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THE SOLUTIONS OF NONLINEAR EQUATIONS OF PLANE DEFORMATION OF THE CRYSTAL MEDIA ALLOWING MARTENSITIC TRANSFORMATIONS: COMPLEX REPRESENTATION FOR MACROFIELD EQUATIONS

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Abstract. Mathematical methods of the solution of the equations of statics of plane nonlinear deformation of the crystal media with a complex lattice allowing martensitic transformations are developed. The equations of a statics represent system of four connected nonlinear equations. The vector of macroshifts is looked in the Papkovish-Neuber form. The system of the connected nonlinear equations is reduced to system of the separate equations. The vector of microshifts can be found from the sine-Gordon equation with variable coefficient (amplitude) before the sine and Poisson equation. The class of doubly periodic solutions expressing in the Jacobi elliptic functions is found for a case of constant amplitude. It is shown that the nonlinear theory possesses a set of solutions which describe fragmentation of the crystal medium, emergence of defects of structure of different types, phase transformations and other topological features of the deformation which are implemented under the influence of intensive power loadings and which can't be described by classical mechanics of the continuous medium. Features of the found solutions are discussed.

Keywords: complex lattice; nonlinear model; plane deformation; complex representation of solution; nonautonomous sine-Gordon equation.

1. Introduction

In recent years nanotechnologies are intensively developed. Practically all modern technologies for metals and alloys with ultrafine-grained structure are based on use of superhigh external impacts on material. The structure of material is significantly changed under the influence of intensive plastic deformations. Medium breaks up on separate nanoscale grains which are variously oriented. Grains divide big-angle borders in which defects like micropores and local consolidations are formed. The superlattice is formed and there are phase transformations.

Modern problems of technologies of obtaining and studying of the new materials set new problems for mechanics of continuous media. New analytical models are necessary for modern technologies of obtaining the new materials with the designated operational properties.

The classical continual model isn't adequate any more to those new problems which have arisen in connection with deep penetration into area the nano-scales. In works [1, 2] nonlinear model of deformation of crystal media with a complex lattice is offered. The offered model allows to describe specific processes of deformation, which are implemented in modern technologies of obtaining new materials.

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2. Nonlinear model of deformation of crystal media with a complex lattice

In nonlinear model [1]–[3] deformation of crystal medium is described by vector of acoustic mode $\mathbf{U}(x,y,z,t)$ and vector of optical mode $\mathbf{u}(x,y,z,t)$. Equations of motion defining $\mathbf{U}(x,y,z,t)$ and $\mathbf{u}(x,y,z,t)$, are derived from Lagrange's variation principle. They have the form:

$$\rho\ddot{U}_i = \sigma_{il,l}, \tag{1}$$

$$\sigma_{il} = \lambda_{ilmn}e_{mn} + C_{ilmn}\varepsilon_{mn} - s_{il}\Phi(u_s), \tag{2}$$

$$\mu\ddot{u}_i = \chi_{il,l} - \mathsf{P}\frac{\partial\Phi(u_s)}{\partial u_i}, \tag{3}$$

$$\chi_{il} = k_{ilmn}\varepsilon_{mn} + C_{ilmn}e_{mn}. \tag{4}$$

In Eqs. (1)–(4) and further the over point denotes time derivative while a comma in indexes defines spatial derivative. Besides, the following designations are introduced: ρ, μ are density and the specified density of mass of couple of atoms, σ_{il}, χ_{il} are tensors of macro- and microstresses, λ_{ilmn}, k_{ilmn}, C_{ilmn} are the coefficients of elasticity, microelasticity and modules of interaction of acoustic and optical modes, e_{il}, ε_{il} are tensors of deformation and microdeformation

$$e_{il} = \frac{1}{2}\left(U_{i,l} + U_{l,i}\right), \quad \varepsilon_{il} = \frac{1}{2}\left(u_{i,l} + u_{l,i}\right), \tag{5}$$

$\Phi(u_s)$ is the energy of interaction of sublattices. In pioneer work [4] and in majority of modern works [5, 6] it is accepted, that

$$\Phi(u_s) = 1 - \cos u_s. \tag{6}$$

The argument is:

$$u_s = \mathbf{B}\cdot\mathbf{u}, \tag{7}$$

where $\mathbf{B}$ is the vector of inverse lattice [7]. For crystal of cubic system with length of elementary cell b one has

$$\mathbf{B} = \frac{1}{b}(\mathbf{i} + \mathbf{j} + \mathbf{k}), \tag{8}$$

$$u_s = \frac{1}{b}(u_x + u_y + u_z). \tag{9}$$

Multiplier $\mathsf{P} = p - s_{il}e_{il}$ is an effective interatomic barrier, where p is half of energy of activation of rigid shift of lattices, and s_{il} is a tensor of nonlinear mechanostriction. For the cubic crystal one has $s_{il} = s\delta_{il}$, and

$$\mathsf{P} = p - s\,\mathrm{div}\mathbf{U}. \tag{10}$$

The material tensors λ_{ilmn}, k_{ilmn}, C_{ilmn} have only three independent components for this crystal. Let they will be in Voigt's designations [8]:

$$\lambda_{11}, \lambda_{12}, \lambda_{44}, \; k_{11}, k_{12}, k_{44}, \; C_{11}, C_{12}, C_{44}.$$

