Rev.Adv.Mater.Sci. 19(2009) 93-97

ON SOLUTION NON-UNIQUENESS IN THE NONLINEAR ELASTICITY THEORY

A.G. Kulikovskii and A.P. Chugainova

Steklov Mathematical Institute, Russian Academy of Sciences, Gubkina st. 8, Moscow, 119991, Russia

Received: August 04,2008

Abstract. The simultaneous effects of dissipation and dispersion on nonlinear wave behavior in elastic media are considered when the effects are small and manifested only in narrow high-gradient regions. If one constructs solutions of self-similar problems in “hyperbolic” approximation using Riemann’s waves and admissible discontinuities (i.e., discontinuities with structures) one obtains many solutions the number of which unlimitedly grows with growing the relative influence of dispersion (as compared to dissipation) in discontinuity structures.
The numerical analysis (based on PDE with dispersion and dissipation) of nonself-similar problems with self-similar asymptotics is performed to determine which of self-similar solutions is an asymptotic form for the nonself-similar solution.

1. INTRODUCTION

The equations of nonlinear elasticity theory are hyperbolic equations expressing the conservation laws. In solutions to these equations discontinuities appear and this is a possible source of solution nonuniqueness. The conservation laws should be satisfied on discontinuities (when there are no external surface effects). It is well known that not all discontinuities which don’t contradict to the conservation laws can be considered as really existing.

We refer to a hyperbolic system of equations together with a particular set of discontinuities (called admissible) as a hyperbolic model. In a reasonable model a set of admissible discontinuities should be given in such a way that it should involve discontinuities which one could consider as realizable under the conditions for which the model was defined. There are widely accepted rules to choose admissible discontinuities: the entropy nondecreasing, *a priori* evolutionness conditions (Lax conditions) and the requirement for a stationary discontinuity structure to exist. The last condition leads to a dependence of the rule to select admissible discontinuities on assumptions concerned with the processes in discontinuity structures.

2. NONLINEAR QUASI-TRANSVERSE SMALL AMPLITUDE WAVES IN ELASTIC MEDIA

One-dimensional unsteady solutions (depending on cartesian coordinate x and time t) to nonlinear elasticity problem are considered. There are three families of characteristics along which perturbations propagate to each of sides. If nonlinearity and anisotropy in planes $x = const$ are small the characteristic velocities c_1 and c_2 corresponding to two quasi-transverse waves appear to be close. To describe small amplitude quasi-transverse waves propagating through the uniform state in positive x–direction the system of equations was obtained [1,2]

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha}\right) = 0, \quad \alpha = 1,2 \qquad (1)$$

Corresponding author: A.G. Kulikovskii, e-mail: kulik@mi.ras.ru

$$R=\frac{1}{2}f\left(u_1^2+u_2^2\right)+\frac{1}{2}g\left(u_1^2-u_2^2\right)-\frac{1}{4}\kappa\left(u_1^2+u_2^2\right)^2,$$

$f, g, \kappa = \text{const.}$

Here $u_\alpha = dw_\alpha/dx$; w_α – the displacements in directions normal to the x – axis, x – the lagrangian coordinate, f – the characteristic velocity in the absence of nonlinearity and anisotropy, g – the anisotropy parameter, κ – the constant to characterize nonlinearity. The function $R(u_1,u_2)$ is determined by medium elastic properties. This expression corresponds to the general case when u_1,u_2 and anisotropy are small.

The conditions on a discontinuities corresponding to system (1) and following from the conservation laws for the transverse momentum are of the form

$$W[u_\alpha]=\left[\frac{\partial R}{\partial u_\alpha}\right], \qquad (2)$$

W is the Lagrangian velocity of the discontinuity.

Square brackets denote a jump of a function. A set of states u_1,u_2 behind various possible discontinuities propagating through a given state is called a shock adiabatic curve. Such a curve is shown in Fig. 1. The state in front of discontinuities is represented by point A.

