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# CRITICAL BEHAVIOR OF THE TWO-DIMENSIONAL RANDOMLY SITE-DILUTED ISING MODEL VIA WANG-LANDAU ALGORITHM

**I. A. Hadjiagapiou, A. Malakis and S. S. Martinos**

Section of Solid State Physics, Department of Physics, University of Athens, Panepistimiopolis, GR 15784 Zografos, Athens, Greece

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**Abstract.** The critical properties of the randomly site-diluted two-dimensional Ising model were studied using the Wang-Landau algorithm. The concentration of nonmagnetic sites was $q$ = 0.1; the remaining sites were occupied by magnetic particles. The study was carried out in the appropriate restricted but dominant energy subspaces. The main effort was focused on the specific heat and magnetic susceptibility by using the density of states for the model for lattices with linear size $L$ = 20-120. The finite size scaling behavior of the specific heat and susceptibility was studied. The ratio of the critical exponents ($\alpha/\nu$) is negative (consistent with the Harris criterion), while the ratio ($\gamma/\nu$) appears to assume its pure Ising model value (weak universality).

## 1. INTRODUCTION

The two-dimensional Ising model, although solved exactly by Onsager, still attracts the interest of physicists by considering various versions for it. Such one, is that with either vacant lattice-sites or occupied by non-magnetic atoms, in the presence of magnetic ones. The effect of randomness on the critical behavior and magnetic phase diagrams of classical random spin systems has attracted much interest in recent years, since an experimental sample usually is not free of the causes bringing about randomness. In general, randomness is encountered in the form of vacancies, variable bonds, impurities, random fields, etc. Its influence on the critical behavior and magnetic properties is a long-standing and still unsettled problem in statistical physics. Various methods were employed for the study of these systems, such as renormalization group, transition matrix and Monte Carlo (MC) simulation. In some cases, conflicting statements resulted revealing the inhibit hidden complexity and the need for more sophisticated approaches. An important question which arises is the extent to which randomness influences the critical behavior in comparison with those of the pure system. The first remarkable criterion for the influence of randomness was given by Harris, [1], according to which randomness changes the critical behavior if the specific heat critical exponent $\alpha$ of the pure system is positive, $\alpha > 0$. In this case, a new critical point with conventional power law scaling and new exponents emerges; however, for the pure two-dimensional Ising model the critical exponent $\alpha$ vanishes ($\alpha = 0$), hence the two-dimensional sitediluted Ising model constitutes a marginal case to the Harris criterion. As a result, many

Corresponding author: I. A. Hadjiagapiou, e-mail: ihatziag@phys.uoa.gr

contradicting suggestions have appeared thus increasing the confusion.

The site-diluted two-dimensional Ising model consists of a lattice with $N$ sites and of linear size $L$ such that $N = L^2$; some of the lattice sites are occupied by magnetic atoms. With each occupied lattice site a spin variable $s_i$, $i = 1,2, \ldots, N$, is associated assuming either value ±1, while the remaining sites are either occupied by nonmagnetic atoms (e.g., Al atoms [2]) or vacant; this system is called two-dimensional randomly sitediluted Ising model (2D RSDIM). For this system, critical exponents associated with the random fixed point have been investigated for dilution-type disorder by theoretical and numerical approaches, which lead to questionable conclusions, [3]. However, these approaches have also shed light on estimates of the critical temperature and nature of the phase transition. In addition, the magnetic phase diagram is strongly influenced, among other factors, by dilution, [4]. The concentration of vacancies (or nonmagnetic particles) shall be denoted by $q$ (dilution), while that of occupied sites (magnetic particles) by $p$ (purity), $q+p = 1$. The vacancies are considered to be quenched, randomly distributed throughout the lattice and uncorrelated, since one can encounter systems where the vacancy locations are correlated, [5,6].

The MC approach has been proved to be a powerful tool for studying difficult problems such as random spin systems; however, its main drawback is that it deals with finite-size systems. In some cases, the simulation method suffers from problems of slow dynamics and new algorithms have been proposed to overcome such difficulties. Wang–Landau (WL) [7] and entropic sampling [8] are examples of such efforts. The critical properties concern always an infinitely bulk system, since the phase transitions appear in such a system; however, from MC simulations the critical behavior is extracted from results obtained on a finite-size system by means of finite-size scaling (FSS), which, since its inception, has been evolved as a very powerful tool for extracting properties of the infinite system near a phase transition by studying finite systems. Although it is as yet not fully completed thus causing, sometimes, ambiguities about its results. When FSS is applied to a random system, randomness brings in additional complexity. Hence, it is of major interest to have a clear theoretical background behind the extrapolation from finite to infinite sizes, which is still lacking. The major goal of the finite-size method is to identify the set of critical exponents that, together with other universal parameters, characterizes the universality class. As these exponents offer the most direct test of universality, their precise calculation is of great importance. However, the experimental devices are finite, consequently, the exponents cannot be measured with infinite precision causing, occasionally, controversies in distinguishing the universality class a specific system belongs, [9]. For more on the theory of finite-size scaling see, *e.g.*, Barber [10], Privman [11], and Binder [12].

