t}{L},\ r = \frac{r'}{L},\ r_0 = \frac{R}{L},\ u = \frac{u'}{L},\ v = \frac{v'}{L},\ H = \frac{H'\mu_e c_1}{cE_*},\ \rho_e = \frac{4\pi\rho'_e L}{\varepsilon E_*},$$

$$E_r = \frac{E'_r}{E_*}, E_\theta = \frac{E'_\theta}{E_*},\ j_r = \frac{j'_r}{\sigma E_*},\ j_\theta = \frac{j'_\theta}{\sigma E_*},\ \eta_e^2 = \frac{\mu_e \varepsilon c_1^2}{c^2}, \gamma = \frac{4\pi\sigma L}{\varepsilon c_1},$$

where $t$ - time; $u$ and $v$, $E_r$ and $E_\theta$, $j_r$ and $j_\theta$ - radial and tangential displacements, components of electric field and current density vectors; $H$ - magnetic field vector nonzero component; $\rho_e$ - surface charge density; $c$ and $c_1$ - speed of light and tension wave propagation; $\varepsilon$ and $\mu_e$ - dielectric and magnetic conductivity factors; $\sigma$ - electric conductivity factor; $L$ and $E_*$ - electric field linear dimension and intensity.

The Ohm's law linearization was performed in regard to initial electromagnetic field (which has a correspondent additional index 0) with the following components:

$$E_{0\vartheta} = E_{0\theta} \equiv 0,\ E_{0r} = E_0(r),\ H_{0r} = H_{0\theta} = H_{0\vartheta} \equiv 0.$$

At the enclosure boundary, the following electric field intensity is specified:

$$E_\theta\big|_{r=r_0} = e_0(\tau,\theta). \tag{3}$$

All the required functions are bounded while the initial conditions are homogeneous:

$$E_r\big|_{\tau=0} = \dot{E}_r\big|_{\tau=0} = E_\theta\big|_{\tau=0} = \dot{E}_\theta\big|_{\tau=0} = H\big|_{\tau=0} = \dot{H}\big|_{\tau=0} = 0. \tag{4}$$

Equations (1), (2) lead to the following equation further used as the basic:

$$\begin{gathered}\eta_e^2\left(\dddot{H} + \gamma\ddot{H}\right) = \Delta\dot{H} - r^{-2}\dot{H}\sin^{-2}\theta + r^{-1}\eta_e^2\left[\left(r\rho_{e0}\dddot{v}\right)_{,r} - \rho_{e0}\dddot{u}_{,\theta}\right],\\ \Delta H = r^{-2}\left[\left(r^2 H_{,r}\right)_{,r} + \sin^{-1}\theta\left(H_{,\theta}\sin\theta\right)_{,\theta}\right]\end{gathered} \tag{5}$$

and the following charge density ratio:

$$\dot{\rho}_e + \gamma\rho_e = -r^{-2}\left(r^2\rho_{e0}\dot{u}\right)_{,r} - r^{-1}\sin^{-1}\theta\left(\rho_{e0}\dot{v}\sin\theta\right)_{,\theta}. \tag{6}$$

## 3. Integral solutions

In order to solve initial boundary value problem (3)-(5) with consideration of (1), functions $E_r$, $\rho_e$, $j_r$, $u$ and $E_\theta$, $H$, $j_\theta$, $v$, $e_0$ are expanded in Legendre $P_n(x)$ and Gegenbauer $C_{n-1}^{3/2}(x)$ polynomials accordingly (two series are given for example):

$$E_r(r,\theta,\tau) = \sum_{n=0}^{\infty} E_{rn}(r,\tau)P_n(\cos\theta),\ H(r,\theta,\tau) = \sin\theta\sum_{n=1}^{\infty} H_n(r,\tau)C_{n-1}^{3/2}(\cos\theta). \tag{7}$$

This leads to the following equations in regard to bounded factors $H_n(r,\tau)$ $(n \ge 1)$:

$$\begin{gathered}\eta_e^2\left(\dddot{H}_n + \gamma\ddot{H}_n\right) = \Delta_n\dot{H}_n + \eta_e^2 l_H\left(\ddot{u}_n, \ddot{v}_n\right),\ \Delta_n H = r^{-2}\left[\left(r^2 H_{,r}\right)_{,r} - mH\right],\\ m = n(n+1),\ l_H(u,v) = r^{-1}\left[\left(r\rho_{e0}v\right)_{,r} + \rho_{e0}u\right].\end{gathered} \tag{8}$$

