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PROPAGATION TECHNIQUE FOR ULTRASHORT PULSES

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Abstract. In this paper, a general propagation equation of ultrashort pulses in an arbitrary dispersive nonlinear medium is derived using a method based on a consistent and mathematically rigorous expansion of the linear dispersion relation including the medium's nonlinear optical response. A specific case of Kerr media is studied. The obtained ultrashort pulse propagation equation which is called Generalized Nonlinear Schrödinger Equation has a very complicated form and looking for its solutions is usually a very difficult task. Theoretical methods to solve this equation are effective only for some special cases. For this reason, several numerical methods of finding approximate solutions are used. We focus mainly on the methods: Split-Step and Runge-Kutta algorithms. Some numerical experiments are implemented for soliton propagation and interacting high order solitons. These algorithms in connection with the variational method give several interesting results concerning solitons in optical nonlinear media. The results from nonlinear optics to atom optics and vice versa could be transferred by analogy between the propagation equation in the Kerr medium and the Gross-Pitajevski equation for Bose-Einstein condensates (BECs).

1. INTRODUCTION

Creation and propagation of ultrashort laser pulses (in fs) in a medium has been intensively researched (both theoretically and experimentally) in the course of the last few years [1-4]. The briefest propagating electromagnetic pulse is a single cycle. One cycle at the wavelength of 800 nm is approximately two fs long. Modern lasers can generate pulses as short as a few optical cycles, with durations of the order of 10-15 seconds. Brief events are generally probed by briefer ones. The short duration of these pulses allows us to look at very fast events, such as vibrating molecules, or charge transfer in biological systems. The shape of the pulse can be also manipulated and used to precisely control the quantum phenomena such as the formation of molecules from cold atoms (noncrystalline structure), or the initiation of a quantum phase transition in a solid. An ultrashort pulse could be used as a photonic reagent in different chemical reactions. A short pulse with a large energy focused by lens gives a very high peak intensity which leads to several potential applications as in creation of unusual states of matter (plasmas) by reaching very high temperatures, or using it as an energy source for X-ray lasers...

Due to some special properties of ultrashort pulses during their propagation in the medium, several new effects have been observed in comparison with the propagation process of short pulses (in ps), namely the effects of dispersion and nonlinear effects of higher orders. Under the influence of these effects, we have complicated changes both in the amplitude and the spectrum of the pulse. It splits into constituents and its spectrum also evolves into several bands which are known as optical shock and self-frequency shift phenomena [1,2,5,6]. These effects should be studied in detail for future concrete applications of ultrashort pulses, especially in the domain of optical communication.

Recently, we have developed a powerful method of deriving a general equation for short-duration intense pulses [7-9]. This method is based on a consistent and mathematically rigorous expansion

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of the nonlinear wave equation which treats the nonlinear processes involved in the problem as perturbations. In Sec. II we will present the basis of the method and we will derive from these considerations a general equation for the pulse propagation process in a nonlinear dispersion medium with all orders of dispersion and nonlinearity. Using this procedure for the Kerr medium in the consideration of the medium's delayed nonlinear response, induced by the stimulated Raman scattering and the characteristic features of both the spectrum and the intensity of the pulse, we will obtain an approximate equation in the most condensed form which describes the propagation of ultrashort pulses, called the generalized nonlinear Schrödinger equation (GNLS). In general, it is very difficult to find analytic solutions for this equation. A review of analytic methods is given in [10]. We consider a normalized form of this equation and demonstrate its general features. We will analyze in detail the influence of the third-order dispersion (TOD), the self-steepening and the self-shift frequency for ultrashort pulses in some special cases. When the higher-order terms are included, the pulse propagation equation becomes very complicated [11]. It is when we should use different numerical methods to solve it. In Sec. IV we present two efficient methods, the Split-Step Fourier and the fourth order Runge-Kutta methods. Sec. V is focused on the exchange of experience between nonlinear optics (optical pulse and beam propagation in nonlinear media) and atom optics (dynamics of coherent waves generated from Bose-Einstein condensates). A common ground here is the nonlinear Schrödinger Equation which, with the proper substitution of variables, describes both types of phenomena. In nonlinear optics it is a light propagation equation that relates the signal at the end of the nonlinear medium to the signal at its input face. In Bose-Einstein condensate dynamics it is called the Gross-Pitaevskii equation. We will discuss various types of soliton formation which have realizations both in nonlinear optics and atom optics. Sec. VI contains conclusions.

2. PROPAGATION EQUATION FOR ULTRASHORT PULSES

2.1. General form of the pulse propagation equation in a nonlinear dispersion medium

The Maxwell equations can be used to obtain the following nonlinear wave equation for the electric field [1,9,12,13]

$$\nabla^2\vec{E}(\vec{r},t)-\nabla\left(\nabla.\vec{E}(\vec{r},t)\right)-\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=\mu_0\frac{\partial^2\vec{P}_l(\vec{r},t)}{\partial t^2}+\mu_0\frac{\partial^2\vec{P}_{nl}(\vec{r},t)}{\partial t^2}, \quad (1)$$

where $\vec{P}_l(\vec{r},t)$ and $\vec{P}_{nl}(\vec{r},t)$ are the linear and nonlinear polarization, respectively.

The electric field $\vec{E}$ is treated as a superposition of monochromatic constituents with different frequencies and wavevectors centered at their central values ω_0 and $\vec{k}_0$. We confine ourselves only to consider the electric field propagation in an arbitrary direction, say O*z* (usually chosen as the direction of $\vec{k}_0$), so we can write

$$\vec{E}(\vec{r},t)=\vec{x}.E(z,t)=\frac{1}{2}\vec{x}\left[A(z,t)e^{-i\omega_0 t+ik_0 z}+c.c\right], \quad (2)$$

where $\vec{x}$ is the unit vector of the x axis perpendicular to the propagation direction, $A(z,t)$ is the complex envelope function, *c.c* denotes the complex conjugate of the first term.

The linear polarization vector of the medium for the homogeneous isotropic medium is expressed as follows

$$\vec{P}_l(\vec{r},t)\equiv\vec{P}_l(z,t)=\vec{x}\vec{P}_l(z,t)=\vec{x}\varepsilon_0\int_{-\infty}^{\infty}\left(\chi^{(1)}(t-t')\times E(z,t')\right)dt'=\vec{x}\varepsilon_0\tilde{\chi}^{(1)}*E, \quad (3)$$

where * denotes the convolution product which displays the causality: the response of the medium in time *t* is caused by the electric field action in all previous times *t'*. The quantity $\chi^{(1)}$ is the susceptibility of the medium. It is a scalar.