For the 2D RSDIM, the nature of the possible phase transition as well as the universality class is not completely understood. The value and even the sign of the specific heat exponent $\alpha$ is still not known. Currently, it seems that two scenarios have prevailed, which, however, are mutually exclusive. According to the first one, the critical exponents are unaffected by disorder, apart from possible logarithmic corrections, while according to the second, the critical exponents vary with disorder, but the exponents' ratios ($\gamma/\nu$), ($\beta/\nu$) remain the same as in the pure model, which is a manifestation of weak universality, [3].

Kim and Patrascioiu [13] studied the 2D RSDIM on a square lattice adopting periodic boundary conditions with dilute site concentration (dilution probability) $q = 1/9,1/4,1/3$ and lattice linear size $L$ up to 600, using MC simulation; their findings are that the specific heat does not diverge for $q = 1/4$, $1/3$ while for $q = 1/9$ it seems to increase as $\varepsilon \to 0$ ($\varepsilon = (T-T_c)/T_c$). The magnetic susceptibility $\chi$ and correlation length $\xi$ fit the pure power law, for which the value of the respective exponents $\gamma$ and $\nu$ increase with $q$ while $\eta = 2-\gamma/\nu$ remains the same as in the pure system. Queiroz and Stinchcombe [14] and Mazzeo and Kühn [15], using transfer-matrix-scaling technique for the same system, came to the same conclusions. Shchur and Vasilyev [3] analyzing the MC data for the magnetic susceptibility critical amplitudes ($\Gamma$,$\Gamma'$) above and below the critical temperature, respectively, for this model with $q \leq 0.25$ and lattice linear size up to $L = 256$, reached the conclusion that the ratio ($\Gamma/\Gamma'$) seems to remain constant for the dilute-site concentrations considered and equal to its pure system value, concluding that the 2D RSDIM belongs to the same universality class as the pure one. Renormalization group works on weakly diluted systems, treating disorder as a perturbation of the pure system, confirmed the existence of a new stable fixed point with new exponents in the case of Ising systems with positive $\alpha$-exponent, [16]. Dotsenko and

Dotsenko [17], Shalaev [18], Shankar [19], Ludwig [20], through field theoretical calculations, showed that randomness is irrelevant (critical exponents are unchanged), only logarithmic corrections might appear in the case of weak dilution; they found for the correlation length, magnetic susceptibility and magnetization, respectively,

$$\xi \propto |\varepsilon|^{-1}\left[1+\lambda \ln\left(\frac{1}{|\varepsilon|}\right)\right]^{1/2}, \quad (1)$$

$$\chi \propto |\varepsilon|^{-7/4}\left[1+\lambda \ln\left(\frac{1}{|\varepsilon|}\right)\right]^{7/8}, \quad (2)$$

$$m \propto |\varepsilon|^{1/8}\left[1+\lambda \ln\left(\frac{1}{|\varepsilon|}\right)\right]^{-1/16}, \quad (3)$$

while the specific heat diverges as ln(ln$L$)

$$C_v \propto |\varepsilon|^{-\alpha} \ln\left[1+\lambda \ln\left(\frac{1}{|\varepsilon|}\right)\right] + C', \quad (4)$$

where $\lambda$ is considered to be a smooth function of $q$ and is the same for these observables, $\alpha = 0$ and $C'$ a constant. The logarithmic correction in (ln[1+ $\lambda$ln(1/|$\varepsilon$|)]) is dominant very close to the critical point whereas it is negligible further away; this yields a crossover of critical behavior from that of the pure Ising system to that of the diluted. Heuer [21] studied the 2D RSDIM on a square lattice with $72 \le L \le 250$ and $0 \le q \le 0.4$, concluding that ($\gamma/\nu$) increases and ($\beta/\nu$) decreases as $q$ increases. Although Ballesteros *et al.* [22], by performing MC simulations in conjunction with finite-size scaling (FSS) for $p$ = 1, 8/9, 3/4, 2/3, observed small deviations of the values of the critical exponents from those of the pure system as varying the concentration, they considered it as a transient effect, since this can be lifted if a pure Ising value for the exponents is combined with logarithmic corrections. Selke *et al.* [23], using MC techniques for lattices with linear size $8 \le L \le 256$ and spin concentration $0.1 \le p \le 1$, concluded that impurities lead to divergence in the specific heat as $C \sim$ ln(ln$L$) on approaching the critical temperature. Tomita and Okabe [24] performed MC simulations on the same system on a square lattice using the probability-changing cluster (PCC) algorithm confirming that randomness is irrelevant and its influence is evident through logarithmic corrections.