3. THE HYPERBOLIC MODEL *NO* 1

Consider a model based on hyperbolic system (1) and a set of discontinuities for which in the framework of the Voight model a stationary viscous structures exist. The equations for quasi-transverse waves are written as follows

$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial u_\alpha}\right)=\mu\frac{\partial^2 u_\alpha}{\partial x^2},\quad \mu=\text{const.} \qquad (3)$$

Eqs. (3) differ from Eqs. (1) by terms on the right hand side. The discontinuity with a structure is looked for in the form of a traveling wave, for which for which $u_\alpha = u_\alpha(\zeta)$; $\zeta = -x + Wt$ take values corresponding to states behind and in front of the discontinuity as $\zeta\to\pm\infty$. It is shown [2,3] that the requirement for a viscous stationary structure to exist is equivalent to the requirement of discontinuity evolutionness with boundary conditions (2) (Lax conditions). This means that one of the following inequality systems should be satisfied

$$c_1^- \le W \le c_2^-,\quad 0\le W < c_1^+, \qquad (4)$$

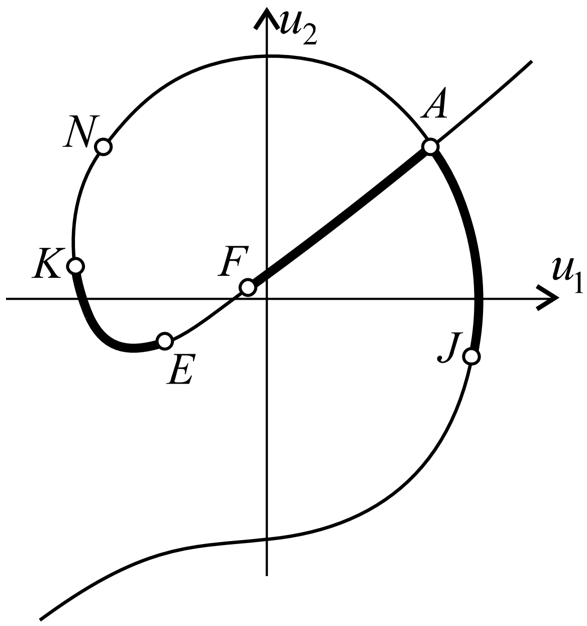

Fig. 1

$$c_2^- \le W,\quad c_1^+ \le W \le c_2^+. \qquad (5)$$

Here c_1 and c_2 are the characteristic velocities of system (3), superscripts "+" and "–" are referred to the states behind and in front of discontinuity, respectively, and W is the discontinuity velocity. In Fig. 1 the intervals AJ and EK of the shock adiabatic curve correspond to states behind fast shock waves (inequalities (5)) while the interval AF corresponds to states behind slow shock waves (inequalities (4)).

The set of discontinuities involving fast and slow shock waves together with Eqs. (1) for continuous solutions form the hyperbolic model *No* 1.

3.1. The problem of arbitrary discontinuity disintegration (the Riemann problem).

The typical problem for hyperbolic equations is a problem of arbitrary discontinuity disintegration. Initial data for $t = 0$ are given in the form $u_\alpha = U_\alpha$ for $x \ge 0$ and $u_\alpha = u^*_\alpha$ for $x \le 0$. One should find the solution of the form $u_\alpha = u_\alpha(x/t)$ involving admissible discontinuities and continuous waves (the Riemann waves).

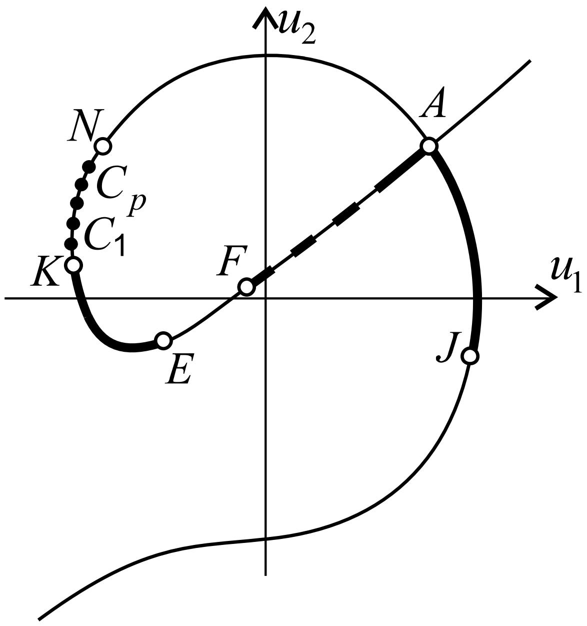

Fig. 2

3.2. Solution nonuniqueness for the model No 1.

It is shown that the region of parameter values U_α; u^*_α exists for which the model *No* 1 makes it possible to construct more than one solutions of the problem of arbitrary discontinuity desientegration [2,4,5].

