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THE EQUILIBRIUM AND STABILITY OF THE NONLINEARLY ELASTIC CYLINDER WITH INTERNAL STRESSES

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Abstract. By using the Saint Venant`s semi-inverse method stress-strain state of stretching nonlinearly elastic cylinder containing screw dislocation was analyzed. The ranges of material parameters when diagram of loading (the relationship between the axial load and the elongation of the cylinder) has a falling segment were defined. The existence of such segments can be treated as a stability loss of stretching cylinder.

To analyze the stability the bifurcation approach was used that based on linearization of the equilibrium equations in the neighborhood of the obtained solutions. The bifurcation point was defined as such value of the "loading" parameter (Burgers vector magnitude, stretch ratio or other strain characteristic) for which the linearized problem has a nontrivial solution. Numerical determination of the bifurcation points was based on the analysis of the homogeneous linear boundary value problem of sixth order whose coefficients expressed through the radial displacement function and its derivative**.** The similar problem of compression was used for verification purposes.

1. Introduction

The concept of internal, or residual, stresses existing in solids that are free from external loads was appeared firstly in the works of V. Volterra [1] at the beginning of the XX century. One particular reason of such stresses could be the existence of isolated linear defects, well known due to A. Love [2] terminology as Volterra dislocations.

The concept of dislocation as a linear defect of the crystal lattice arose in physics much later – in the thirties of the last century [3]. The concept of disclinations (rotational defects or rotary dislocations) appeared even later though having found practical confirmation not only in lattices but in different various material structures either [4-5].

Simulation of dislocation within the continuum description is quite wide and rapidly developing branch of modern mechanics. A significant contribution to its development was made by the Rostov-on-Don school of mechanics, some results of the work of which had been presented in [6], particularly in matters related to the generalization of the theory of elastic dislocations and disclinations to the nonlinear case.

Isolated screw dislocation was the object rather "convenient" for the study within the framework of the nonlinear continuum mechanics, since the corresponding stress-strain state is described by a function of the radial coordinate, namely the function of radial displacement of the points of the cylinder. Various aspects of this problem, including the elimination of singularities at the axis of dislocation, the existence of discontinuous solutions etc. for incompressible media were considered, for example, in [7, 8]. In this paper, we consider the equilibrium and stability of nonlinear elastic cylinder with a screw dislocation in the case of a compressible material. The influence of defect formation on the length of the load-free cylinder was studied. Some questions of the stability of the expansion and contraction processes were discussed.

2. The equilibrium of the cylinder with a screw dislocation

The appearance of a screw dislocation in the cylinder is described by the following semi-inverse representation:

$$R = P(r),\ \Phi = \varphi + \psi z,\ Z = \gamma z + a\varphi, \tag{1}$$

where $\{R, \Phi, Z\}$, $\{r, \varphi, z\}$ – cylindrical coordinates of the actual and reference configuration, respectively, stretch ratio γ describes changing of the cylinder length, $a = |\mathbf{b}|/2\pi$ – dislocation parameter, $\mathbf{b}$ – Burgers vector, $P(r)$ – function of radial displacement of points of the cylinder. Since the formation of dislocation may be accompanied by twisting [9, 10], parameter ψ – twist angle per unit length of the cylinder – was introduced in the semi-inverse representation (1).

Given a semi-inverse representation (1) all tensorial characteristics of strain could be determined, namely deformation gradient $\mathbf{C}$, Cauchy-Green strain measure $\mathbf{G}$, and its invariants $I_k, k = 1,2,3$ [11]. After setting up the specific potential energy function W, the equilibrium equations for Piola stress tensor $\mathbf{D}$ can be written as follows

$$div\mathbf{D} = 0. \tag{2}$$

We will limit our considerations by the simple boundary conditions on the lateral surface of the cylinder

$$\mathbf{e}_r \cdot \mathbf{D} = 0, \tag{3}$$

meaning no applied loads there; $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$ – orthonormal basis in a cylindrical coordinate system of reference configuration. By using (1) problem (2)–(3) is reduced to a boundary value problem for an ordinary differential equation of second order for the function $P(r)$.