Due to the fact that conflicting viewpoints concerning the dependence of the critical exponents on randomness of this model have arisen, we shall study the critical properties of the 2D RSDIM using the MC method in conjunction with FSS. The data was generated by extensive MC calculations using WL sampling method to estimate the density of states (DOS) $g(E)$ [7]. WL sampling performs a random walk with an acceptance ratio $P(E_i \to E_j) = \min\{1, [g(E_i)/g(E_j)]\}$, $E_i$ and $E_j$ are the energies before and after the transition, respectively, and aiming at sampling a flat histogram in energy. The WL algorithm overcomes difficulties, such as critical slowing down and long relaxation times in systems with complex energy landscape, appearing in other MC processes. This algorithm has been applied to a wide range of systems spanning from the Ising model [25-27], 3D conserved-order-parameter Ising model [28], competing interactions [29], Potts [30], to glassy systems [31], polymers [32], DNA [33] and the Baxter-Wu model [34]. The DOS $g(E)$ is not constant during the random walk, it is modified according to the rule $g(E) \to (g(E)f)$; the modification factor $f$ varies as $f_{j+1} = \sqrt{f_j}$, $j$ being the order of iteration. In the current investigation the WL algorithm was performed for 24 iterations and the initial value for $f$ was $f = e$. In the MC methods nonbased on the calculation of $g(E)$, one has to perform, for each temperature, a large number of MC steps for a long time period to estimate the proper quantities, a time consuming procedure.

In this analysis, the major goal is to estimate the critical temperature and exponents of the 2D RSDIM for various values of the spin concentrations to assess whether it belongs to the Ising universality class or not. This is achieved by studying the finite-size behavior of the specific heat and magnetic susceptibility. We associate, for every value of $L$, a dilution-dependent temperature $T_C^*(q,L)$, called "pseudocritical temperature", with maximum value $C_L^*(q)$ for the specific heat and one with the respective maximum value for the magnetic susceptibility $\chi_L^*(q)$, the latter denoted by $T_\chi^*(q,L)$. In each case, a sequence of "pseudocritical temperatures" is formed converging to the critical temperature $T_c(q)$ of the infinite system as $L \to \infty$.

The layout of the paper is as follows. In the next section we discuss the Wang-Landau algorithm and our numerical implementation related to the system's energy space. In section 3 we apply the FSS to the Ising model under consideration and close with the conclusions and discussions in section 4.

## 2. THE SIMULATION METHOD

The Hamiltonian of a two-dimensional site-diluted Ising model, in the absence of any external field, takes the form,

$$H = -J\sum_{<ij>} c_i c_j S_i S_j ,$$
$$S_i = \pm 1, \qquad (5)$$

where $J>0$ is the interaction constant, ferromagnetic interactions. $c_i$'s, called occupation variables, are quenched, uncorrelated random variables chosen to be equal to 1 with probability $p$, when the $i$-site is occupied by a magnetic atom and 0, with probability $q$=1-$p$, otherwise; that is, we have as probability distribution $P(c_i)= p\delta(c_i-1)+q\delta(c_i)$. The summation extends over all nearest-neighbor pairs of the square lattice, of linear size $L$ with periodic boundary conditions. The WL algorithm was implemented on lattices with $20\le L\le 120$ and the density of states was stored as a function of the energy. The dilution $q$ can vary from 0.0 ($p$ = 1) to the percolation threshold $q_c^{\mathrm{PERC}}$ = 0.407255 ($p_c^{\mathrm{PERC}}$ = 0.592745(2)), [35]; we consider only the case $q$ = 0.1.

The presence of randomness is evident in the way the averaging processes are carried out for an observable $X$. First, the usual thermal average is performed (for a specific realization of the dilution) and after completion of the $M$ realizations of randomness, the disorder average is carried out over the $M$ realizations, denoted by square brackets, [ ]. The observable $X$ assumes a different value for each of the $M$ random realizations of the disorder corresponding to the same value of dilution $q$.

