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A NEW REPRESENTATION FOR THE PROPERTIES OF ANISOTROPIC ELASTIC FIBER REINFORCED COMPOSITE MATERIALS

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Abstract. A new procedure based on constructing orthonormal tensor basis using the form-invariant expressions which can easily be extended to any tensor of rank *n*. A new decomposition, which is not in literature, of the stress tensor is presented. An innovational general form and more explicit physical property of the symmetric fourth rank elastic tensors is presented. A new method to measure the stiffness and piezoelectricity in the elastic fiber reinforced composite and piezoelectric ceramics materials using the norm concept on the crystal scale. This method will allow to investigate the effect of fiber orientaion, number of plies, material properties of matrix and fibers, and degree of anisotropy on the stiffness of the structure. The results are compared with those available in the literature for semiconductor compounds, piezoelectric ceramics and reinocored composite materials.

1. INTRODUCTION

Most of the elastic materials in engineering are, with acceptable accuracy, considered as anisotropic materials; metal crystals (due to the symmetries of the lattice), fiber-reinforced composites, polycrystalline textured materials, biological tissues, rock structure etc. can be considered as orthotropic materials. In recent years fiber reinforced composite materials have been paid considerable attentions due to the search for materials of light weight, great strength and stiffness. Consequently the determination of their mechanical properties, i.e. stiffness effect, becomes important. Piezoelectric materials nowadays have been widely used to manufacture various sensors, conductors, actuators, and have been, extensively, applied in electronics, laser, ultrasonics, naval and space navigation as well as biologics, smart structures and many other high-tech areas.

Historically, the study of anisotropic elastic materials has been synonymous with study of crystals. For a deep understanding of the physical properties of these anisotropic materials use of tensors is inevitable. Tensors are the most apt mathematical entities to describe direction-dependent-physical properties of solids, and the tensor components characterizing physical properties which must be specified without reference to any coordinate system. Specifying the values of the tensor components which represent physical properties of crystals, as

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Nowack [1] points out, do not determine the material constants directly since their values vary with the direction of the coordinate axes. It is, therefore, natural to seek to characterize physical properties of crystals by constants whose values do not depend upon the choice of the coordinate system, i.e. constants which are invariant under all coordinate transformations. Some of such invariants have been obtained using different decomposition methods in the case of photoelastic coefficients (Srinivasan *et al.* [2]), piezoelectric coefficients (Srinivasan [3]) and elastic stiffness coefficients (Srinivasan *et al.* [4-6]).

A physical property is characterized by *n* rank tensor that has two kinds of symmetry properties. The first kind is due to an intrinsic symmetry derives from the nature of the physical property itself, and this can be established by the thermodynamical arguments or from the indispensability of some of the quantities involved. The second kind of symmetry is due to the geometric or crystallographic symmetry of the system described. The decomposition methods of tensors have many applications in different subjects of engineering. In the mechanics of continuous media i.e. in elasticity studies; so far, the stress and strain tensors are decomposed into spherical (hydrostatic) and deviatoric parts, the hydrostatic pressure is connected to the change of volume without change of shape, whereas the change of shape is connected to the deviatoric part of the stress.

The constitutive relation for linear anisotropic elasticity is the generalized Hooke's law

$$\sigma_{ij} = C_{ijkm}\varepsilon_{km} \tag{1}$$

which is the most general linear relation between the stress tensor whose components are σ_{ij} and the strain tensor whose components are ε_{km}. The coefficients of linearity, namely C_{ijkm}, are the components of the fourth rank elastic stiffness tensor.

In anisotropic solids, the direct piezoelectric effect comprises a group of phenomena in which the mechanical stresses or strains induce in crystals an electric polarization (electric field) proportional to those factors. Besides, the mechanical and electrical quantities are found to be linearly related as following

$$P_i = d_{ijk}\sigma_{jk}\,, \tag{2}$$

where P_i and σ_{ij} denote the components of the electric polarization vector and the components of the mechanical stress tensor, respectively, and d_{ijk} are the piezoelectric coefficients forming a third rank tensor. The elastic properties of crystals appear to be well described in terms of symmetry planes. Symmetry planes (i.e. planes of mirror symmetry) were defined, for example, by Spencer [7], Cowin *et al.* [8] classified the known elastic symmetries of materials and ordered materials on the basis of symmetry planes. Cowin *et al.* [8] and Hue *et al.* [9] list ten symmetry classes.

The purpose of this work is to develop the existing methods of decomposing Cartesian tensors into orthonormal basis using invariant-form to decompose some well known tensors into orthonormal tensor basis. Next, as an outcome of these decompositions, to investigate the contributions to the formulation of the physical properties of elastic stress, strain, piezoelectric and elastic stiffness anisotropic materials. Finally, the concept of norm is introduced to measure the overall effect of material properties and, then, numerical engineering applications are introduced for several engineering materials like semiconductor compounds, piezoelectric ceramics and fiber reinforced composites.