The nonlinear polarization vector is generally expressed as follows

$$\vec{P}_{nl}(r,t)=\varepsilon_0\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\chi^{(2)}(t-t_1,t-t_2):\vec{E}(\vec{r},t_1)\times\vec{E}(\vec{r},t_2)\right)dt_1\,dt_2+\int_{-\infty}^{\infty}\int\int\left(\chi^{(3)}(t-t_1,t-t_2,t-t_3):\vec{E}(r,t_1)\vec{E}(r,t_2)\vec{E}(r,t_3)\right)dt_1\,dt_2\,dt_3+\ldots\right], \quad (4)$$

where $\chi^{(n)}(t-t_1, t-t_2, \ldots, t-t_n)$ is the n-order nonlinear susceptibility. For the homogeneous isotropic medium, the elements of the even- order nonlinear susceptibility $\chi^{(2k)}(t-t_1, \ldots, t-t_{2k})$ disappear due to the spatial inversion symmetry [1,12,13]. In the

expression (5) we have only the nonlinear polarizations of odd orders. It is only the third-order nonlinear susceptibility (the Kerr medium) that is considered in detail. Then, the tensor $\chi^{(3)}$ has $3^4 = 81$ elements (as a matrix with 3 lines and 27 columns), but only 21 of its elements are different from zero and three are independent [1]. Therefore, we have

$$\vec{P}_{nl}(\vec{r},t) \equiv \vec{P}_{nl}(z,t) = \vec{x}P_{nl}(z,t) = \vec{x}.\varepsilon_0 \int_{-\infty}^{\infty}\int\int \left(\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)\times E(z,t_1)E(z,t_2)E(z,t_3)\right) \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3. \quad (5)$$

In the hierarchy of magnitudes, the nonlinear polarization is much smaller than the electric field and the linear polarization $\left|\vec{P}_{nl}(z,t)\right| << \left|\vec{P}_l(z,t)\right|$, $\left|\vec{P}_{nl}(z,t)\right| << \varepsilon_0\left|\vec{E}(z,t)\right|$, so it can be considered as a perturbation and we have the approximate formula [9]: $\nabla.\vec{E}(z,t) \approx 0$. Substituting these results into (1) we obtain the following scalar wave equation:

$$\Delta E(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E(z,t) + \tilde{\chi}^{(1)} * E\right) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 P_{nl}(z,t)}{\partial t^2}. \quad (6)$$

By transforming Eq. (6) to the Fourier space and by using the properties of the Fourier Transform that concern the convolution and the derivatives of transformed functions, we obtain the algebraic equation for the monochromatic part ω of the pulse as follows:

$$\left[k^2 - \left(\frac{n(\omega)\omega}{c}\right)^2\right]E(k,\omega) - \frac{\omega^2}{\varepsilon_0 c^2}P_{nl}(k,\omega) = 0, \quad (7)$$

where $n(\omega) = \sqrt{1+\chi^{(1)}(\omega)}$ is the refractive index of the medium calculated at the frequency ω. We can write this equation in another form

$$\left[k + \sqrt{\beta^2(\omega) + \frac{\omega^2}{\varepsilon_0 c^2}\frac{P_{nl}(k,\omega)}{E(k,\omega)}}\right]\times \left[k - \sqrt{\beta^2(\omega) + \frac{\omega^2}{\varepsilon_0 c^2}\frac{P_{nl}(k,\omega)}{E(k,\omega)}}\right]E(k,\omega) = 0, \quad (8)$$

with the notation

$$\beta = \beta(\omega) = \frac{n(\omega).\omega}{c}$$

as the wave number of the part ω in the medium. The signs - and + before the square root sign describe the wave propagating in or oppositely to the positive direction of axis Oz. We are interested only in the propagation in the positive direction, so we will consider only the equation in the second square parenthesis.

As $P_{nl}(k,\omega)$ is a perturbation in comparison with the field $E(k,\omega)$, the nonlinear term in the square root is small and the Taylor expansion for this term can be performed [9]:

$$\left[-k + \beta(\omega)\right]E(k,\omega) + \frac{\omega^2}{2\beta(\omega)\varepsilon_0 c^2}P_{nl}(k,\omega) + \sum_{j=2}^{\infty}\frac{(2j-3)!!}{(-1)^{j-1}(2j)!!}\frac{(\omega/c)^{2j}}{\beta^{2j-1}(\omega)\varepsilon_0^j}\frac{\left[P_{nl}(k,\omega)\right]^j}{\left[E(k,\omega)\right]^{j-1}} = 0. \quad (9)$$

As the frequencies ω of the monochromatic parts of the pulse concentrate around the central frequency ω_0, we change the variables $\omega \to \omega_0 + \omega$, $k \to k_0 + k$ in the above equation and expand around ω_0. Next, we should perform the inverse Fourier Transform of the obtained equation to obtain the pulse propagation function in the medium, [7]. Finally, we obtain

$$\left[i\frac{\partial}{\partial z} + i\beta'(\omega_0)\frac{\partial}{\partial t} - \frac{\beta''(\omega_0)}{2}\frac{\partial^2}{\partial t^2} + \sum_{p=3}^{\infty}\frac{i^p}{p!}\left(\frac{\partial^p\beta(\omega)}{\partial\omega^p}\right)_{\omega_0}\frac{\partial^p}{\partial t^p}\right]\left\{E(z,t)e^{i\omega_0 t - ik_0 z}\right\} + \left[1 + i\left(\frac{1}{\omega_0} - \frac{n'(\omega_0)}{n(\omega_0)}\right)\frac{\partial}{\partial t} + \left(\frac{n'(\omega_0)}{\omega_0 n(\omega_0)} - \left(\frac{n'(\omega_0)}{n(\omega_0)}\right)^2 + \frac{n''(\omega_0)}{2n(\omega_0)}\right)\frac{\partial^2}{\partial t^2} + \sum_{q=3}^{\infty}\frac{i^q}{q!}\frac{\beta(\omega_0)}{(\omega_0/c)^2}\left(\frac{\partial^q}{\partial\omega^q}\frac{(\omega/c)^2}{\beta(\omega)}\right)_{\omega_0}\frac{\partial^q}{\partial t^q}\right]\times \frac{(\omega_0/c)^2}{2\beta(\omega_0)\varepsilon_0}\left\{P_{nl}(z,t)e^{i\omega_0 t - ik_0 z}\right\} + \sum_{q=0}^{\infty}\sum_{m=2}^{\infty}\left(\frac{i^q(2m-3)!!}{(-1)^{m-1}\varepsilon_0^m q!(2m)!!}\left(\frac{\partial^q}{\partial\omega^q}\frac{(\omega/c)^{2m}}{\beta^{2m-1}(\omega)}\right)_{\omega_0}\times \frac{\partial^q}{\partial t^q}\left\{\phi^m(z,t)e^{i\omega_0 t - ik_0 z}\right\}\right) = 0 \quad (10)$$