To describe the mechanical properties of the cylinder we will use two models of compressible medium, i.e. two specific energy functions.

$$W = \lambda \frac{1}{2} I_1^2(\mathbf{U} - \mathbf{E}) + \mu I_1\left[(\mathbf{U} - \mathbf{E})^2\right], \tag{4}$$

and

$$W = \mu \frac{1}{2}(1-\beta)\left[I_2 I_3^{-1} + \frac{1}{\alpha}\left(I_3^{\alpha} - 1\right) - 3\right] + \mu \frac{1}{2}\beta\left[I_1 + \frac{1}{\alpha}\left(I_3^{-\alpha} - 1\right) - 3\right]. \tag{5}$$

Model (4) is known as harmonic material, while Eq. (5) presents Blatz and Ko material. In (4)–(5) $\mathbf{U} = \mathbf{G}^{1/2}$ – distortion tensor, $\lambda, \mu, \beta, \alpha$ – material parameters. In the case of small strains, parameter α is associated with Poisson ratio by relation $\alpha = \nu/(1-2\nu)$.

Investigation of the stability of the cylinder under tension or compression should obviously begin with an analysis of the "proper" length of the cylinder, due to the formation of dislocations. Following the scheme presented in [12], it is convenient to introduce following representations of axial force Q and twisting moment M in the form:

$$Q = \iint_S D_{zZ}\, dS, \tag{6}$$

$$M = \iint_S D_{z\Phi} R\, dS. \tag{7}$$

Consider firstly the case of non-twisted cylinder assuming $\psi = 0$ in (1).Then, following the scheme in [12], from the condition $Q = 0$ we obtain the dependence between the stretch factor γ and dislocation parameter a. For the case of harmonic material (4) numerical calculations show that the dislocation formation in the cylinder always leads to its shortening. For the model (5) the situation is more complicated: the cylinder can be shortened or stretched depending on the parameter β. These results are consistent with the asymptotic formulas given in [12].

To analyze the cylinder with free ends both parameters γ and ψ should be considered as varying, wherein to determine these parameters it is necessary to vanish the axial force (6) and twisting moment (7).

Figure 1. shows that change of length is not monotonic for values α close to 0.5, which corresponds to a Poisson ratio $\nu = 1/4$; the cylinder is shortened for all other considered values of parameter α.

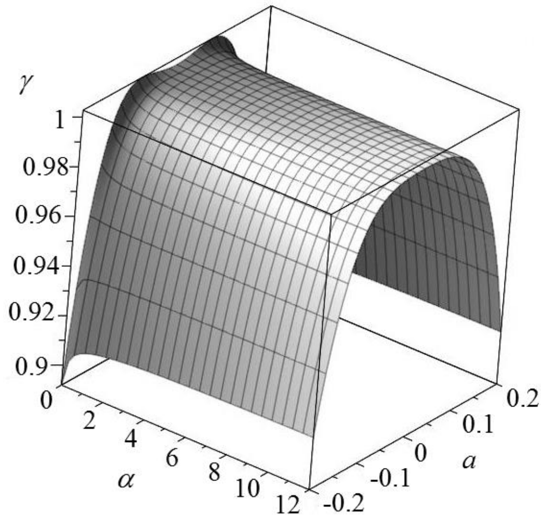

Fig. 1. Change of the length of the cylinder due to screw dislocation (material model (5), $\beta = 0$).

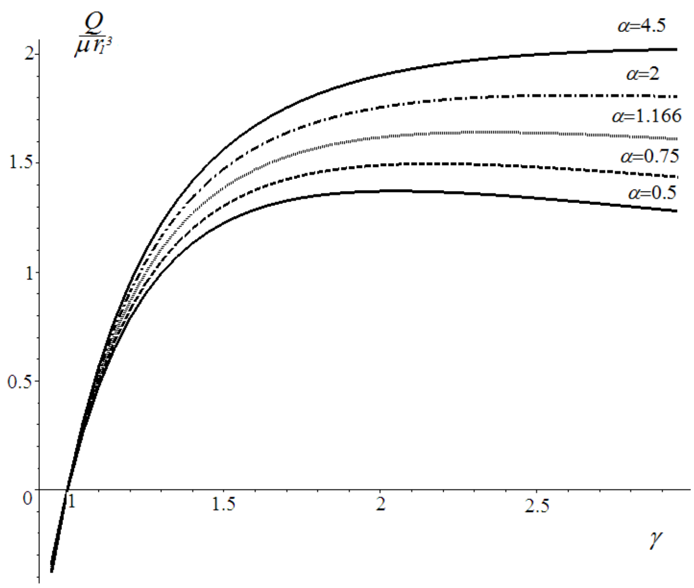

Fig. 2. Loading diagram of the cylinder with a screw dislocation (material model (5), $\beta = 0$).