2. FORM INVARIANTS AND ORTHONORMAL BASIS ELEMENTS

The symmetry properties of the material, due to the geometric or crystallographic symmetry, may be defined by the group of orthonormal transformations which transform any of these triads ν_a into its equivalent positions. For the monoclinic symmetric second rank tensor, for instance, the basis elements can be found depending on the form invariant for the monoclinic system. Its form invariant expression, with ν_2 normal to the $\nu_1\nu_3$ plane, can be written as

$$\begin{aligned}\sigma_{ij} &= A_{11}\nu_{1i}\nu_{1j} + A_{22}\nu_{2i}\nu_{2j} + \\ &A_{33}\nu_{3i}\nu_{3j} + A_{31}(\nu_{3i}\nu_{1j} + \nu_{1i}\nu_{3j})\,,\end{aligned} \tag{3}$$

where ν_{ai} are the components of the unit vectors ν_a (a = 1,2,3) along the material directions axes. The corresponding reciprocal triads satisfy the relations [10]

$$\nu^{ai}\nu_{aj} = \delta_{ij} \tag{4}$$

using (4) and orthonormalization by the well known Gram-Schmidt scheme, the four basis elements of the monoclinic system are obtained [10]:

$$T^{I}_{ij} = \frac{1}{\sqrt{4}}\delta_{ij}\,,$$

$$T^{II}_{ij} = \frac{1}{\sqrt{2}}\left[2\delta_{1i}\delta_{1j} + \delta_{3i}\delta_{3j} - \delta_{ij}\right],$$

$$T_{ij}^{III} = -\frac{1}{\sqrt{6}}\left[3\delta_{3i}\delta_{3j} - \delta_{ij}\right], \qquad (5)$$

$$T_{ij}^{IV} = \frac{1}{\sqrt{2}}\left[\delta_{3i}\delta_{1j} + \delta_{1i}\delta_{3j}\right],$$

It is well known that for a symmetric second order tensor is of dimension six; an orthonormal basis set of six elements can be constructed. By taking cyclic permutation of {1,2,3}; the elements V and VI can be generated from IV in (5) as

$$T_{ij}^{V} = \frac{1}{\sqrt{2}}\left[\delta_{1i}\delta_{2j} + \delta_{2i}\delta_{1j}\right],$$
$$T_{ij}^{VI} = \frac{1}{\sqrt{2}}\left[\delta_{2i}\delta_{3j} + \delta_{3i}\delta_{2j}\right]. \qquad (6)$$

A complete orthonormal basis for the second rank symmetric tensor will be the set {I, II,…,VI}. The decomposition σ_{ij} of is given in terms of these basis elements as

$$\sigma_{ij} = \sum_k (\sigma, T_{ij}^k) T_{ij}^k, \; (k = I, II, \ldots, VI), \qquad (7)$$

where (σ, T_{ij}^k) represents the inner product of and the k^{th} elements, T_{ij}^k, of the basis. Hence, the second rank symmetric tensor is decomposed into six orthonrmal terms expressed in matrix form:

$$\sigma_{ij} = \begin{bmatrix} \frac{1}{3}\sigma_{pp} & 0 & 0 \\ 0 & \frac{1}{3}\sigma_{pp} & 0 \\ 0 & 0 & \frac{1}{3}\sigma_{pp} \end{bmatrix} +$$
$$\begin{bmatrix} \frac{1}{2}(2\sigma_{11} + \sigma_{33} - \sigma_{pp}) & 0 & 0 \\ 0 & \frac{1}{2}(2\sigma_{11} + \sigma_{33} - \sigma_{pp}) & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$
$$\begin{bmatrix} \frac{1}{6}(-3\sigma_{33} - \sigma_{pp}) & 0 & 0 \\ 0 & \frac{1}{6}(-3\sigma_{33} - \sigma_{pp}) & 0 \\ 0 & 0 & \frac{1}{6}(-3\sigma_{33} - \sigma_{pp}) \end{bmatrix} +$$
$$\begin{bmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_{23} \\ 0 & \sigma_{23} & 0 \end{bmatrix} \qquad (8)$$

From (8), the second rank symmetric tensor, σ_{ij}, is decomposed into six terms, each of which has a physical meaning. Also, the second rank symmetric tensor is virtually decomposed into two parts:

$$\sigma_{ij} = \frac{1}{3}\sigma_{pp}\delta_{ij} + \left(\sigma_{ij} - \frac{1}{3}\sigma_{pp}\delta_{ij}\right). \qquad (9)$$

From (9), it is clear that the symmetric second rank stress tensor is decomposed into spherical (hydrostatic pressure) part, $1/3\sigma_{pp}\delta_{ij}$, which is the first term of (8), and the deviatoric part, σ_{ij}-$1/3\sigma_{pp}\delta_{ij}$, which is the sum of the other five terms of (8). It is shown that the method is able to decompose the symmetric second rank stress (and strain) tensors into the spherical part which is connected to the change of volume without change of shape, and into deviatoric part, which is connected to the change of shape. This result is very well known in the literature. On the other hand, this method is introducing a new form of decomposition, which has a more featured and transparent physical information. It is easily verified that the sum of the six decomposed tensors is the symmetric second rank tensor, σ_{ij}. Physically, each of the six tensor parts is associated with a distinct type of deformation; the first part of (8) represents the spherical (hydrostatic pressure) effect, the second and third parts represent combined simple extension or contraction along the various symmetry axes, the second term could be, for example, stress of a non-uniform distribution of pure shear stress, which occurs in a long rod subjected to pure torsion, while the last three parts represent simple shearing in the symmetry planes. Besides, the deviatoric part of the stress tensor is decomposed into traceless tensors each of them is related to shearing which represents a general symmetric second rank tensor (stress and strain tensors). It agrees with previous studies considered special cases of this general case, for instance, Blinowski *et al.* [11] have decomposed a tensor of only shear into exactly identical forms to the last three terms of (8) for this specific case.