Take for a measure of anisotropy of a cubic crystal [9]:

$$a_1 = \frac{2\lambda_{44}}{\lambda_{11} - \lambda_{12}}, \; a_2 = \frac{2k_{44}}{k_{11} - k_{12}}, \; a_3 = \frac{2C_{44}}{C_{11} - C_{12}}. \tag{11}$$

For isotropic medium we have:

$$a_1 = a_2 = a_3 = 1. \tag{12}$$

Material ratios for the media of cubic system have the form:

$$\sigma_{il} = \begin{cases} (\lambda_{11} - \lambda_{12})e_{il} + (C_{11} - C_{12})\varepsilon_{il} + (\lambda_{12}e + C_{12}\varepsilon - s\Phi(u_s))\delta_{il} \ (i = l), \\ 2\lambda_{44}e_{il} + 2C_{44}\varepsilon_{il} \ (i \neq l), \end{cases} \tag{13}$$

$$\chi_{il} = \begin{cases} (k_{11} - k_{12})\varepsilon_{il} + (C_{11} - C_{12})e_{il} + (k_{12}\varepsilon + C_{12}e)\delta_{il} \ (i = l), \\ 2k_{44}\varepsilon_{il} + 2C_{44}e_{il} \ (i \neq l), \end{cases} \tag{14}$$

$$e = e_{xx} + e_{yy} + e_{zz}, \ \ \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}. \tag{15}$$

3. Plane deformation. Statics equations

We will call the deformed state as statically plane and parallel to an axis x_3 if

$$U_x = U_x(x, y), \ \ U_y = U_y(x, y), \ \ U_z = 0, \tag{16}$$

$$u_x = u_x(x, y), \ \ u_y = u_y(x, y), \ \ u_z = 0. \tag{17}$$

Taking into account (16), (17) in material ratios (13) and (14), and substituting the expressions for σ_{il} and χ_{il} in Eqs. (1) and (3), we find the equations of static in movements for plane deformation of nonlinear model:

$$\lambda_{44}\Delta\mathbf{U} + (\lambda_{12} + \lambda_{44})\mathrm{grad}\,\mathrm{div}\mathbf{U} + C_{44}\Delta\mathbf{u} + (C_{12} + C_{44})\mathrm{grad}\,\mathrm{div}\mathbf{u} - s\,\mathrm{grad}\Phi(u_s) = 0, \tag{18}$$

$$k_{44}\Delta\mathbf{u} + (k_{12} + k_{44})\mathrm{grad}\,\mathrm{div}\mathbf{u} + C_{44}\Delta\mathbf{U} + (C_{12} + C_{44})\mathrm{grad}\,\mathrm{div}\mathbf{U} - \mathbf{B}(p - s\,\mathrm{div}\mathbf{U})\sin u_s = 0. \tag{19}$$

In Eqs. (18) and (19) one has:

$$\mathbf{U} = U_x\mathbf{i} + U_y\mathbf{j}, \ \ \mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}, \ \ \mathbf{B} = \frac{\mathbf{i} + \mathbf{j}}{b}, \ \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and, besides, restrictions (12) are accepted.

3.1. General solution of the equations of statics. The equations of static (18), (19) are a system of four coupled nonlinear equations. We will seek a vector of macroshifts $\mathbf{U}$ in Papkovish-Neuber form:

$$\mathbf{U} = a\mathbf{A} + \mathrm{grad}\,\chi. \tag{20}$$

here ($a, \chi, \mathbf{A}$) are an arbitrary constant, a scalar function $\chi(x, y)$ and a vector function $\mathbf{A}(x, y)$. If one substitutes the expression (20) in Eq. (18), then it will be solved if vectors $\mathbf{A}$ and $\mathbf{u}$ satisfy the equation

$$a\lambda_{44}\Delta\mathbf{A} + C_{44}\Delta\mathbf{u} = 0, \tag{21}$$

and the scalar function χ is a solution of the Poisson equation:

$$\Delta\chi = \frac{1}{\lambda_{12} + 2\lambda_{44}}\left[s\Phi(u_s) - a(\lambda_{12} + \lambda_{44})\mathrm{div}\mathbf{A} - (C_{12} + C_{44})\mathrm{div}\mathbf{u}\right]. \tag{22}$$