3.3. Nonself-similar problems with self-similar asymptotics for a viscous elastic media.

For system (3) the problem with initial data $u^0_\alpha(x)$ in the form of continuous functions tending to U_a as $x\to+\infty$ and to u^*_α as $x\to-\infty$ is analyzed. For high values of time the size l of a region where initial data differ essentially from constant values appears to be small as compared to the region size L where perturbations propagate. Therefore the solution considered as a function of x/t and t should tend to a self-similar limit determined by the hyperbolic model.

The above formulated nonself-similar problem was numerically analyzed [6,7]. It is shown that when the model *No* 1 has a nonunique solution the nonself-similar solutions tends to one or other asymptotic depending on the form of functions $u^0_\alpha(x)$ which smooth the discontinuity in initial conditions. The dependence of the arising self-similar asymptotics on the choice of these functions was qualitatively analyzed.

4. THE HYPERBOLIC MODEL *NO* 2

A set of admissible discontinuities is defined as a set of discontinuities with a stationary structure described by the system of equations [8]

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial R}{\partial u_1}\right) = \mu \dfrac{\partial^2 u_1}{\partial x^2} + m \dfrac{\partial^2 u_2}{\partial x^2} \\ \dfrac{\partial u_2}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{\partial R}{\partial u_2}\right) = -m \dfrac{\partial^2 u_1}{\partial x^2} + \mu \dfrac{\partial^2 u_2}{\partial x^2} \end{cases}, \qquad (6)$$

$\mu, m = \text{const}.$

The so defined set of admissible discontinuities together with Eqs. (1) from hyperbolic model *No* 2.

This system of equations describes quasi-transverse waves for some types of elastic composites in the case of viscous dissipation. Terms with the coefficient m simulate wave dispersion. If the value m/μ is rather high the discontinuity structure is of oscillatory form. The admissible discontinuity set on the shock adiabatic curve takes the form shown in Fig. 2 by solid intervals and separate points. The appearance of separate points C_i corresponding to special discontinuities with discrete set of discontinuity velocities W_i, $i = 1, \ldots, n$ is an essential peculiarity of the discontinuity set when dispersion effects in structures are taken into account. The velocities W_i are such that the special discontinuities don't catch up fast waves (continuous waves or shocks), but no slow waves catch them up. The number of various types of special discontinuities and the number of separate intervals on the interval *AF* of the shock adiabatic curve (corresponding to admissible shocks) depends on the ratio m/μ and grows unlimitedly with the growth of m/μ.

4.1. Solution nonuniqueness for the model No 2.

For given states to the right and to the left of the initial discontinuity there is a possibility to construct solutions involving fast and slow waves (continuous waves or shocks), and between them a special discontinuity corresponding to any of points C_i ($i = 1, \ldots, n$) can be situated. Besides, in the solution several special discontinuities can follow each other. This means that the number of possible solutions of the problem depends on m/μ and grows unlimitedly when the ratio m/μ grows.

4.2. Nonself-similar problems with self-similar asymptotics for viscous elastic media with dispersion.

Problems with initial conditions in the form of a step smoothed in a various way were numerically analyzed [8]. It was shown that by choosing way to smooth one could obtain any of possible asymptotics described in point 4.1. For the same conditions at infinity the solutions can tend to different asymptotics for various smoothing functions in initial data.

Qualitative conclusions were made on conditions for nonself-similar solutions to tend to one or other asymptotic when time growing.

5. THE HYPERBOLIC MODEL *NO* 3

Longitudinal nonlinear waves in rods with complicated nonlinearity are considered. The equation of small amplitude long waves propagating in the positive x–direction can be written as follows

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial t} = 0,\ \ c(u) = \frac{\partial \varphi(u)}{\partial u}. \qquad (7)$$

The function $\varphi(u)$ is determined by the dependence of a rod elongation on its tension and the plot of the function is assumed to have two inflection points. The hyperbolic model *No* 3 involves Eqs. (7) for continuous solutions and a set of discontinuities with stationary structures described by equation [9]

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} - m\frac{\partial^3 u}{\partial x^3}. \qquad (8)$$

Here $u = \partial w/\partial x, w$ - the longitudinal displacement of rod points, μ and m are the viscosity and dispersion coefficients, respectively. Dispersion is a result of non-onedimensional motions of rod points. Previously this model was considered for the case without dispersion ($m = 0$) [10].