Ratios for other electromagnetic field component expansion factors follow from formulas (1) and (6):

$$\eta_e^2\left(\dot{E}_{\theta n}+\gamma E_{\theta n}\right)=-r^{-1}\left(rH_n\right)_{,r}-\eta_e^2\dot{v}_n\left(n\geq 1\right),\ \eta_e^2\left(\dot{E}_{rn}+\gamma E_{rn}\right)=mr^{-1}H_n-\eta_e^2\dot{u}_n; \tag{9}$$

$$\dot{\rho}_n+\gamma\rho_n=-l_{n\rho}\left(\dot{u}_n,\dot{v}_n\right),\ l_{n\rho}\left(u,v\right)=r^{-2}\left(r^2\rho_{e0}u\right)_{,r}+mr^{-1}\rho_{e0}v. \tag{10}$$

In the last ratio, $\rho_n$ are the factors of series (7) for function $\rho_e$.

Boundary and initial conditions (3) and (4) with consideration of the first equality in (9) is transformed as follows:

$$r^{-1}\left(rH_n\right)_{,r}\Big|_{r=r_0}=-\eta_e^2h_0\left[v_n\left(r_0,\tau\right),e_{0n}\left(\tau\right)\right]\left(n\geq 1\right),\ h_0\left(v,e\right)=\dot{v}+\dot{e}+\gamma e, \tag{11}$$

$$E_{nr}\Big|_{\tau=0}=\dot{E}_{nr}\Big|_{\tau=0}=0\ \left(n\geq 0\right),\ E_{n\theta}\Big|_{\tau=0}=\dot{E}_{n\theta}\Big|_{\tau=0}=H_n\Big|_{\tau=0}=\dot{H}_n\Big|_{\tau=0}=0\ \left(n\geq 1\right). \tag{12}$$

Next, we apply Laplace time transform $\tau$ ($s$ means its parameter while index $L$ points to its view) [3] to ratios (8)-(11) with consideration of conditions (12):

$$s_e^2\eta_e^2H_n^L=\Delta_nH_n^L+\eta_e^2sl_H\left(u_n^L,v_n^L\right)\ \left(n\geq 1\right),\ s_e=\sqrt{s\left(s+\gamma\right)}; \tag{13}$$

$$\eta_e^2\left(s+\gamma\right)E_{\theta n}^L=-r^{-1}\left(rH_n^L\right)_{,r}-\eta_e^2sv_n^L\left(n\geq 1\right),\ \eta_e^2\left(s+\gamma\right)E_{rn}^L=r^{-1}mH_n^L-\eta_e^2su_n^L; \tag{14}$$

$$\left(s+\gamma\right)\rho_n^L=-sl_{n\rho}\left(u_n^L,v_n^L\right); \tag{15}$$

$$r^{-1}\left(rH_n^L\right)_{,r}\Big|_{r=r_0}=-\eta_e^2h_0^L\left[v_n^L\left(r_0,s\right),e_{0n}^L\left(s\right)\right]\left(n\geq 1\right),\ h_0^L\left(v,e\right)=sv+\left(s+\gamma\right)e. \tag{16}$$

It is convenient to represent the solution for boundary-value problem (13), (16) as integrals:

$$H_n^L(r,s)=-\eta_e^2s\int_{r_0}^{\infty}G_{Hn}^L(r,\xi,s)l_H\left[u_n^L(\xi,s),v_n^L(\xi,s)\right]d\xi-\eta_e^2G_{Hn0}^L\left(r,s\right)h_0^L\left[v_n^L(r_0,s),e_{0n}^L(s)\right]. \tag{17}$$

Here, $G_{Hn}^L\left(r,\xi,s\right)$ and $G_{Hn0}^L\left(r,s\right)$ are the Green's functions, i.e. bounded solutions of the problems ($\delta\left(x\right)$ - Dirac delta function [3]):

$$\Delta_nG_{Hn}^L-s_e^2\eta_e^2G_{Hn}^L=\delta\left(r-\xi\right),\ r^{-1}\left(rG_{Hn}^L\right)_{,r}\Big|_{r=r_0}=0; \tag{18}$$

$$\Delta_nG_{Hn0}^L-s_e^2\eta_e^2G_{Hn0}^L=0,\ r^{-1}\left(rG_{Hn}^L\right)_{,r}\Big|_{r=r_0}=1. \tag{19}$$