As is well-known, for the pure Ising model the specific heat and magnetic susceptibility in the thermodynamic limit diverge; in a finite lattice, the divergence is rounded off and manifests itself by a maximum exhibited in the above quantities. This maximum increases gradually with $L$ and ultimately tends to infinity as $L\to\infty$. For the 2D RSDIM, in attempting to detect the maximum $C^*(q,L)$ of the specific heat and the respective temperature $T^*_C(q,L)$ (pseudocritical temperature), we considered two routes. Let $C_m(q,L)$ be the specific heat for a particular realization $m$, out of $M$ realizations, for a specific dilution $q$. In the first route, we estimated, initially, the maximum value $C^*_m(q,L)$ together with respective pseudocritical temperature $T^*_{C,m}(q,L)$ for every realization $m$ of the disorder; then, we considered the sample average of the individual specific heat maximum $[C^*(q,L)]$ and pseudocritical temperature $[T^*_C(q,L)]$, for the $M$ realizations,

$$[C^*(q,L)]_{sum} = \frac{1}{M}\sum_{m=1}^{M} C^*_m(q,L),$$
$$[T^*_C(q,L)] = \frac{1}{M}\sum_{m=1}^{M} T^*_{C,m}(q,L). \qquad (6)$$

In the second route, following Rieger and Young, the sample summation for the specific heat for the totality of $M$ realizations was considered [36],

$$[C(q,L)]_{sum} = \frac{1}{M}\sum_{m=1}^{M} C_m(q,L). \qquad (7)$$

In (7), the resulting specific heat curve is very complex with many local maxima, reflecting the strong pseudocritical temperature fluctuations in the ensemble of random realizations; the more realizations the smoother the specific heat curve becomes. From these maxima we selected the absolute one, indicated by $[C(q,L)]^*_{\mathrm{sum}} \equiv \max[C(q,L)]_{\mathrm{sum}}$, occurring at the pseudocritical temperature $T^*_{C,\mathrm{sum}}(q,L)$. In general, $[C^*(q,L)]\neq [C(q,L)]^*_{\mathrm{sum}}$ and $[T^*_C(q,L)]\neq T^*_{C,\mathrm{sum}}(q,L)$. The same procedure was also followed for the magnetic susceptibility $\chi$, the resulting quantities are denoted by $[\chi^*(q,L)]$, $[T^*_\chi(q,L)]$, $[\chi(q,L)]^*_{\mathrm{sum}}$, $T^*_{\chi,\mathrm{sum}}(q,L)$.

An important issue arising in a system with randomness is that of self-averaging, that is, to what extent properties of the system depend on the particular realization of the quenched random variables. Let $X$ be a thermodynamic quantity, $[\overline{X}]$ its thermal (denoted by an overline) and random–sample average over dilution and $V_X = [(\overline{X} - [\overline{X}])^2]$ the respective variance. If the ratio $R_X(L)=V_X/[\overline{X}]^2$ tends to zero as $L\to\infty$ then the random system is self-averaging, [37]. This implies that the thermal average of the observable $X$ does not depend on random samples as the system size becomes infinite.

To investigate the critical behavior of the system, we performed extensive MC simulations to calculate the density of states $g(E)$ for each value of $L$ and dilution $q$ through the WL algorithm. In calculating $g(E)$, temperature is not needed to be specified a priori, the main advantage of the WL algorithm. After estimating the density of states $g(E)$, one can proceed to the calculation of the necessary thermodynamic quantities, such as the energy $E$, specific heat $c_L(T)$, magnetization $M$, and susceptibility $\chi_L(T)$ for further use in order to identify the probable universality class. However, be-

fore proceeding to this end, we outline a new method for speeding up the numerical calculations; this method is called "Critical Minimum Energy Subspace" (CrMES) technique, [26-28,34]. Let's begin by considering the specific heat per site for a lattice of linear size $L$ at temperature $T$,

$$c_L(T) = L^{-d}T^{-2}\left\{Z^{-1}\sum_{E_{min}}^{E_{max}} E^2 \exp[S(E) - \beta E] - \right.$$
$$\left.(Z^{-1}\sum_{E_{min}}^{E_{max}} E \exp[S(E) - \beta E])^2\right\}, \quad (8)$$

where the Boltzmann constant was set $k_B = 1$, thus $\beta = 1/T$, $d$ is the spatial dimension ($d = 2$) and $Z$ the "partition function" of the system,