The quantities

$$\phi^m(z,t) = F^{-1}\left\{\frac{P_{nl}^m(k+k_0,\omega+\omega_0)}{E^{m-1}(k+k_0,\omega+\omega_0)}\right\} = F^{-1}\left\{\frac{\left[F\{P_{nl}(z,t)\}\right]^m}{\left[F\{E(z,t)\}\right]^{m-1}}\right\}, \quad (11)$$

are higher-order perturbations, F and F^{-1} denote the Fourier and inverse Fourier Transforms. The notations $\beta'(\omega_0)$; $\beta''(\omega_0)$; $n'(\omega_0)$; $n''(\omega_0)$… are first-order and second-order derivatives of the respective functions, calculated at the value ω_0.

Eq. (10) with the concrete form for the nonlinear polarization (5) and the initial condition for the input pulse permits us to consider the pulse evolution in the propagation within the medium. It is the most general form for a one-dimensional case as it contains all orders of dispersion and nonlinearity. This equation is very complicated and finding a general analytic method (given for example in [10]) for this equation is practically a "mission impossible", so we should reduce it to a simpler approximate form.

It may be worthwhile to look at some simple solutions to the general nonlinear partial differential equation (NPDE) before starting the big machine of any general analytic or numerical scheme for solving it. Without a detailed study of symmetries, we may expect inter alia that our integrable equations will have solutions in the form of a traveling wave. The traveling wave is a solution of the form

$$u(x,t) = U(z), \quad z = x - Vt \quad (12)$$

for equations in which the variable may be interpreted as a real wave function. When the wave consists of a single traveling bump (a displacement from an unperturbed state) or a traveling shock (kink) we call it solitary wave. This name is extended to the solutions of equations like the nonlinear Schrödinger equation (NLS) which describe the evolution of an envelope of fast oscillations and it is only an envelope of a (usually complex) wave function which has the form of a traveling wave (12). Finally, two or more solitary waves may travel at the same time with different velocities; if they travel towards each other, they would "collide" sooner or later. If they get through the collision unchanged (except for a possible shift in their positions and phases), they are called solitons.

For a special case when the medium is isotropic and the medium is of the Kerr type the cubic nonlinear Schrödinger (NLS) equation is obtained which describes the propagation of light pulses in fibers [7,13]. An optical soliton in fiber exists because of the exact balancing between the group velocity dispersion (GDV) and its counterpart self-phase modulation (SPM). SPM is a nonlinear effect due to the lowest dominant nonlinear susceptibility in silica fibers. One of the most famous physicists working in this domain has written that the parameters of fiber are a gift from God and that it is a sin not to use solutions in telecommunication! However, in general we should take into account the higher order contributions [7,8,14].

In some other cases we can also find soliton solutions in an analytic way [10]. In our works on the equations which describe the interaction of higher harmonics with the fundamental mode in a laser beam [15,16] we have applied the Hirota scheme to the systems of equations

$$\begin{aligned} &iU_{,z} + U_{,tt} + U^*W = 0, \\ &iW_{,z} + PW_{,tt} \pm U^2 = 0, \end{aligned} \quad (13)$$

for the 2nd harmonic and

$$\begin{aligned} &iu_z + u_{xx} - u + \left[(1/9)|u|^2 + 2|w|^2\right]u + (1/3)u^{*2}w = 0, \\ &i\sigma\omega_z + w_{xx} - \alpha u + \left(9|u|^2 + 2|w|^2\right)w + (1/9)u^3 = 0, \end{aligned} \quad (14)$$

for the 3rd, where U and u are the amplitudes of fundamental frequency modes, while W and ω are the amplitudes of the 2nd and 3rd harmonics, respectively (all of them rescaled to reduce the number of coefficients). The equations describe the propagation of these nonlinearly interacting modes along a waveguide.

We have found that the Hirota scheme worked merely for the exact resonance cases, i.e. not only had the frequencies of the higher harmonics been found to be multiplies of the fundamental frequency, but also the ratio of the dispersion coefficients had to be equal to the ratio of frequencies. Moreover, the only solitary wave solutions of that type were single traveling waves. For the amplitudes of 2nd harmonic we have found a new equation of the NLS type which they satisfy, namely

$$iU_z + U_{tt} \pm \sqrt{2}|U|U = 0. \quad (15)$$

In the case of the ultrashort pulses, we can simplify Eq. (10) with the use of specific properties of their spectrum and intensity through neglecting the higher order nonlinear perturbations and merely preserving the linear and nonlinear terms with their lower-order derivatives. Before doing this we should

consider in more detail the medium's nonlinear polarization in the propagation of the ultrashort pulses in the next subsection.

2.2. Nonlinear polarization of the medium. Raman response function

The medium's nonlinear polarization is given by (5), where the medium's property is characterized by the quantity $\chi^{(3)}_{xxxx}(t - t_1, t - t_2, t - t_3)$. Apart from being dependent on the molecule microscopic structure and molecule ordering in the medium, it depends also on the characteristics of the propagating pulses. The microscopic processes have usually the characteristic time of femtoseconds (the characteristic time for the electron response is of the order 0.1 fs and 10 fs for the nuclei and lattice [3]). The medium's nonlinear response for picosecond pulses can be considered as instantaneous. In this case the nonlinear susceptibility can be written as follows [2,3,12]

$$\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3) = \chi^{(3)}\delta(t-t_1)\delta(t-t_2)\delta(t-t_3). \quad (16)$$

Here, $\chi^{(3)}$ is a real constant of the order 10^{-22} m/V^2, and $\delta(t - t_i)(i = 1,2,3)$ are the Dirac functions. The reduced equation obtained in this case from (10) is the well-known NLS equation [1,5,12,13]. It describes perfectly the experimental observations for propagation process.