Similar (17) representations for functions $E_{rn}^L\left(r,s\right)$ and $E_{\theta n}^L\left(r,s\right)$ may be obtained from equalities (14). Original representation (17) has the following view (asterisk stands for time convolution):

$$H_n(r,\tau)=-\eta_e^2\int_{r_0}^{\infty}G_{Hn}(r,\xi,\tau)*l_H\left[\dot{u}_n(\xi,\tau),\dot{v}_n(\xi,\tau)\right]d\xi-\eta_e^2G_{Hn0}(r,\tau)*h_0\left[v_n(r_0,\tau),e_{0n}(\tau)\right]. \tag{20}$$

Similarly, functions $E_{rn}\left(r,\tau\right)$ and $E_{\theta n}\left(r,\tau\right)$ may be represented. The formula for the surface charge expansion factors follows from (15):

$$\rho_n\left(r,\tau\right)=-l_{n\rho}\left[\dot{u}_n\left(r,\tau\right),\dot{v}_n\left(r,\tau\right)\right]*e^{-\gamma\tau}. \tag{21}$$

**4. Green's functions**

The solutions for boundary-value problems (18) and (19) have the following view ($h(x)$ - Heaviside function [3]):

$$G_{Hn}^{L}(r,\xi,s)=\xi^{2}\left[\tilde{G}_{Hn}^{L}(r,\xi,s)h(\xi-r)+\tilde{G}_{Hn}^{L}(\xi,r,s)h(r-\xi)\right],$$
$$\tilde{G}_{Hn}^{L}(r,\xi,s)=\eta_e s_e S_{en}(\eta_e r_0 s_e,\eta_e r s_e)Z_{1n}(\eta_e \xi s_e)Y_{3n}^{-1}(\eta_e r_0 s_e); \quad (22)$$

$$G_{Hn0}^{L}(r,s)=-\eta_e^{-1}s_e^{-1}Z_{1n}(\eta_e r s_e)Y_{3n}^{-1}(\eta_e r_0 s_e), \quad (23)$$

where

$$Z_{1n}(z)=z^{-1/2}K_{n+1/2}(z),\ Z_{2n}(z)=z^{-1/2}I_{n+1/2}(z),\ S_{en}(x,y)=Z_{1n}(y)Y_{4n}(x)-Y_{3n}(x)Z_{2n}(y),$$
$$Y_{3n}(z)=-z^{-3/2}\left[(n+1)K_{n+1/2}(z)-zK_{n+3/2}(z)\right],\ Y_{4n}(z)=-z^{-3/2}\left[(n+1)I_{n+1/2}(z)+zI_{n+3/2}(z)\right].$$

Here, $K_{n+1/2}(z)$ and $I_{n+1/2}(z)$ are modified Bessel functions.

When building formulas (22) and (23), the Bessel function expressions through elementary functions [4] were used from which follow the below listed equalities for $Z_{1n}(z)$, $Z_{2n}(z)$ and $Y_{3n}(z)$, $Y_{4n}(z)$:

$$Z_{1n}(z)=z^{-n-1}R_{n0}(z)e^{-z}\sqrt{\pi/2},\ Z_{2n}(z)=(-1)^n z^{-n-1}(2\pi)^{-1/2}\left[R_{n0}(-z)e^{z}-R_{n0}(z)e^{-z}\right],$$
$$Y_{3n}(z)=z^{-n-2}R_{n3}(z)e^{-z}\sqrt{\pi/2},\ Y_{4n}(z)=(-1)^n z^{-n-2}(2\pi)^{-1/2}\left[R_{n3}(-z)e^{z}-R_{n3}(z)e^{-z}\right],$$

where

$$R_{n0}(z)=\sum_{k=0}^{n}A_{nk}z^{n-k},\ A_{nk}=\frac{(n+k)!}{2^k(n-k)!k!},\ R_{n3}(z)=R_{n1}(z)-R_{n0}(z),\ R_{n1}(z)=R_{n+1,0}(z)-nR_{n0}(z).$$