$$Z = \sum_{E_{min}}^{E_{max}} \exp[S(E) - \beta E]. \quad (9)$$

The latter expression is the partition function in case $g(E)$ is the exact DOS of the system and properly normalized, [38]. In practice, the DOS, resulting from WL simulations, is an approximate result whose accuracy depends on that of the simulation. In the expression (8), the calculation of the specific heat in the critical region can be speeded up by restricting the energy interval if we use the CrMES technique. Let $\tilde{E}$ be the energy corresponding to the maximum term $\exp[S(E)-\beta E]$ of the partition function (9) for the temperature at hand. Because of the sharpness of the energy distribution, the energy interval $(E_{min},E_{max})$ in the summation (8) is replaced by a smaller one $(\tilde{E}_-, \tilde{E}_+)$ around $\tilde{E}$ corresponding to a predefined accuracy $r$ for the specific heat expressed as, $|[c_L(\tilde{E}_-,\tilde{E}_+)/c_L(E_{min},E_{max})]-1| \le r$, where $r = 1\cdot10^{-6}$ and $\tilde{E}_\pm = \tilde{E} \pm \Delta_\pm$, $\Delta_\pm \ge 0$. The induced errors are much smaller than the ones in determining the DOS, for more see [26]. The magnetic properties were obtained using the final stages of the WL algorithm, see [39].

## 3. FINITE-SIZE SCALING ANALYSIS. RESULTS

The FSS is based on the assumption that the free energy of a system of linear size $L$ and in the absence of an external magnetic field scales as,

$$F(L,t) = L^{-\psi}F_0(tL^{\theta}), \quad (10)$$

where $\psi = (2-\alpha)/\nu$ and $t = |(T-T_c)/T_c|$. The scaling of the correlation length $\xi = \xi_0 t^{-\nu}$ suggests that $\theta = \nu^{-1}$. The scaling function $F_0(x)$ is universal, in that, it is independent of the lattice size.

In the general contemplation of critical phenomena and specifically in FSS, an important task is the determination of the critical temperature $T_c$. We, therefore to this end, analyze the pseudocritical-temperature sequences for the specific heat and susceptibility according to FSS.

First, we focus on the specific heat. Following Rieger and Young [36], the maximum $[C(q,L)]^*_{sum}$ and its corresponding pseudocritical temperature $T^*_{C,sum}$ obey the scaling laws,

$$[C(q,L)]^*_{sum} = p_0 + q_1 L^{\alpha/\nu_{SpH}},$$
$$T^*_{C,sum}(q,L) = T_{c,SpH} + b_1 L^{-1/\nu_{SpH}}. \quad (11)$$

We also adopt the same scaling laws for the sample averages of the specific heat maxima and the respective pseudocritical temperatures, see (6), but with different symbols for the critical exponents because of the finiteness of lattices, that is,

$$[C^*(q,L)] = \tilde{p}_0 + \tilde{q}_1 L^{\tilde{\alpha}/\tilde{\nu}_{SpH}},$$
$$[T^*_C(q,L)] = \tilde{T}_{c,SpH} + \tilde{b}_1 L^{-1/\tilde{\nu}_{SpH}}. \quad (12)$$

In addition, we considered the FSS for both susceptibility averages. The maximum $[\chi(q,L)]^*_{sum}$ and its corresponding pseudocritical temperature $T^*_{\chi,sum}$, obey the scaling laws,

$$[\chi(q,L)]^*_{sum} = r_0 + s_1 L^{\gamma/\nu_\chi},$$
$$T^*_{\chi,sum}(q,L) = T_{c,\chi} + c_1 L^{-1/\nu_\chi}, \quad (13)$$

while for the corresponding sample averages,

$$[\chi^*(q,L)] = \tilde{r}_0 + \tilde{s}_1 L^{\tilde{\gamma}/\tilde{\nu}_\chi},$$
$$[T^*_\chi(q,L)] = \tilde{T}_{c,\chi} + \tilde{c}_1 L^{-1/\tilde{\nu}_\chi}. \quad (14)$$

Initially, we focus on estimating the critical temperature $T_c(q)$ of the diluted system by fitting the pseudocritical temperatures of specific heat and susceptibility for both averaging processes, see (6,7), to the respective scaling laws in (11)-(14), see Table 1. In this estimation, one calculates simultaneously the n-exponent, as well. The results of the fitting procedure for the scaling laws (11) and (13) appear in Figs. 1 and 2. The chi-square value of the fit is $1.1597\cdot10^{-6}$ and $8.3462\cdot10^{-6}$, respectively. Similar plots result for the scaling laws (12) and (14). The fitting processes lead to the fol-