When the input pulses are shorter than 4-5 ps (tens or hundreds of fs) the assumption of the medium's instantaneous response is no longer valid because the time width of the pulses is comparable with the characteristic times of microscopic processes. Some additional terms describing the delayed response of the medium should be included in expression (16). This delayed response is related to the reduced Raman scattering on the medium's molecules [4,9]. Using the Lorentz atomic model in the adiabatic approximation [1,3,9] we can present the medium's nonlinear susceptibility in the form [2,3]:

$$\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3) = \chi^{(3)}\left[(1-f_R)\delta(t-t_1) + f_R h_R(t-t_1)\right]\times \delta(t-t_2)\delta(t-t_3). \quad (17)$$

In the nonlinear susceptibility expression (17) we have two contributions, one of the electron layer and one of the nuclei with the crystal lattice. The electron response is considered as instantaneous, and the delayed response of the nuclei and the lattice is characterized by the function $h_R(t)$ called the Raman response function. It has the following form [3,9,12]:

$$h_R(t) = \frac{\tau_1^2 + \tau_2^2}{\tau_1\tau_2^2} e^{-t/\tau_2}\sin(t/\tau_1). \quad (18)$$

The Raman response function satisfies the normalization condition $\int_0^\infty h_R(t)\,dt = 1$. The constants f_R, τ_1 and τ_2 depend on the medium. These parameters for the material applied usually in the production of optical fibers SiO_2 have been measured [12,17]: $\tau_1 \approx 12.2$ fs, $\tau_2 \approx 32$ fs, $f_R \approx 0.18$ [11].

The Fourier Transform of the $h_R(t)$ (called also the Raman response function, but at the frequency ω) has the following form

$$g_R(\omega) = \frac{1/\tau_1^2 + 1/\tau_2^2}{-\omega^2 - 2i\omega/\tau_2 + \left(1/\tau_1^2 + 1/\tau_2^2\right)}. \quad (19)$$

The imaginary part of $g(\omega)$ is called the Raman amplification function [3,4,18]. It can be seen from the above formula that the Raman amplification function of the medium has a very broad support, especially it does not disappear in the low frequencies. This fact has an important influence on the propagation process of ultrashort pulses. The Raman scattering leads to a continuous downshift in the pulse's spectrum. This so called self- shift frequency [4,12] is considered in detail later.

2.3. Generalized nonlinear Schrödinger equation

Substituting expression (17) by (5), after having expanded the terms containing the the electric field intensity powers and neglecting the high-order harmonics (as the phase-matching condition is not fulfilled), we obtain the following expression for nonlinear polarization:

$$P_{nl}(z,t) = \frac{3\varepsilon\chi^{(3)}}{8}\left[(1-f_R)|A(z,t)|^2 A(z,t) + f_R A(z,t)\int_{-\infty}^{t} h_R(t-t_1)|A(z,t)|^2\,dt_1 + c.c\right]. \quad (20)$$

The physical properties of the medium do not depend on the choice of the time scale beginning, so the second term in (20) can be rewritten as:

$$\int_{-\infty}^{t} h_R(t-t_1)|A(z,t_1)|^2\,dt_1 = \int_0^\infty h_R(t_1)|A(z,t-t_1)|^2\,dt_1. \quad (21)$$

Expanding to the first order of of the envelope module square under the integral sign in (21) and using the normalization condition for the function $h_R(t)$ lead to the result

$$\int_0^\infty h_R(t_1)\left|A(z,t-t_1)\right|^2 \mathrm{d}t_1 \approx$$

$$\left|A(z,t)\right|^2 - \frac{T_R}{f_R}\frac{\partial\left|A(z,t)\right|^2}{\partial t}, \tag{22}$$

where T_R is the characteristic time for the Raman scattering effect:

$$T_R = f_R\int_0^\infty t h_R(t)\,\mathrm{d}t. \tag{23}$$

From these results the nonlinear polarization can be written in the form:

$$P_{nl}(z,t) = \frac{3\varepsilon_0\chi^{(3)}}{8}\left[A(z,t)\left|A(z,t)\right|^2 + T_R A(z,t)\frac{\partial\left|A(z,t)\right|^2}{\partial t} + c.c\right]. \tag{24}$$

As has been recognized above, the general equation (10) is very complicated, so we should reduce it into an approximate form. It is worth noting that the time and intensity characters of ultrashort pulses are quite different in comparison to those of short pulses. It follows that their spectrum is much broader and the pulse power is larger, so in Eq. (10) we should consider the third-order dispersion terms [2,5,12] and the first-order term of the Kerr nonlinearity [1,13].

Substituting the expression for the nonlinear polarization (24) by (10), having omitted the fast oscillating terms and having neglected the high-order derivatives of the nonlinear term the following simplest approximate pulse propagation equation is obtained:

$$i\frac{\partial A(z,t)}{\partial z} + i\beta'(\omega_0)\frac{\partial A(z,t)}{\partial t} - \frac{\beta''(\omega_0)}{2}\times \frac{\partial^2 A(z,t)}{\partial t^2} - \frac{i\beta'''(\omega_0)}{6}\frac{\partial^3 A(z,t)}{\partial t^3} + \left(\left|A(z,t)\right|^2 A(z,t) + i\tau_s\frac{\partial\left|A(z,t)\right|^2 A(z,t)}{\partial t} - T_R A(z,t)\times\frac{\partial\left|A(z,t)\right|^2}{\partial t}\right) = 0 \tag{25}$$

where

$$\gamma = \frac{3}{8}\frac{\chi^{(3)}\omega_0}{n(\omega_0)c},\quad \tau_s = \frac{1}{\omega_0}\frac{n'(\omega_0)}{n(\omega_0)} \approx \frac{1}{\omega_0}. \tag{26}$$

Using the new parameters and variables

$$L_D = \frac{\tau_0^2}{\left|\beta''(\omega_0)\right|},\ L_N = \frac{1}{\gamma P_0},\ N^2 = \frac{L_D}{L_N},$$
$$\delta_3 = \frac{\beta'''(\omega_0)}{6\left|\beta''(\omega_0)\right|\tau_0},\ S = \frac{\tau_s}{\tau_0},\ \tau = \frac{t-\beta'(\omega_0)z}{\tau_0}, \tag{27}$$
$$\xi = \frac{z}{L_D},\ U(\xi,\tau) = \frac{1}{\sqrt{P_0}}A(z,t),$$

where t_0 and P_0 stand respectively for the time width and the maximal power at the top of the envelope function, we can rewrite Eq. (25) in a normalized form:

$$\frac{\partial U}{\partial\xi} = -\operatorname{sign}(\beta''(\omega_0))\frac{i}{2}\frac{\partial^2 U}{\partial t^2} + \delta_3\frac{\partial^3 U}{\partial\tau^3} + iN^2\left(\left|U\right|^2 U + iS\frac{\partial}{\partial\tau}\left(\left|U\right|^2 U\right) - \tau_R U\frac{\partial\left|U\right|^2}{\partial\tau}\right). \tag{28}$$

Eq. (28) is a lowest-order approximate form when the higher- order dispersion and nonlinearity effects in the general propagation equation (10) are considered. It is one of the most useful approximate forms describing the propagation process of ultrashort pulses, called the generalized nonlinear Schrödinger equation [2,3,5,6]. Some general remarks concerning the application of this equation will be given in the next Section.