After substitution Eq. (20) in Eq. (19), we can see that this equation will be solved if

$$\mathrm{div}\mathbf{A} = \mathrm{div}\mathbf{u}, \ \ a = \frac{k_{12} + k_{44} - C(C_{12} + C_{44})}{(\lambda_{12} + \lambda_{44})C - (C_{12} + C_{44})}, \ \ C = \frac{C_{12} + 2C_{44}}{\lambda_{12} + 2\lambda_{44}}, \tag{23}$$

$$k_{44}\Delta\mathbf{u} + aC_{44}\Delta\mathbf{A} + sC\,\mathrm{grad}\,\Phi(u_s) - \mathbf{B}(p - s\,\mathrm{div}\mathbf{U})\sin u_s = 0. \tag{24}$$

One can exclude $\Delta\mathbf{A}$ from Eq. (24) with help (21). Then the components of microshift vector (u_x, u_y) are the solutions of the equations:

$$K\Delta u_x + sC\frac{b^2}{2}\sin u_s\frac{\partial u_s}{\partial x} - \frac{b}{2}(p - s\mathrm{div}\mathbf{U})\sin u_s = 0, \tag{25}$$

$$K\Delta u_y + sC\frac{b^2}{2}\sin u_s\frac{\partial u_s}{\partial y} - \frac{b}{2}(p - s\mathrm{div}\mathbf{U})\sin u_s = 0, \tag{26}$$

$$K = \frac{b^2}{2}\left(k_{44} - \frac{C_{44}^2}{\lambda_{44}}\right).$$

Eqs. (25), (26) can be transformed to more simple form. If to summarize (25) and (26) and to express $\mathrm{div}\mathbf{U}$ from $\mathrm{div}\mathbf{u}$ and $\Phi(u_s)$ using Eqs. (20), (22) and (23) then we find the equation for u_s:

$$K\Delta u_s = P\sin u_s, \tag{27}$$

$$P = p - \frac{s}{\lambda_{12}+2\lambda_{44}}\left[\mathrm{div}\mathbf{u}_+ + s\Phi(u_s)\right],$$

$$\mathbf{u}_+ = (a\lambda_{44} - C_{12} - C_{44})\mathbf{u} + \frac{b^2}{2}(C_{12}+2C_{44})\mathbf{B}u_s.$$

If to subtract (26) from (25) then one has the equation for $u_m = (u_x - u_y)/b$:

$$\Delta u_m + f = 0, \tag{28}$$

$$f = sC\frac{b}{2K}\left(\frac{\partial u_s}{\partial x} - \frac{\partial u_s}{\partial y}\right)\sin u_s = sC\frac{b^2}{2K}\mathbf{k}\,\mathrm{rot}[\mathbf{B}\Phi(u_s)].$$

So the nonlinear model is reduced to the equation (27). This is the sine-Gordon equation. It differs from classical in the fact that coefficient before a sine (amplitude) is a variable. If function u_s is found, then u_m is a solution of Poisson equation as it follows from (28). The values of u_s and u_m unambiguously define a vector of microshifts $\mathbf{u}$.

Vector of the macroshifts $\mathbf{U}$ is calculated by formula (20). Vector $\mathbf{A}$ is found from (21) and function χ is found from the Poisson equation:

$$\Delta\chi = \frac{1}{\lambda_{12}+2\lambda_{44}}\left\{s\Phi(u_s) - [(C_{12}+C_{44}) + a(\lambda_{12}+\lambda_{44})]\mathrm{div}\mathbf{u}\right\}. \tag{29}$$

One can find total solution for macroshifts vector $\mathbf{U}$ and stress tensor σ_{ik} in the form of complex variable functions.