**Table 1.**

| $L$ | $T^*_{c,sum}$ | $T^*_{\chi,sum}$ | $[C(q,L)]^*_{sum}$ | $[\chi(q,L)]^*_{sum}$ |
|---|---|---|---|---|
| 10 | 2.010(0.003) | 2.170(0.003) | 0.759(0.003) | 2.613(0.109) |
| 20 | 1.965(0.001) | 2.035(0.002) | 1.166(0.003) | 8.395(0.400) |
| 30 | 1.946(0.002) | 1.991(0.002) | 1.333(0.005) | 19.235(1.252) |
| 40 | 1.936(0.001) | 1.975(0.001) | 1.424(0.007) | 32.261(1.767) |
| 50 | 1.930(0.001) | 1.960(0.001) | 1.469(0.005) | 47.118(2.385) |
| 60 | 1.926(0.001) | 1.953(0.001) | 1.509(0.005) | 64.521(3.276) |
| 70 | 1.924(0.001) | 1.946(0.001) | 1.534(0.009) | 84.079(9.365) |
| 80 | 1.923(0.001) | 1.939(0.001) | 1.545(0.009) | 103.494(11.356) |
| 90 | 1.921(0.001) | 1.932(0.001) | 1.555(0.006) | 123.571(15.828) |
| 100 | 1.920(0.001) | 1.929(0.001) | 1.570(0.007) | 145.043(16.380) |
| 120 | 1.917(0.001) | 1.927(0.001) | 1.578(0.006) | 195.812(22.032) |

lowing asymptotic values for the critical temperature for the dilution $q = 0.1$,

$$T_{c,SpH} = 1904 \pm 0.002,$$
$$\tilde{T}_{c,SpH} = 1910 \pm 0.001,$$
$$T_{c,\chi} = 1906 \pm 0.004, \qquad (15)$$
$$\tilde{T}_{c,\chi} = 1895 \pm 0.003.$$

Although the asymptotic values of the critical temperature for the various estimating processes are different, their relative difference is small and that between the two extreme values is roughly 0.8%. For low impurity concentrations $q$, as in the current case, the critical temperature for the diluted system can be estimated by the formula $T_c(q)=T_c(0)(1-1.565q)$, $T_c(0)$ is the pure system critical temperature, see Stinchcombe [40]; thus for $q = 0.1$ it yields $T_c(q = 0.1) = 1.9141$; comparing this value with the ones in (15) (calculated by FSS) we see a very good agreement. Specifically, the asymptotic critical temperature $\tilde{T}_{c,SpH}$, resulting as sample average of the specific heat pseudocritical-temperatures, differs from $T_c(q = 0.1)$ 0.2%. This

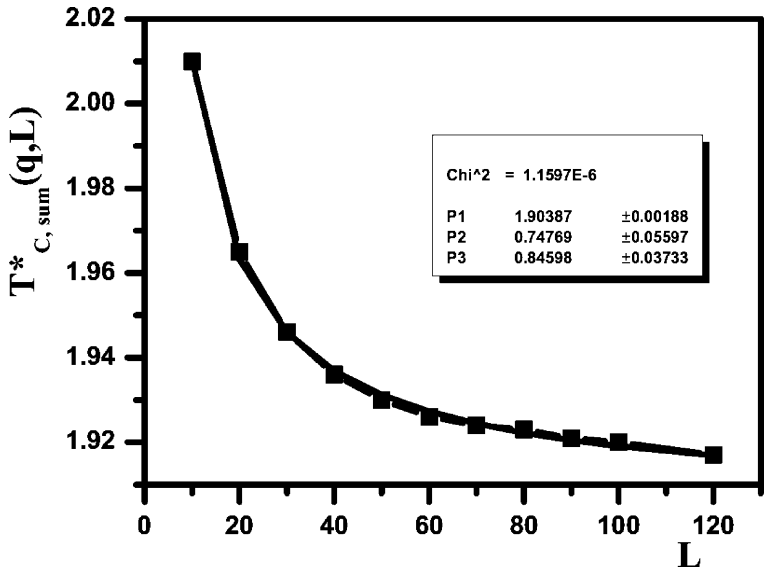


**Fig. 1.** The specific heat pseudocritical temperatures $T^*_{C,sum}$ vs. lattice size $L$ for $q = 0.1$. The solid curve is the best-fit using the respective scaling expression in (11). The parameters $P_1$, $P_2$, $P_3$ in the inlet represent the respective coefficients in the scaling law (11).