3. THE CONCEPT OF NORM

The comparison of magnitudes of the norms can give valuable information about the origin of the physical property under examination. Since the norm is invariant in the material, the norm of a Cartesian tensor may be used as the most suitable representing and comparing the overall effect of a certain property of anisotropic materials of the same or different symmetry or the same material with different phases based on the crystallographic level [7,10,12-15]. The large the norm value, the more effective the property is. Generalizing the concept of the modulus of a vector, a norm of a Cartesian tensor is defined as the square root of the contracted product over all the indices with itself [10,12,15]. Since the constructed basis in this method is orthonormal and C_{ijkm} is in the space spanned by that orthonormal basis, the norm for the elastic stiffness, for example, is given by:

$$N = \left\{ \sum_k (C, T_{ijkm})^2 \right\}^{\frac{1}{2}}. \tag{10}$$

4. APPLICATIONS

Among semiconductors crystals, a family of wurtzite- type belongs to the 6mm class, which is piezoelectric active. The material properties [16] and the norm calculations are in Table 1. From the table, the most piezoelectric effective among the five materials is Cds which has a very important feature in the thin films of semiconductors. Piezoelectric ceramic is the most potential piezoelectric material because of its higher sterngth, high rigidity and more importantly, the better piezoelectricity. Table 2 includes the piezoelectric coefficients [17] and calculated norms for transversely isotropic materials. From the Table, the most effective piezoelectric among the seven ceramics is PZA-8. The extremely important applications for engineering purposes are the class of orthotropic fiber reinforced composite materials, and under specific couplings of their elastic constants, a family of orthotropic materials degenerates into the class of either transversely isotropic or isotropic media. Most of the engineering composites, especially the fiber-reinforced, are of transversely isotropic media. Hence, for different composites, the norms for each material [18] in are calculated Table 3. From the Table, it is concluded that the strongest stiffness effect among the five composites is the B(4)/N5505.

Table 1. The constants and norms of piezoelectric semiconductors [10^{-12} C N^{-1}].

MATERIAL	d_{11}	d_{33}	d_{15}	N
BeO	-0.12	0.24		0.29
ZnO	-5.0	12.4	-8.3	18.48
CdS	-5.2	10.3	-14.0	23.50
CdSe	-3.9	7.8	-10.0	17.07

Table 2. The constants and norms of Piezoelectric ceramics [10^{-12} C N^{-1}].

MATERIAL	d_{11}	d_{33}	d_{15}	N
PZT-4	-5.2	15.1	12.7	24.59
PZT-5	-5.4	15.8	12.3	24.71
BaTi O3	-4.35	17.5	11.4	18.82
PZT-5H	-6.5	23.3	17	34.72
PZT-6B	-0.9	7.1	4.6	9.71
PZT-8	-4.0	23.3	10.4	28.13
C-24	1.51	8.53	3.89	10.37

5. CONCLUSIONS

Any physical property is characterized by n rank tensors, and this method is capable for decomposing these tensors with intrinsic symmetry, which is derived from the nature of the physical property itself, of any rank into orthonormal tensor basis. This method is capable to decompose tensors with non-intrinsic symmetry of rank n, as well. The decomposition developed in this work for tensors has many engineering applications in anisotropic elastic materials which are, both qualitatively and quantitatively, different from isotropic materials. A new innovational

Table 3. The elastic constants and norms for transversely isotropic materials, GPa.

MATERIAL	C_{11}	C_{22}	C_{12}	C_{23}	C_{44}	C_{55}	NORM
T300/5208	184.60	13.94	5.88	7.06	3.44	7.17	174.06
B(4)/N5505	208.08	25.04	95.72	12.70	6.17	5.59	284.62
AS/H3501	141.80	12.20	85.08	6.21	3.00	7.10	222.11
E-glass/Epoxy	41.12	11.57	21.38	6.04	2.77	4.14	62.58
Kev 49/Epoxy	78.66	7.53	53.49	3.86	1.83	2.30	132.92

of general and more explicit physical property for the symmetric second rank stress and strain tensors is introduced. The results are compared to be identical to the special cases available in literature [7-13,15]. Nevertheless, this method is introducing a new form of decomposition that has a more featured and transparent physical information. Criteria to measure the overall effect of the material properties proposed and the norms which represent the piezoelectricity and stiffness effect in the material like fiber-reinforced composites and piezoceramics, respectively, are calculated. Through this method it is possible to study the effect of angle orientation of fibers and the material properties of fiber and matrix on the stiffness of the composite.

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