3.2. Complex representation of the general solution for macrofield equations. Instead of material variables (x, y) we will introduce complex variables $z = x + iy$, $\overline{z} = x - iy$. The equation of statics (18) in the new variables will take form:

$$\frac{\partial}{\partial z}\left[2\lambda_{44}\frac{\partial\overline{W}}{\partial\overline{z}} + (\lambda_{12}+\lambda_{44})\mathrm{div}\mathbf{U} + 2C_{44}\frac{\partial\overline{w}}{\partial\overline{z}} + (C_{12}+C_{44})\mathrm{div}\mathbf{u} - s\Phi(u_s)\right] = 0. \tag{30}$$

Here

$$W = U_x + iU_y, \quad w = u_x + iu_y, \quad \mathrm{div}\mathbf{U} = \frac{\partial W}{\partial z} + \frac{\partial\overline{W}}{\partial\overline{z}}, \quad \mathrm{div}\mathbf{u} = \frac{\partial w}{\partial z} + \frac{\partial\overline{w}}{\partial\overline{z}},$$

and the line from above designates complex conjugation. After integration of the Eq. (30) in z variable, we will find:

$$2\mu\frac{\partial\overline{W}}{\partial\overline{z}} + (\lambda+\mu)\mathrm{div}\mathbf{U} + 2C_{44}\frac{\partial\overline{w}}{\partial\overline{z}} + (C_{12}+C_{44})\mathrm{div}\mathbf{u} - s\Phi(u_s) = (1+\chi)\overline{\phi'(z)}. \tag{31}$$

In formula (31) new designations for modules of macroelasticity were introduced

$$\lambda_{44} = \mu, \qquad \lambda_{12} = \lambda,$$

and arbitrary harmonious function of $\overline{z}$ in the right part is presented in form of derivative of arbitrary analytical function with a multiplier $(1+\chi)$, where $\chi = (\lambda+3\mu)/(\lambda+\mu)$. So the solutions of nonlinear model for case $\mathbf{u} = 0$ are transformed into Muskhelishvili's formulas of the classical theory of elasticity.

The complex conjugate equation for Eq. (31) is:

$$2\mu\frac{\partial W}{\partial z} + (\lambda+\mu)\mathrm{div}\mathbf{U} + 2C_{44}\frac{\partial w}{\partial z} + (C_{12}+C_{44})\mathrm{div}\mathbf{u} - s\Phi(u_s) = (1+\chi)\phi'(z). \tag{32}$$

If to summarize the equations (31) and (32), then we will find:

$$\operatorname{div}\mathbf{U} = \frac{1}{\lambda+2\mu}\left[s\Phi(u_s) + \frac{1+\chi}{2}\left(\phi'(z) + \overline{\phi'(z)}\right) - (C_{12}+C_{44})\operatorname{div}\mathbf{u}\right]. \tag{33}$$

We exclude $\operatorname{div}\mathbf{U}$ from Eq. (32) with help (33) and find the equation for W:

$$2\mu\frac{\partial W}{\partial z} + 2C_{44}\frac{\partial w}{\partial z} = \chi\phi'(z) - \overline{\phi'(z)} + Q(z,\overline{z}), \tag{34}$$

$$Q(z,\overline{z}) = \frac{\chi-1}{\chi+1}s\Phi(u_s) + \left[\frac{1-\chi}{1+\chi}(C_{12}+C_{44}) - C_{44}\right]\operatorname{div}\mathbf{u}.$$

From this equation we will find:

$$2\mu W + 2C_{44}w = \chi\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} - \frac{\chi-1}{\chi+1}\cdot\frac{1}{\pi}\iint_S \frac{Q(\xi,\eta)}{\overline{\zeta}-\overline{z}}d\xi d\eta. \tag{35}$$

Here $\zeta = \xi + i\eta$, $\psi(z)$ is arbitrary analitical function and S is the domain of integration. The formula (35) expresses the vector of macroshifts (U_x, U_y) through harmonious functions. We can see that $Q(\xi,\eta)$ plays a role of volume sources for the field of macroshifts.

Through harmonic functions it is possible to express also the tensor of macrostresses σ_{ik} if to write the ratio (13) in complex variables $z,\overline{z})$:

$$\begin{aligned} &\sigma_{xx}+\sigma_{yy} = 2(\lambda+\mu)\left(\frac{\partial W}{\partial z}+\frac{\partial\overline{W}}{\partial\overline{z}}\right) + (C_{11}+C_{12})\left(\frac{\partial w}{\partial z}+\frac{\partial\overline{w}}{\partial\overline{z}}\right) - 2s\Phi(u_s),\\ &\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = -\mu\frac{\partial\overline{W}}{\partial z} - (C_{11}-C_{12}+2C_{44})\frac{\partial\overline{w}}{\partial z}. \end{aligned} \tag{36}$$

Solutions (35), (36) trasform in Muskhelishvili's formulas if $\mathbf{u}=0$, i.e. when the optical mode isn't excited.