3. IMPACT OF DISPERSION AND HIGHER ORDER NONLINEAR EFFECTS

The propagation equation for ultrashort pulses (28) has a more complicated form than the nonlinear Schrodinger equation describing the propagation of short pulses [1,5,12,13,] as it contains higher-order dispersive and nonlinear terms. The parameters characterizing these effects: δ_3, S, τ_R govern the effects of TOD, self-steepening and the self-shift frequency, respectively. In can be seen in formulas (27) that when τ_0 decreases, i.e. the pulse is shorter and the magnitude of these parameters increases, the higher-order effects should be considered.

Both the pulse shape and the spectrum change in a complicated way under the influence of TOD. When the propagation distance is larger, the envelope function's oscillation is stronger, creating a long trailing edge to the later time, and the spectrum is broadened to two sides and splits into several peaks [5,12].

The pulse self-steepening leads to the formation of a steep front in the trailing edge of the pulse, resembling the usual shock wave formation. This effect is called the optical shock. The pulse becomes more asymmetric in the propagation and its tail finally breaks up [1,5,6,13].

In the stimulated Raman scattering it is the Stokes process that is more effective than the anti-Stokes process [4,12]. This fact leads to the so-called self-shift frequency of the pulse. As a result, the spectrum is shifted down to the low-frequency region. In other words, the medium "amplifies" the long wavelength parts of the pulse. The pulse loses its energy and undergoes a complex change when it enters deeply into the medium.

For the ultrashort pulses with the width of $\tau_0 \approx 50$ fs and the carrier wavelength of $\lambda_0 \approx 1.55$ mm, during their propagation in the medium SiO_2 the higher order parameters in (27) have the values of $\delta_3 \approx 0.03$, $S \approx 0.03$, $\tau_R \approx 0.1$. These values are smaller than one, hence the higher-order effects are considered as perturbations in comparison with the Kerr effect. Therefore, when the pulse propagates in a silica optical fiber, the self-shift frequency effect dominates over the TOD and self-steepening for the pulses with the width of hundreds and tens of femtoseconds. The self-steepening becomes important only for pulses of nearly 3 fs [5,12].

When τ_0 has the value of picoseconds or larger, the values of δ_3, S, and τ_R are very small and they can be neglected. Eq. (28) is reduced to the well known NLS equation for the short pulses [1,12,13]. As has been recognized above, NLS can be solved by the Inverse Scattering Method [10], but this Method cannot be applied to Eq.(28) any more. The problem of finding a general analytic method for this equation is practically a "mission impossible". Therefore several methods of finding approximate solutions of this equation are used. Two important algorithms [11], namely the Split-Step Method and the Runge-Kutta Method are considered in the next section.

4. NUMERICAL METHODS TO SOLVE THE PULSE PROPAGATION EQUATION

4.1. Split-Step Algorithm of second order

Firstly, the Split-Step Algorithm for finding approximate solutions of the pulse propagation equation is presented. Eq.(28) can be written in the form:

$$\frac{\partial U}{\partial \xi} = \left(\hat{L} + \hat{N}(U)\right)U, \tag{29}$$

where $\hat{L}$ and $\hat{N}$ are linear and nonlinear operators, respectively acting on the envelope function:

$$\hat{L} = \frac{i}{2}\frac{\partial^2}{\partial \tau^2} + \delta_3 \frac{\partial^3}{\partial \tau^3}, \tag{30}$$

$$\hat{N}(U) = iN^2\left(|U|^2 + iS\frac{1}{U}\frac{\partial(U)|U|^2}{\partial \tau} - \tau_R \frac{\partial |U|^2}{\partial \tau}\right). \tag{31}$$

[2] is obtained by integrating Eq.(29) over the variable ξ in the interval $\xi \to \xi + \Delta\xi$:

$$U(\xi + \Delta\xi, \tau) = \exp(A + B)U(\xi, \tau) \tag{32}$$

with

$$A = \int_{\xi}^{\xi+\Delta\xi} \hat{L}\, d\xi' = \hat{L}\int_{\xi}^{\xi+\Delta\xi} d\xi' = \Delta\xi\hat{L}, \tag{33}$$

$$B = \int_{\xi}^{\xi+\Delta\xi} \hat{N}(U(\xi', \tau))\, d\xi' \approx \Delta\xi\hat{N}(U(\xi, \tau)). \tag{34}$$

When the step-size $\Delta\xi$ of the propagation distance is pretty small, using the Baker-Campbell-Hausdorff formula for the exponential operator involved in (32) we can present its approximate form as follows [2,13,14,19]:

$$\exp(A + B) \approx \exp\left(\frac{A}{2}\right)\exp(B)\exp\left(\frac{A}{2}\right). \tag{35}$$

In this approximation it is assumed that the operators A and B are commutative between themselves when $\Delta\xi$ is small. The error of formula (35) is of the order of $(\Delta\xi)^2$.

The following formula describing Split-Step algorithm for problem (29) are obtained by substituting the expressions given above by Eq. (32):

$$U(\xi + \Delta\xi, \tau) \approx \exp\left(\frac{\Delta\xi}{2}\hat{L}\right)\times$$
$$\exp\left(\Delta\xi\hat{N}(U(\xi, \tau))\right)\exp\left(\frac{\Delta\xi}{2}\hat{L}\right)U(\xi, \tau). \tag{36}$$

This expression permits us to specify the approximate value of the envelope function in the location $\xi + \Delta\xi$ from its value in the ξ.

To calculate the envelope function value by (36) we should know how the action of the linear and nonlinear operators on the envelope function is

calculated. As these operators contain time partial derivatives they can be calculated just with the Fourier Transform.