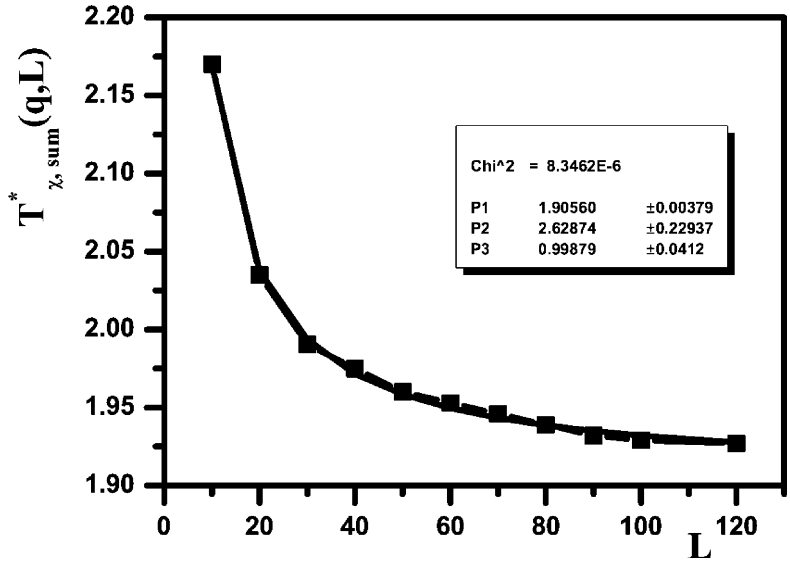


**Fig. 2.** The susceptibility pseudocritical temperatures $T^*_{\chi,\mathrm{sum}}$ vs. $L$ for $q = 0.1$. The solid curve is the best-fit using the respective scaling expression in (13). The parameters $P_1$, $P_2$, $P_3$ in the inlet represent the respective coefficients in the scaling law (13).

implies that the specific heat can estimate the critical temperature without the need to use very large lattices, while such lattices are needed for calculating it from the susceptibility, since in this case the difference between $T_c(q = 0.1)$ and $\tilde{T}_{c,\chi}$ is 1%. Although, the other two estimates, $T_{c,SpH}$ and $T_{c,\chi}$, result from summing the respective curves, where some peaks are either wiped out or enhanced that might conceal the true critical temperature, they are close to $T_c(q = 0.1)$. The mean value of the above critical-temperature estimations is $T_{av}$=1.904±0.002; this agrees with that estimated by Tomita and Okabe, $T_c$ = 1.9022(6) [24], by Heuer, $T_c$ = 1.90045 [21], and consistent with that by Kim and Patrascioiu, [13], using the same dilution.

In addition, from the scaling laws in (11)-(14), one can read off the correlation length exponent ν, thus

$$\begin{aligned}
\nu^{-1}_{SpH} &= 0.846 \pm 0.037, \\
\tilde{\nu}^{-1}_{SpH} &= 0.955 \pm 0.021, \\
\nu^{-1}_{\chi} &= 0.999 \pm 0.041, \\
\tilde{\nu}^{-1}_{\chi} &= 0.947 \pm 0.027.
\end{aligned} \tag{16}$$

The mean-value of the above estimations for the inverse correlation–function exponent is $\nu^{-1}$ = 0.937±0.032 and the resulting value is ν= 1.067±0.035, consistent with that in [15].

In the next step, we analyze, by applying the FSS, the size dependence of the maximum of the specific heat for both averaging processes, for deciding whether it diverges or not. According to the respective expression in (11) and (12), one can also read off the exponent ratio (α/ν) as well as the asymptotic value $p_0$, in case it assumes a finite value. The results of the fitting procedure for $[C(q,L)]^*_{\mathrm{sum}}$, (see (11)), appear in Fig. 3 and the chi-square value of the fit is 0.00009; a similar plot results for $[C^*(q,L)]$, (see (12)). The corresponding estimates are,

$$\begin{aligned}
\frac{\alpha}{\nu_{SpH}} &= -0.866 \pm 0.036, \\
\frac{\tilde{\alpha}}{\tilde{\nu}_{SpH}} &= -0.932 \pm 0.029, \\
p_0 &= 1700 \pm 0.016, \\
\tilde{p}_0 &= 1719 \pm 0.013.
\end{aligned} \tag{17}$$

From (17) we conclude that the ratio (α/ν) is negative in both processes, in conformity with the Harris criterion [1], although they assume different numerical values. This implies that the specific heat tends to a limit value given by that of the constant term ($p_0$,$\tilde{p}_0$) in each expression, see Fig. 3.