3.3. Solutions of the equations of optical mode and structures of microdeformation corresponding to them. In literature there are no analytical methods for solution of sine-Gordon (SG) equation with a variable amplitude. Functionally invariant solutions of the (2+1)- and (3+1)-dimensional SG equations are constructed for a wide, but specific type of amplitudes in [10]–[12]. The Eqs.(25) - (26) can be reduced to well studied cases if to make some restrictions for model or for the field of microdeformations. So, if not to consider dependence of potential of interaction of sublattices on deformation of the medium, i.e. to accept $s=0$, then Eq. (28) becomes the SG equation with constant coefficients (K, p)

$$K\Delta u_s = p\sin u_s. \tag{37}$$

Also it will be if

$$\operatorname{div}\mathbf{u}_+ + \frac{s}{\lambda_{12}+2\lambda_{44}}\Phi(u_s) = 0. \tag{38}$$

In literature the method of the solution of the SG equation (37) based on substitution

$$u_s = 4\operatorname{arctg}(G(x,y)), \quad G(x,y) = \Phi_1(x)\Phi_2(y), \tag{39}$$

is widely known. The solution (39) assumes that $K>0$. If $K<0$, then solution of Eq. (37) is

$$u_s = \pi + 4\operatorname{arctg}(G(x,y)). \tag{40}$$

The solution (39) is connected with G. L. Lamb Jr. [13], though the first time it was used by Steuerwald [14].

The functions $\Phi_1(x)$ and $\Phi_2(y)$ can be found by inversion of the corresponding elliptic integrals by Legendre's method. The Legendre method is rather difficult. In [15] the

method of finding of functions $\Phi_1(x)$ and $\Phi_2(y)$ is based on the differential equations which satisfy elliptic functions of Jacobi (the modified Lamb method). This offered approach allows to receive a wide class of doubly periodic solutions of the SG equation. Two solutions from this class are given below

$$1)\ A\frac{\mathrm{cn}(\xi,\nu_1)}{\mathrm{cn}(\eta,\nu_2)},\quad \begin{aligned} H^2 &= \frac{\nu_1^2(1+A^2)\left[(1+A^2)\nu_2^2-1\right]}{A^4\nu_2^2}\mathrm{K}^2(\nu_1),\\ B^2 &= \frac{(1+A^2)\left[(1+A^2)\nu_2^2-1\right]}{A^2}\mathrm{K}^2(\nu_2),\\ A^4 &= \frac{\nu_1^2(1-\nu_2^2)}{\nu_2^2(1-\nu_1^2)}, \end{aligned} \tag{41}$$

$$2)\ \frac{A}{\mathrm{cn}(\xi,\nu_1)}\frac{\mathrm{sn}(\eta,\nu_2)}{\mathrm{dn}(\eta,\nu_2)},\quad \begin{aligned} H^2 &= \frac{(1-\nu_1^2)(\nu_2^2-A^2)(A^2+1-\nu_2^2)}{A^4}\mathrm{K}^2(\nu_1),\\ B^2 &= \frac{(\nu_2^2-A^2)(A^2+1-\nu_2^2)}{A^2}\mathrm{K}^2(\nu_2),\\ A^4 &= \frac{\nu_2^2(1-\nu_1^2)(1-\nu_2^2)}{\nu_1^2},\quad \nu_1^2+\nu_2^2>1, \end{aligned} \tag{42}$$

In examples (41), (42) values $\mathrm{K}(\nu_1)$, $\mathrm{K}(\nu_2)$ are full elliptic integrals of the first sort and variables are:

$$\xi = \frac{x}{lH}\mathrm{K}(\nu_1),\ \eta = \frac{y}{lB}\mathrm{K}(\nu_2),\ l = \sqrt{K/p}. \tag{43}$$

In Fig. 1 and Fig. 2 microdeformations corresponding to the solutions (41) and (42) are shown.