We take the time variable in the finite interval [*a*,*b*] which is so large that its borders do not have any influence on the final results of the calculations. We assume now the periodic condition on the borders that $U(\xi,a) = U(\xi,b)$ for $\xi \in [0, \xi_0]$. For convenience, we change the variable in (36) in such a way that it normalizes the interval [*a*,*b*] into the interval $[0,2\pi]$ and we divide this interval into *N* points with distance between them $\Delta\tau = 2\pi/N$. We denote these points as $\tau_j = 2\pi j/N$, $j = 0, 1, 2, ...,N$. Then, we have the Discrete Fourier Transform of the series $U(\xi, \tau_j)$ as follows:

$$U(\xi,\omega_k) = F_k\left[U(\xi,\tau_j)\right] = \frac{1}{N}\sum_{j=0}^{N-1} U(\xi,\tau_j)\times \exp\left(i\omega_k\tau_j\right),\ -\frac{N}{2} \le \omega_k \le \frac{N}{2}-1 \tag{37}$$

The Inverse Fourier Transform is defined as follows:

$$U(\xi,\tau_j) = F_j^{-1}\left[U(\xi,\omega_k)\right] = \sum_{k=-N/2}^{N/2-1} U(\xi,\omega_k)\times \exp\left(i\omega_k\tau_j\right),\ \ j = 0,1,2,\ldots,N. \tag{38}$$

F here denotes the Fourier Transform and F^{-1} denotes its inverse transform. Calculations in (37) and (38) are made effective by the fast algorithm FFT [20]. The time partial derivatives of the envelope function in both the linear and nonlinear operators (30) and (31) can be easily calculated by multiplying the Fourier coefficients $U(\xi,\omega_k)$ by the powers of $-i\omega_k$ corresponding to the order of derivative and then taking the Inverse Fourier Transform. For example, the second-order derivative of the envelope function in the point (ξ,τ_j) can be calculated as $F_j^{-1}\left[-\omega_k^2 F_k\left[U(\xi,\tau_j)\right]\right]$.

4.2. Fourth order Runge-Kutta algorithm

Eq. (28) can be solved using the Runge-Kutta algorithm. In this method the time discretization and calculations of time partial derivatives are the same as in the previous subsection, but the spatial derivatives are calculated by the Runge-Kutta algorithm. What is applied here is the fourth order Runge-Kutta algorithm, very popular for solving differential equations [6,20-22].

The Fourier Transform having been used to calculate the time partial derivatives as above, Eq. (28) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(F[U]\right) = \left((-i\omega)^2\frac{i}{2} + (-i\omega)^3\delta_3\right)F[U] + iN^2\left[(1+iS(-i\omega))F\left[|U|^2U\right] - \tau_R F\left[UF^{-1}\left[(-i\omega)F\left[|U|^2\right]\right]\right]\right]. \tag{39}$$

Denoting

$$= \exp\left(\left(\frac{i\omega^2}{2} - i\omega^3\delta_3\right)\xi\right)F[U], \tag{40}$$

Eq. (28) can be rewritten in the form:

$$\frac{\mathrm{d}V}{\mathrm{d}\xi} = f(\xi,U), \tag{41}$$

where

$$f(\xi,U) = iN^2\exp\left(\left(\frac{i\omega^2}{2} - i\omega^3\delta_3\right)\xi\right)\times \left[(1+S\omega)F\left[|U|^2U\right] + \tau_R F\left[UF^{-1}\left[(i\omega)F\left[|U|^2\right]\right]\right]\right]. \tag{42}$$

Using the fourth order Runge-Kutta algorithm for Eq. (41) we calculate the function *V* in the location $\xi + \Delta\xi$ [20]:

$$V(\xi+\Delta\xi) = V(\xi) + \frac{1}{6}\left[K_1 + 2(K_2+K_3) + K_4\right], \tag{43}$$

where the coefficients K_i are calculated as follows:

$$K_1 = \Delta\xi.f(\xi,U(\xi,\tau)), \tag{44}$$

$$K_2 = \Delta\xi.f\left(\xi+\frac{\Delta\xi}{2}, U(\xi,\tau)+\frac{1}{2}K_1\right), \tag{45}$$

$$K_3 = \Delta\xi.f\left(\xi+\frac{\Delta\xi}{2}, U(\xi,\tau)+\frac{1}{2}K_2\right), \tag{46}$$

$$K_3 = \Delta\xi.f(\xi+\Delta\xi, U(\xi,\tau)+K_3). \tag{47}$$

The envelope function in the location *x*+*Dx* is obtained from (40) and (43):

$$U(\xi+\Delta\xi) = F^{-1}\left[V(\xi+\Delta\xi)\times \exp\left(\left(-\frac{i\omega^2}{2} + i\omega^3\delta_3\right)(\xi+\Delta\xi)\right)\right]. \tag{48}$$

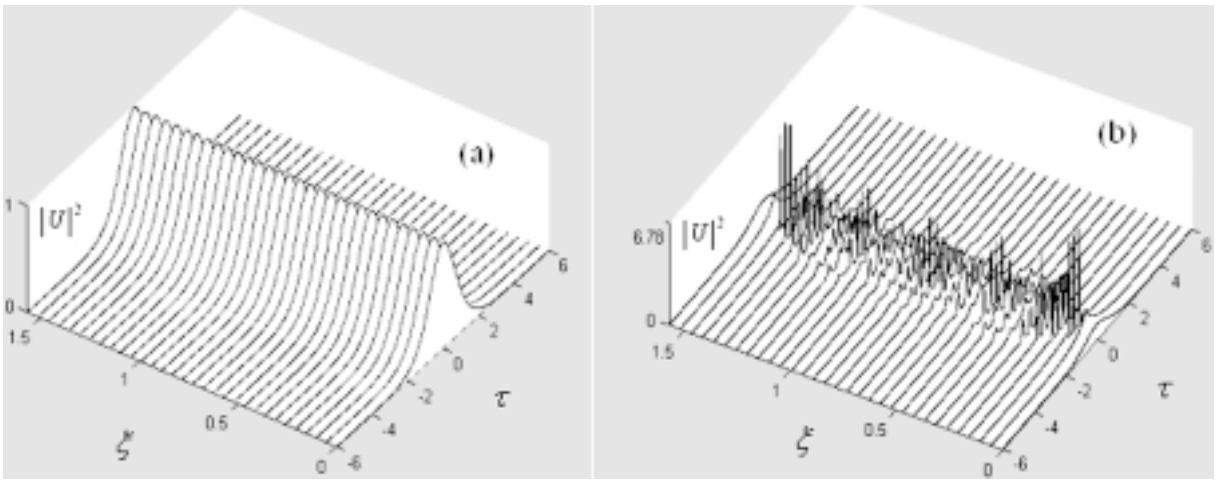

Fig. 1. Change of pulse intensity in the propagation process for the case of fundamental (a) and tenth-order solitons (b) over one soliton period $\xi = \pi/2$.

Errors in applying (48) are of orders of $(\Delta\xi)^5$. Formula (48) has a higher accuracy in comparison to the calculations performed by (36), although the computational time is longer as the number of calculation steps in (42) and (47) is very large.