Eq. (8) (as well as (7)) expresses the conservation law, therefore the relation on a discontinuity can be written in the form

$$W = \frac{[\varphi(u)]}{[u]}. \qquad (9)$$

Here W is the discontinuity velocity, square brackets denote a difference between values of functions behind and in front of the discontinuity.

A complicate nonlinearity provides the existence of three different discontinuities propagating through the same state at the same velocity. If effects of small scale dispersion are essential (as compared to viscous effects) the stationary discontinuity structure becomes to be oscillatory. As in the case of quasi-transverse waves the set of admissible discontinuities becomes complicate and special discontinuities appear. This leads to multiple nonuniqueness of solutions in the hyperbolic model.

5.1. Nonself-similar solutions with self-similar asymptotic described by the hyperbolic model No 3.

Numerical solutions to Eq. (8) with initial data in the form of a smoothed step shows [11] that one can choose a smoothing function in such a way that the solution tends to asymptotic represented by any of self-similar solutions of the hyperbolic model. However, if initial data are monotonically smoothed the asymptotic arises which involves the special discontinuity with the simplest structure in all cases when the condition at infinity admits such a self-similar solution. But a self-similar solution with such a special discontinuity exists not for any conditions for $x \to +\infty$. In the cases when there is no such a self-similar solution, there are self-similar solutions involving special discontinuities with more complicate structures. In these cases when solving nonself-similar problems for Eq. (8) with initial data in the form of monotonically smoothed step we obtain quite different asymptotic involving the single shock wave with unsteady structure inside of which periodic nondecaying in time oscillations occur. Discontinuities with such structures should be included into the set of admissible discontinuities.

6. CONCLUSIONS

It is shown that the hyperbolic models can be put in correspondence to complicate equations with dissipation and dispersion taken into account (Eqs. (3), (6), (8)). In this case one should often (but not always – see point 5a) choose discontinuities with stationary structures as admissible ones. Hyperbolic models make it possible to construct self-similar solutions representing asymptotics to solutions of original complete equations. In many cases such solutions appear to be nonunique, the nonuniqueness being multiple when effects of dispersion are more essential then effects of dissipation. For appropriate initial conditions these solutions are stable asymptotics to solutions of original complicate equations. To which of self-similar solutions of a hyperbolic model the solutions of complete equations tend depends on such details of the problem formulation which are missing from hyperbolic models.

ACKNOWLEDGEMENTS

This research is supported partly by the Russian Foundation for Basic Research (08-01-00618) and by the programs "Nonlinear dynamics" of Russian Academy of Sciences and "Leading Scientific Schools" of Russian Federation (the grant NSh-1959.2008.1).

REFERENCES

[1] A.G.Kulikovskii // *J. Appl. Maths. Mechs.* **50** (1986) 455.

[2] A.G. Kulikovskii and E.I. Sveshnikova, *Nonlinear waves in elastic media*(Boca Raton, New York, London, Tokyo: CRC Press, 1995).

[3] A.G.Kulikovskii and E.I.Sveshnikova // *J. Appl. Maths. Mechs.* **51** (1987) 711.

[4] A.G.Kulikovskii, E.I.Sveshnikova // *J. Appl. Maths. Mechs.* **49** (1985) 214.

[5] A. G. Kulikovskii, A. P. Chugainova and E. I. Sveshnikova // *J. Engrg. Math.* **55** (2006) 97.

[6] A.P.Chugainova // *J. Appl. Maths. Mechs.* **52** (1988) 541.

[7] A.P.Chugainova // *J. Appl. Maths. Mechs.* **57** (1993) 375.

[8] A. P. Chugauinova // *Teoret. Mat. Fiz.* **147** (2006) 240, In Russian; translated in: *Theoret. and Math. Phys.* **147** (2006) 646.

[9] A.G.Kulikovskii // *Soviet Physics Doklady* **275** (1984) 1349.

[10] O.A.Oleinik // *USSR Mathematical Surveys* **14** (1959) 59.

[11] A. G. Kulikovskii and A. P. Chugauinova // *Zh. Vychisl. Mat. Mat. Fiz.* **44** (2004) 1119, In Russian); translated in: *Comput. Math. Math. Phys.* **44** (2004) 6.