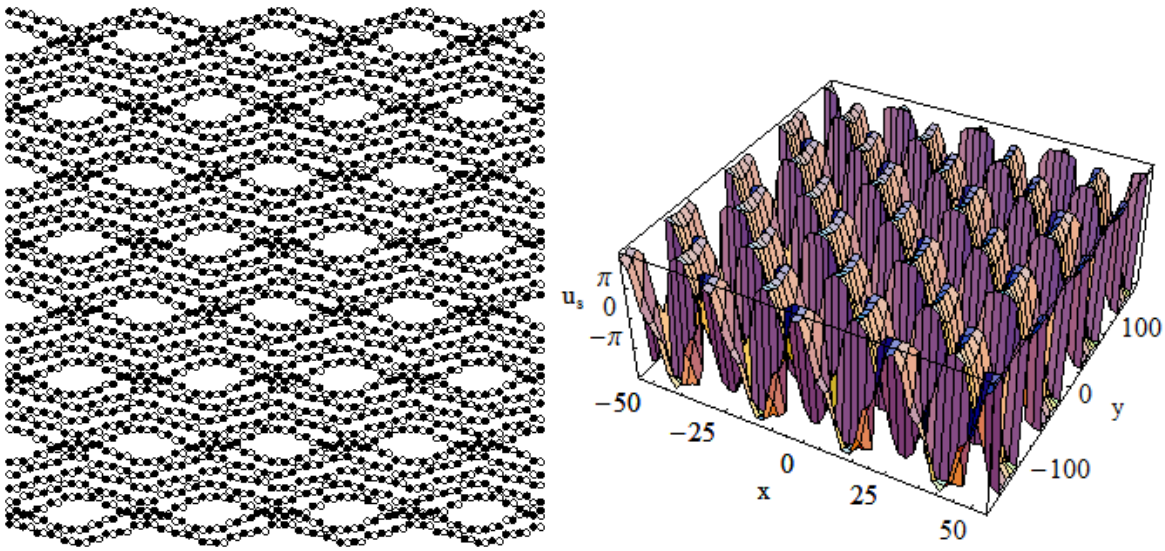

Fig. 1. Microdeformations of crystal and solution (41) for $\nu_1 = 0.99999$, $\nu_2 = 0.9$.

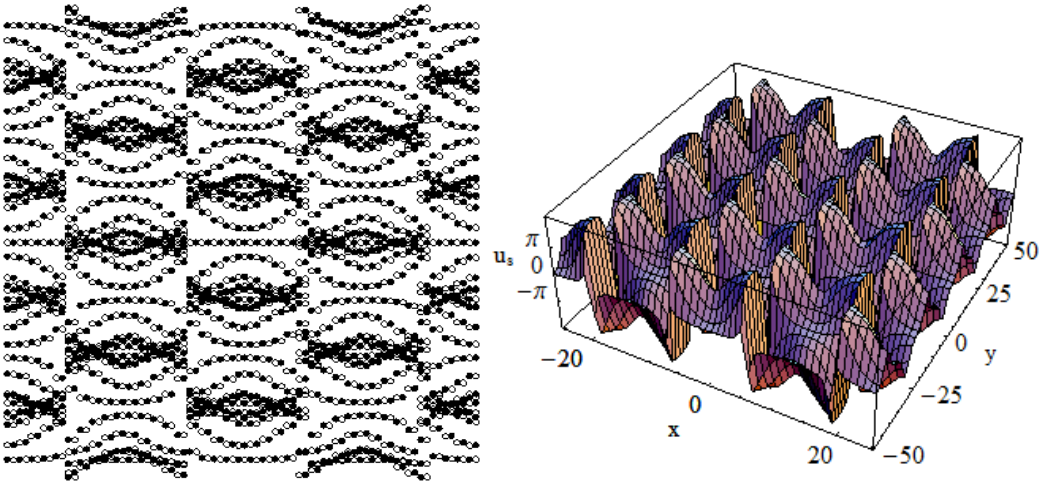

Fig. 2. Microdeformations of crystal and solution (42) for $\nu_1 = \nu_2 = 0.999$.

As we can see, the solution (41) describes creating in crystal lattice system of regularly located micropores, and (42) describes system of regularly located microconsolidations.

The modified method allows to construct solutions which are expressed through circular or hyperbolic functions from solutions which are expressed through elliptic functions of Jacobi. It can be done, if to use the known limit ratios

$$\begin{aligned} &1. \quad \nu \to 0,\ \ \mathrm{sn}(u,\nu) \to \sin u,\ \mathrm{cn}(u,\nu) \to \cos u,\ \ \mathrm{dn}(u,\nu) \to 1, \\ &2. \quad \nu \to 1,\ \ \mathrm{sn}(u,\nu) \to \mathrm{th}u,\ \ \mathrm{cn}(u,\nu) \to \frac{1}{\mathrm{ch}u},\ \ \mathrm{dn}(u,\nu) \to \frac{1}{\mathrm{ch}u}. \end{aligned} \tag{44}$$

Using (44), one finds:

$$G = \begin{cases} \mathrm{tg}\psi \dfrac{\mathrm{sh}(x\cos\psi)}{\mathrm{sh}(y\sin\psi)} \\ \mathrm{tg}\psi \dfrac{\mathrm{ch}(x\cos\psi)}{\mathrm{ch}(y\sin\psi)}, \\ \dfrac{1}{\mathrm{th}\psi} \dfrac{\cos(x\mathrm{sh}\psi)}{\mathrm{sh}(y\cosh\psi)}, \end{cases} \tag{45}$$

where ψ is an arbitrary constant.

In Fig. 3 – 5 microdeformations corresponding to the solutions (45) are shown.