Loading diagrams of the cylinder with a screw dislocation $a = 0.01$ for different values of parameter α are presented on Fig. 2. It is seen that each curve has the maximum point, followed by a decreasing segment. Such segment may indicate a stability loss of the cylinder at tension.

2. Stability analysis

Let us give small displacements to all points of the cylinder from the known equilibrium state by changing the semi-inverse representation (1):

$$\begin{cases} R = P(r) + \varepsilon U_1(r,\phi,z), \\ \Phi = \phi + \psi z + \varepsilon U_2(r,\phi,z), \\ Z = \gamma z + a\phi + \varepsilon U_3(r,\phi,z), \end{cases} \tag{9}$$

ε – small parameter, $U_k, k = 1,2,3$ – new unknown functions. The linearization process is reduced to computation following expressions for all strain characteristics

$$\overset{\circ}{\mathbf{F}} = \frac{d}{d\varepsilon}\mathbf{F}(\mathbf{R} + \varepsilon\mathbf{w}). \tag{10}$$

Here $\mathbf{R}$ – the radius vector of the known equilibrium position, $\mathbf{w}$ –vector of small displacements expressed in terms of the unknown functions. Finally, by linearizing Piola stress tensor we change the original nonlinear problem (2)–(3) by its linearized version:

$$div\,\overset{\circ}{\mathbf{D}} = 0, \tag{11}$$

$$\mathbf{e}_r \cdot \overset{\circ}{\mathbf{D}} = 0. \tag{12}$$

Equations (11) are partial differential equations of second order with respect to the unknown functions U_k. System (11)–(12) admits solution in the form

$$\begin{aligned} U_1(r,\varphi,z) &= u_1(r)\cos(n\varphi + bz), \\ U_2(r,\varphi,z) &= u_2(r)\sin(n\varphi + bz), \\ U_3(r,\varphi,z) &= u_3(r)\sin(n\varphi + bz), \end{aligned} \tag{13}$$

where $b = \frac{\pi m}{l}; n, m \in N;$ l – initial length of the cylinder.

The substitution (13) turns the system (11)–(12) into a linear boundary value problem for a system of three ordinary differential equations of second order in relation to $u_k(r)$. Detailed scheme of analysis of the existence of non-trivial solutions for such systems was described in [13].

Typical bifurcation curves corresponding to the case of Blatz and Ko material (5) are shown in Fig. 3: a) for compression, b) for tension. Symbol $\gamma*$ identifies critical value of the stretch ratio corresponding to the first encountered mode of the stability loss.

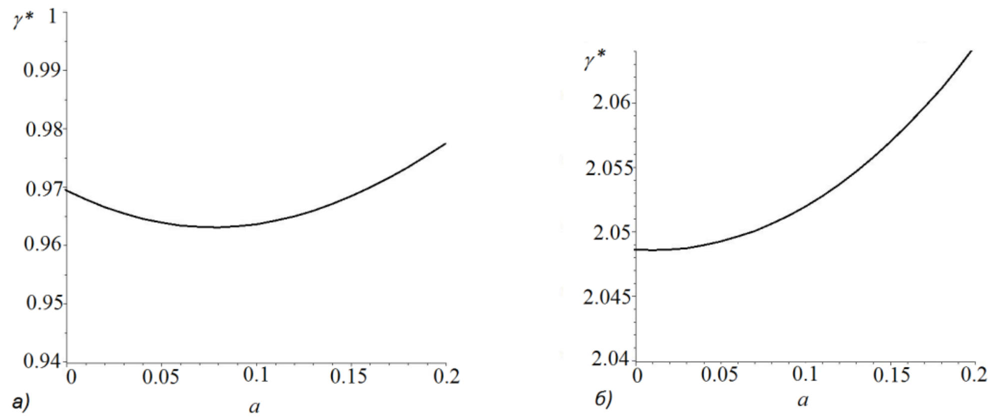

Fig. 3. Critical values γ^* (material model (5), $\beta = 0$, the thickness of the cylinder 0.1, length $l = 10$).

Instability of sufficiently long cylinder at compression occurs by the mode (n,m) = (1, 1), at tension – by the mode (n,m) = (0, 1), i.e. by axially symmetric mode. It can be seen in particular that the effect of dislocation on buckling during compression is much more important than in tension. Non-monotonic character of the curve on Fig. 3a appears to be connected with the inverse Poynting effect [11].

Acknowledgements

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