In this case, Green's functions in (22) and (23) obtain the following view:

$$\tilde{G}_{Hn}^{L}(r,\xi,s)=\sum_{k=1}^{2}\tilde{G}_{Hn}^{(0k)L}(r,\xi,s)e^{-\tau_{e0k}(r,\xi)s_e},\ G_{Hn0}^{L}(r,s)=r_0^{n+2}r^{-n-1}G_{Hn0}^{(02)L}(r,s)e^{-\tau_{e02}(r,r_0)s_e},$$

where

$$\tilde{G}_{Hn}^{(01)L}(r,\xi,s)=(-1)^{n+1}R_{n0}(\eta_e\xi s_e)R_{n0}(-\eta_e r s_e)D_n^{-1}(r\xi,\eta_e s_e),$$
$$\tilde{G}_{Hn}^{(02)L}(r,\xi,s)=(-1)^{n}R_{n0}(\eta_e\xi s_e)R_{n0}(\eta_e r s_e)R_{n3}(-\eta_e r_0 s_e)D_n^{-1}(r\xi,\eta_e s_e),$$
$$G_{Hn0}^{(02)L}(r,s)=-R_{n0}(\eta_e r s_e)R_{n3}^{-1}(\eta_e r_0 s_e),$$
$$D_n(x,y)=2x^{n+1}y^{2n+1},\ \tau_{e01}(r,\xi)=\eta_e(\xi-r),\ \tau_{e02}(r,\xi)=\eta_e(r+\xi-2r_0).$$

Their originals may be analytically found using operator calculus theorems. However, since for real mediums $\eta_e << 1$, serious challenges arise during calculations associated with the small parameter. That is why quasistatic analogues of Green's functions with $\eta_e = 0$ will be further used.

**5. Quasistatic solution**

In this version, equations in boundary problems (18) and (19) are simplified and have the following fundamental system of solutions: $r^{-(n+1)}$, $r^n$. Then, similar to p. 3, we come to the following formulas for Green's functions:

$$G_{Hn}^{L}(r,\xi,s)=G_{Hn}^{c}(r,\xi)=\xi^{2}\left[\tilde{G}_{Hn}^{c}(r,\xi)h(\xi-r)+\tilde{G}_{Hn}^{c}(\xi,r)h(r-\xi)\right],$$
$$\tilde{G}_{Hn}^{c}(r,\xi)=-\frac{(n+1)r_{0}^{2n+1}+nr^{2n+1}}{n(2n+1)r^{n+1}\xi^{n+1}},\ G_{Hn0}^{L}(r,s)=G_{Hn0}^{c}(r)=-\frac{r_{0}^{n+2}}{nr^{n+1}}. \tag{24}$$

Then, formula (17) is also simplified by substituting kernels $G_{Hn}^{L}(r,\xi,s)$ and $G_{Hn0}^{L}(r,s)$ with functions $G_{Hn}^{c}(r,\xi)$ and $G_{Hn0}^{c}(r)$. Its original (20) is substituted with the following equality:

$$H_{n}(r,\tau)=-\eta_{e}^{2}\int_{r_{0}}^{\infty}G_{Hn}^{c}(r,\xi)l_{H}\left[\dot{u}_{n}(\xi,\tau),\dot{v}_{n}(\xi,\tau)\right]d\xi-\eta_{e}^{2}G_{Hn0}^{c}(r)h_{0}\left[v_{n}(r_{0},\tau),e_{0n}(\tau)\right]. \tag{25}$$

The algorithm is followed by the summation of series presented as (7) and the identification of the current density component vector using formulas (2).

## 6. Example

Let's assume that the space material is aluminium ($\eta_e=0{,}111\cdot10^{-4}$; $\gamma=5{,}06$) while the other initial values are represented as: $r_0=1$, $\rho_{e0}(r)=2ar^{-3/2}$, $u=\tau_+^2/2\cos\theta$, $v=-\tau_+^2/2\sin\theta$, $e_0\equiv0$.

Then, the electromagnetic field components calculated using formulas (7), (21), (24) and (25) are determined as follows:

$$H(r,\theta,\tau)=\eta_{e}^{2}wr^{-2}\left[r_{0}^{3}+\frac{2a}{3}\left(2r^{3/2}+r_{0}^{3/2}\right)\right]\tau h(\tau)\sin\theta,$$
$$\rho_{e}(r,\theta,\tau)=3\gamma^{-2}awr^{-5/2}\left(\gamma\tau-1+e^{-\gamma\tau}\right)h(\tau)\cos\theta.$$

## 7. Conclusions

An analytical solution of the axisymmetric non-stationary problem is presented related to the identification of the components of the electromagnetic field in the space filled with homogeneous isotropic conductor with spherical enclosure. Its movement law and field distribution at the cavity boundary are predetermined. It is shown that the kernels of the integral representations built may be substituted with their quasistatical analogues, which significantly simplifies the evaluation of the correspondent integrals. The solution obtained may serve as the basis for the research of problems with more complex geometry.

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