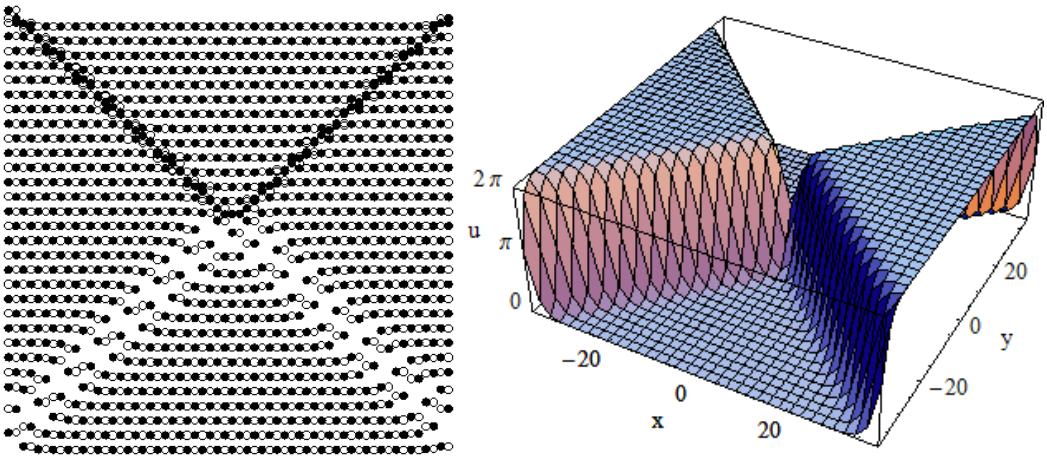

Fig. 3. Microdeformations and solutions for the first case in (45) with $\psi = \pi / 4$.

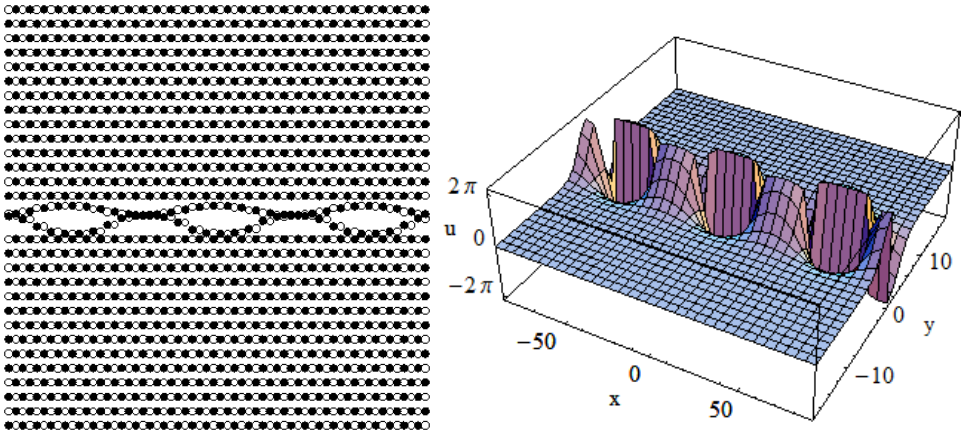

Fig. 4. Microdeformations and solutions for the second case in (45) with $\psi = \pi / 4$.

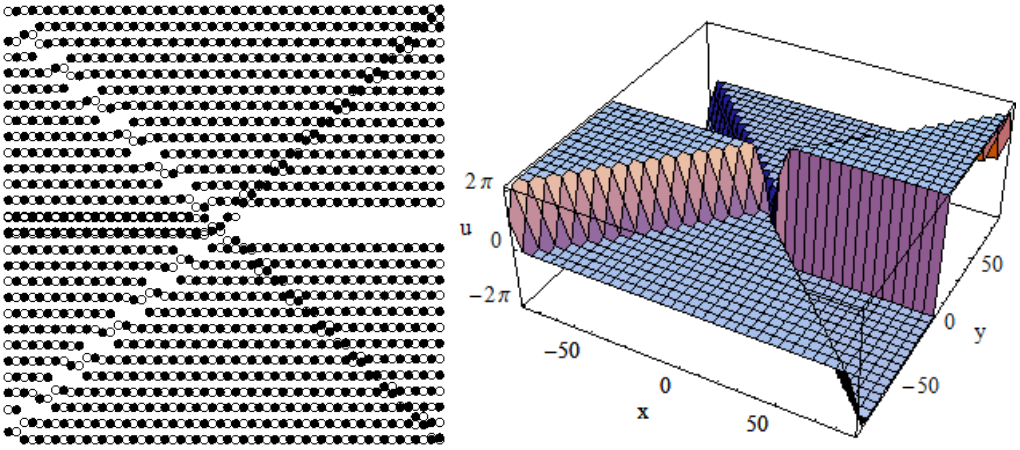

Fig. 5. Microdeformations and solutions for third case in (45) with $\psi = 0.1$.