In the simulations performed below both the algorithms presented above have been used and the obtained results have been compared. They are almost the same when the interval $\Delta\xi$ is relatively small.

At first, the numerical simulations performed by using the algorithms introduced above are compated with the analytical results obtained in some special cases. In this way, the accuracy of these numerical algorithms is tested. As we have not known any analytical solution of the GNLS equation in literature to date, our results will be compared with the NLS equation results for the picosecond pulses case. A very important phenomenon: propagation of the solitons [5,12] is considered.

According to Inverse Scattering Transform Method, when the higher-order parameters δ_3, S and τ_R in Eq. (28) equal zero and the initial shape of the pulses is the function of a secant hyperbolic form, the equation will have the soliton solutions [23,24]. These solitons exhibit the periodic feature with a characteristic period during propagation. With the exception of the case of the first-order (temporal) soliton (called the fundamental soliton) when the amplitude of the envelope function remains unchanged during propagation, higher-order solitons change in shape and spectrum in a complicated manner, but their shape follows a periodic pattern, so that the input shape is recovered at the propagation period $\xi = \pi/2$. The order of soliton is determined by the parameter N in (28). When the value of N is larger (higher-order solitons), the envelope changes in a more complicated way over one soliton period.

We have simulated the pulse evolution for the first-order and tenth-order (N=10) solitons over one soliton period with the input pulse having an initial amplitude [12]:

$$U(0,\tau) = N\,\mathrm{sech}(\tau). \qquad (49)$$

Fig. 1 shows these results by plotting the pulse intensity $|U(\xi,t)|^2$. In Fig. 1a the envelope function of the pulse has an unchanged shape in the propagation process conserving the initial form (49). In Fig. 1b the envelope function has a complex evolution in propagation, but it comes back to the initial shape in the end of the period and this process repeats itself in the next periods. These results are in good agreement with analytical predictions about the periodic feature in envelope function evolution. Analytical expressions for the higher-order solitons are very complicated and it is only in the case of the second- and third-order that they are explicitly given in literature [5,23,24], but for the tenth-order soliton considered above the expression is presented only by numerical results.

Moving on, the case of multiple soliton propagation is considered. The input amplitude for a soliton pair entering the medium is expressed by:

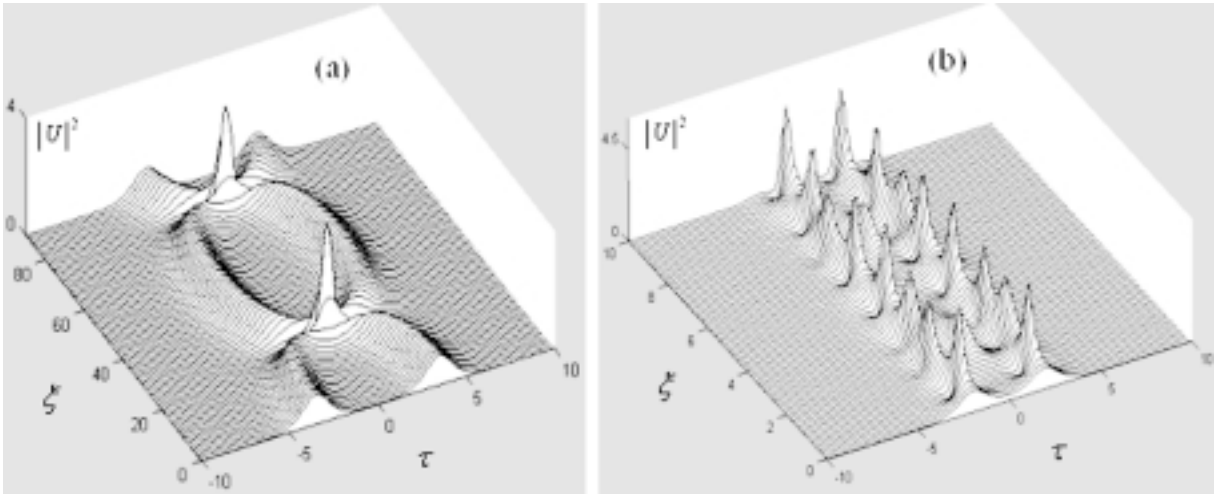

Fig. 2. Collision between two fundamental solitons over the propagation distance $\xi = 90$ (a) and between two second-order solitons over the propagation distance $\xi = 10$ (b).

$$U(0,\tau) = \operatorname{sech}(\tau - \tau_1) + r\operatorname{sech}[r(\tau + \tau_2)]\exp(i\theta), \quad (50)$$

where r is the relative amplitude of the two solitons and θ is the relative phase between them [5,12,21,25]. Analytical results [23,24] show that neighboring solitons either come closer or move apart due to a nonlinear interaction between them. The time of soliton collisions depends on both the relative phase θ and the amplitude ratio r. Solitons collide periodically along the distance of propagation, the collision period usually is much longer than the soliton period. After a collision the shape of wave amplitudes remains unchanged and stable. This effect is similar to a collision of rigid particles, so the name ”soliton” reflects the particle feature of nonlinear waves [5,12].

The following calculations are performed for a collision between fundamental solitons and higher-order solitons. The parameters in (50) are chosen as $r = 1$, $\theta = 0$ (equal-amplitude and in-phase case) and $\tau_1 = \tau_2$ (initial spacing). Numerical results are displayed in Fig. 2.

Fig. 2a displays the collision process between two fundamental solitons, where $\tau_1 = \tau_2 = 3.5$ and the propagation distance $\xi = 90$. During the propagation, the two solitons attract each other at first, then they approach each other with increasing intensity in a location where two solitons are nearest to themselves, the intensity is four times greater than the initial value. After that, the solitons repel each other and their spacing increases with the distance, and the intensity decreases to the initial value. This process of attracting and repelling is repeated periodically because of the dispersion and nonlinear effects, after each such collision the shape of the envelope function remains unchanged. In Fig. 2b the collision of two second-order solitons is considered. The values of the parameters are chosen as $\tau_1 = \tau_2 = 2$ and the propagation distance $\xi = 10$. As the distance between two solitons is smaller than in the previous case the time collision is much faster. The solitons attract and repel each other in the same periodical manner but the envelope function change is more complicated. Similar numerical calculations ate performed for higher-order solitons and the obtained results show that the complexity of the envelope function change increases with the order of the solitons. These observations are in good agreement with the calculations in [5,25].