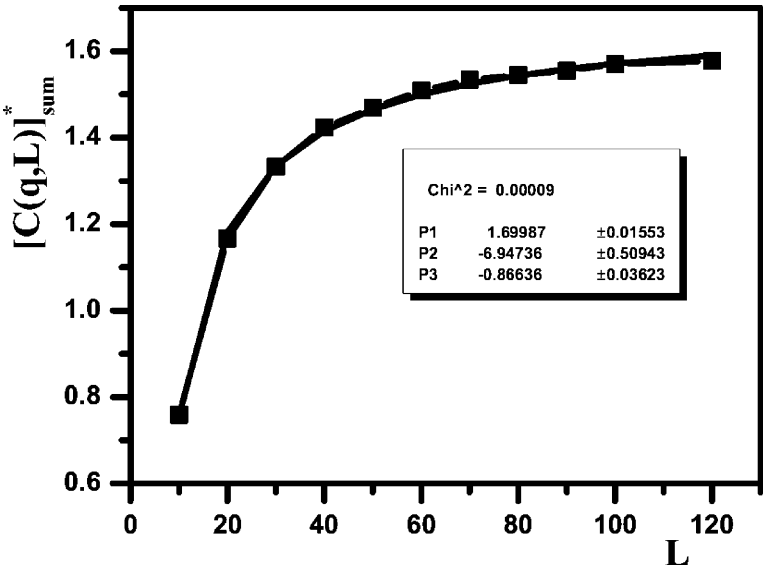


**Fig. 3.** The specific heat maxima $[C(q,L)]^*_{sum}$ vs. lattice size $L$ for $q = 0.1$. The solid curve is the best-fit using the respective scaling expression in (11). The parameters $P_1$, $P_2$, $P_3$ in the inlet represent the respective coefficients in the scaling law (11).

Next, we analyze the susceptibility MC results, for both averaging processes, by fitting them to the respective scaling laws in (13,14); the log-log plot for $[\chi(q,L)]^*_{sum}$ appears in Fig. 4 and the chi-square value of the fit is 0.00256; a similar plot results for $[\chi^*(q,L)]$, (see (14)). The resulting value of the critical exponents ratio ($\gamma/\nu$) are,

$$\frac{\gamma}{\nu_\chi} = 1752 \pm 0.021\,, \qquad \frac{\tilde{\gamma}}{\tilde{\nu}_\chi} = 1745 \pm 0.019\,. \tag{18}$$

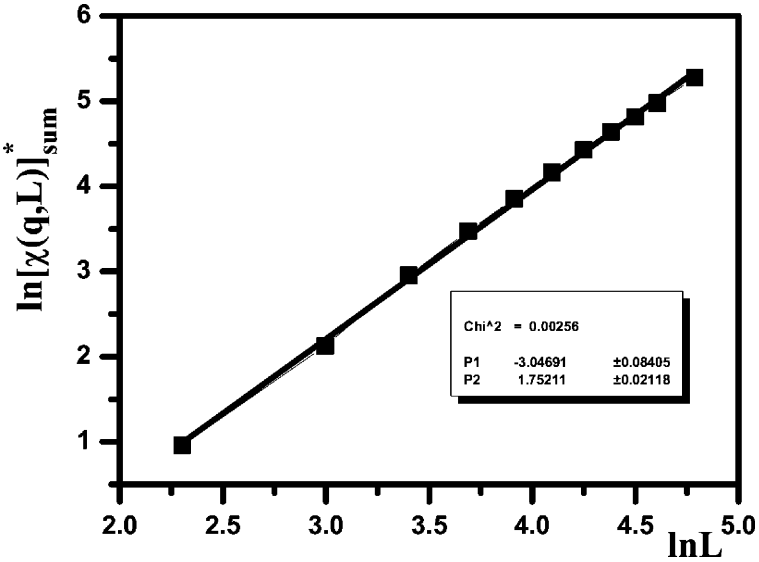


**Fig.4.** The susceptibility maxima $\ln[\chi(q,L)]^*_{sum}$ vs. $\ln L$ for $q = 0.1$. The solid curve is the best-fit for the law $\ln[\chi(q,L)]^*_{sum} = P_1 + P_2 \ln L$ for obtaining the ratio $P_2 = \gamma/\nu_\chi$.

The results in (18) imply that the ratio ($\gamma/\nu$) assumes its pure Ising model value ($\gamma/\nu$) = 1.75 (weak universality) for the weak disorder $q$ = 0.1 as well as the exponent $\eta$= 2-$\gamma/\nu$, that is, $\eta$ = 1/4.

## 4. CONCLUSIONS AND DISCUSSIONS

The Wang-Landau algorithm was applied to the two-dimensional Ising model with random site dilution. We analyzed the MC data for the specific heat and susceptibility following two distinct averaging processes. We have found that the critical behavior is clearly affected by disorder for the aforementioned quantities as well as the correlation length. Our data provides enough evidence that the 2D RSDIM cannot be described by the pure Ising critical exponents with additional logarithmic corrections, but instead the invariance of ($\gamma/\nu$) supports the scenario of weak universality at least for the ratio of the exponents $\gamma$ and $\nu$. The constancy of the ratio $\gamma/\nu$ was also verified in Refs. [13-15]. The important effect of disorder occurs in the specific heat whose exponent is negative, thus the Harris criterion appears to apply also in this random system. In addition, data imply that the specific heat tends to a finite limit rejecting, in this way, the slow divergence implied by ln(ln$L$), while susceptibility and correlation length diverge.

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