For the first solution the plane of $y = 0$ is the plane with defects. On half-plane $x > 0$ defect like the main crack is formed, half-plane $x < 0$ contains the defects caused by introduction of the excess crystal planes. It is visible that defects are also the inclined planes. For the second solution the plane of $y = 0$ is not the singular plane. The third solution (45) describes system of micropores which are located in $y = 0$ plane.

The given examples show that the nonlinear model describes features of deformation which are implemented in the field of big external tension, are observed on experience and not described by classical mechanics of the continuous medium.

4. Conclusion

The plane deformation plays an extremely important role in classical mechanics of continuous medium. It is caused by the fact that effective analytical and numerical methods are developed for the solution of problems of plane deformation. The problems of plane deformation can be reduced to boundary problems of theory of the functions of complex variable. It allowed for their solution to apply both methods of classical mathematical physics, and the methods of the theory of functions of complex variable (conformal mappings, Riemann problem, theory of singular integrable equations, etc.) As a result exact analytical solutions of a large number of concrete cases of deformation of one- and multiply connected domains have been found [16]. The found solutions have formed a scientific basis of modern materials science.

However the linear classical model of the continuous medium doesn't meet inquiries of

modern technologies of materials with internal structure. It doesn't describe processes of deformation and specific nanoscale changes (defects, phase transformations, fragmentation, etc.) which are implemented in the field of intensive plastic deformations. More adequately these processes can be described by the nonlinear theory of deformation of crystal media with a complex lattice. However, the development of mathematical methods for realization of the nonlinear model is necessary so that it becomes a scientific basis for engineering calculations in modern technologies.

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References

[1] E.L. Aero, Microscale deformations in a two-dimensional lattice: Structural transitions and bifurcations at critical shear // *Phys. Solid State* **42** (2000) 1147–1153.
[2] E.L. Aero, Significantly nonlinear micromechanics of the medium with changeable periodic structure // *Uspekhi Mekhaniki* **1** (2002) 130–176.
[3] E.L. Aero, A.N. Bulygin, Yu.V. Pavlov, Nonlinear model of deformation of crystal media with complex lattice: mathematical methods of model implementation // *Math. Mech. Solids* **21** (2016) 19–36.
[4] J. Frenkel, T. Kontorova, On the theory of plastic deformation and twinning // *ZhETF* **8** (1938) 89.
[5] O.M. Braun, Y.S. Kivshar, *The Frenkel-Kontorova Model. Concepts, Methods, and Applications* (Springer, New York, 2004).
[6] A.V. Porubov, E.L. Aero, G.A. Maugin, Two approaches to study essentially nonlinear and dispersive properties of the internal structure of materials // *Phys. Rev. E* **79** (2009) 046608.
[7] M.P. Shaskol'skaya, *Crystallography* (Vysshaya Shkola, Moscow, 1984). (In Russian).
[8] W. Voigt, *Lehrbuch der Kristalphysik* (Leipzig, 1910).
[9] C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1956).
[10] E.L. Aero, A.N. Bulygin, Yu.V. Pavlov, New approach to solution of sine-Gordon equation with variable amplitude // *Proc. Int. Conf. "Days on Diffraction 2010"*, p. 10–15.
[11] E.L. Aero, A.N. Bulygin, Yu.V. Pavlov, New approach to the solution of the classical sine–Gordon equation and its generalizations // *Differential Equations* **47** (2011) 1442–1452.
[12] E.L. Aero, A.N. Bulygin, Yu.V. Pavlov, Solutions of sine-Gordon equation with variable amplitude // *Theor. Math. Phys.* **184** (2015) 961–972.
[13] G.L. Lamb, Jr., Analytical Descriptions of Ultrashort Optical Pulse Propagation in a Resonant Medium // *Rev. Mod. Phys.* **43** (1971) 99–124.
[14] R. Steuerwald, Über Enneper'sche Flächen und Bäcklund'sche Transformationen // *Abh. Bayer. Akad. Wiss., N. F.* **40** (1936) 1–105.
[15] E.L. Aero, A.N. Bulygin, Yu.V. Pavlov, Mathematical methods for solution of nonlinear model of deformation of crystal media with complex lattice // *Proc. Int. Conf. "Days on Diffraction 2015"*, p. 8–13.
[16] N.I. Muskhelishvili, *Some base problems of the theory of elasticity* (Nauka, Moscow, 1966). (In Russian).