5. SOLITONS IN NONLINEAR OPTICS AND NONLINEAR ATOM OPTICS

It is well known that a great number of physical processes involved in a given nonlinear problem can be understood in terms of the formation of spatial, temporal or spatio-temporal localized structures or solitons, which has been defined above. In our recent paper we concentrate on these subjects and review some of our latest results. The study of these localized waves is strongly complicated by the fact that the nonlinear partial differential equations (PDEs) of a given system are usu-

ally not integrable. By investigating the integrability of a nonlinear PDE, important insight is gained into the structure of the equation and the nature of its solutions. With the exception of some analytical solutions obtained by well-known methods (inverse scattering method, Hirota's method) [10], the solitary wave solutions have to be determined numerically [26].One of the most effective method is the Fourier Split- Step method which is used in studying dynamics of short-pulse splitting in dispersive nonlinear media [11,27]. In some cases analytical and numerical methods should be used simultaneously [28].

As has been recognized before [29,30] the search for spatiotemporal solitons in optical media, alias "light bullets" (LBs) [31], is a challenge to the fundamental and applied research in nonlinear optics (see [32-34]). Stationary solutions for LBs can be found from the cubic ($\chi^{(3)}$) multi-dimensional nonlinear Schrödinger (NLS) equation [31] which is a well-known asymptotic model governing the slow evolution of the local amplitude of an electromagnetic field [7]. However, they are unstable against a spatiotemporal collapse [35]. The problem may be avoided by introducing milder nonlinearities, saturable [36], cubic-quintic [37], or quadratic ($\chi^{(2)}$) [32,33,38].

Despite a considerable progress in theoretical studies, three-dimensional (3D) LBs in a bulk medium have not been observed in an experiment yet. The only successful experimental finding reported thus far has been a stable quasi-2D spatiotemporal soliton in $\chi^{(2)}$ crystals [33]. The tilted-wavefront technique [39] has been used in that paper to introduce the strong artificial group-velocity dispersion (GVD). This has precluded achieving a self-confinement in one transverse direction. On the other hand, it has bee predicted [38] that a spatial cylindrical soliton may be stabilized in a bulk medium composed of layers with alternating signs of the Kerr coefficient.

In the 2D case, we have proposed a scheme for stabilizing spatiotemporal (STS) solitons in Kerr media with a layered structure. We consider an experimentally realistic possibility, viz., periodic reversal of the GVD sign, which resembles known dispersion-management (DM) schemes in fiber optics. First, we have developed a variational approximation (VA) based on the Gaussian ansatz for 2D STS. In the 2D case, simulations of the resulting systems of coupled variational equations reveal well-defined stability regions in the relevant parameter space. In [30] we have verified these results by direct simulations by the Split- Step Fourier Method mentioned above using the MATLAB program. We have again obtained a new stable object in the form of a periodically oscillating bound state of two subpulses. As a byproduct, we have concluded that the program in MATLAB is twice faster than the program in FORTRAN used before.

In our further work [40], we have proposed a possibility to stabilize spatiotemporal solitons ("light bullets") in three-dimensional self-focusing Kerr media by means of dispersion management (DM) which means that the local group-velocity dispersion coefficient alternates between positive and negative values along the propagation direction, z. The model is based on a scalar equation of the NLS type which can be derived in paraxial approximation for the slowly varying amplitude of a linearly polarized electromagnetic wave. We have previously shown that that the DM alone can stabilize solitons in 2D (planar) waveguides, but the "bullets" in a bulk (3D) DM medium are unstable. In [40] we have demonstrated that a complete stabilization can be provided if the longitudinal DM is combined with a periodic modulation of the refractive index in one transverse direction (y), with a period much longer than the carrier wavelength. A stability area for the light bullets have been identified in the model parameter space. Its salient features are a necessary minimum strength of the refractive index's transverse modulation, and the minimum and maximum values $E_{\min,\max}$ of the soliton energy. The former feature can be accurately predicted in an analytical form from the evolution equation for the soliton width in the y-direction. Also, similar to the case of the DM solitons in fibers, we have found that the stability area extends into a region of *normal* average dispersion [41]. On the other hand, the existence of E_{max} can be understood similarly to the way it has been done in the 2D counterpart of the present model (the strong transverse lattice can squeeze the system to a nearly 2D shape).

The model opens a way to address further issues, such as collisions between the LBs, and the existence and stability of solitons with different symmetries (for instance, LBs which are odd in the longitudinal and/or transverse directions).

As has been recognized in our previous paper [42], the results from nonlinear optics to atom optics and vice versa [28,43] could be transferred by analogy between the propagation equation in the Kerr medium and the Gross-Pitajevski equation for Bose-Einstein condensates (BECs). A similar stabilization mechanism has been predicted for 2D solitons in Bose-Einstein condensates (BECs), with

the coefficient in front of the cubic nonlinear term subjected to a periodic modulation in time via the Feshbach resonance in an external AC magnetic field [44-46]. However, no stable 3D soliton could be predicted in either realization of this setting (optical or BEC).

In [47] we have investigated the stability properties of breather solitons in a threedimensional Bose-Einstein Condensate with "Feshbach Resonance Management" of the scattering length and confined only by a one dimensional optical lattice. We have compared regions of stability in the parameter space obtained from a fully 3D analysis with those from quasi two-dimensional treatment. For moderate confinement we have discovered a new island of stability in the 3D case, not present in the quasi 2D treatment. Stable solutions from this region have nontrivial dynamics in the lattice direction, hence, they fully describe 3D breather solitons. We have demonstrated these solutions in direct numerical simulations and more importantly, we have suggested a way of creating robust 3D solitons in experiments in a Bose-Einstein Condensate in a one-dimensional lattice.

6. CONCLUSIONS

In this paper we have derived the generalized nonlinear Schrödinger (GNLS) equation for the propagation process of ultrashort pulses in the Kerr medium. The influence of the higher-order dispersive and nonlinear effects, especially the nonlinear effect induced by the stimulated Raman scattering, have been considered in detail.

As the GNLS equation is strongly nonlinear, the problem of solving it is a difficult task. Until now we have not been able to find any exact analytical solution for this equation in the general case. Several approximate methods of solving the problem are applied. We have presented two of them.

The results from nonlinear optics to atom optics and vice versa could be transferred by analogy between the propagation equation in the Kerr medium and Gross-Pitajevski equation for Bose-Einstein condensates (BECs). Thus, the algorithms presented here could be very useful as a byproduct in consideration of a fifth state of matter: Bose-Einstein Condensates. Various types of phenomena which have realizations both in the nonlinear optics and atom optics have been discusses. Our considerations have focused on soliton formation, which is a challenge to the fundamental and applied research in these two domains of physics.

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