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INFLUENCE OF THE DISPERSION ON THE EQUATIONS FOR NONHOMOGENEOUS MECHANICS

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Abstract. The consequences from the calculation of an angular momentum in an elementary volume of gases, liquids or solids are discussed. The modified laws of conservation for gases, fluids, and solids were received for the particles without structure. The equations for the gases follow from the modified Boltzmann equation. Usually the law of angular momentum is postulated in the form of the symmetric stress tensor in spite of the fact that in general case the movement of particles is non-inertial. Taking into account the angular moment law, a nonsymmetrical stress tensor is received. The method for calculation of nonsymmetrical part is suggested. Besides, the local equilibrium distribution function f_0, as the basis in the solution of the Boltzmann equation by the Chapman-Enskog method, is verified. Steady motion of conducting fluids in pipes under transverse magnetic fields is investigated for the modified equations. Other examples are discussed.

1. INTRODUCTION

Mathematical modeling of different phenomena, roughly speaking, is reduced to solution of two problems: physical one, i.e. creation of an adequate model and mathematical one, i.e. the formulation of a problem and the development of a solution method. The most brightly the interaction of these aspects appears for the Boltzmann equation both for classical one and modified. We discuss the problems that can be appearing when considering an angular moment variation in an elementary volume. The theory by Bogolubov [1] has difficulties to write the Boltzmann equation for a nonhomogeneous gas. In this work the influence of great gradient density, velocity or temperature is investigated. The modified conservation laws were suggested in [2-7]. Under deduction of the Boltzmann equation one has a truncation error which is proportional to the product of a free path for the particle velocity by a time increment and by gradients of the distribution function. By using an average free path, we have the macroscopic equation for velocity of molecules, but one must use the velocity of a considered molecule. Another problem for solving this equation is asymptotical methods. It is essential to select the local equilibrium distribution function f_0 as the basis for solution of the Boltzmann equation by the Chapman-Enskog method with macroscopic parameters from the Euler equations. The macroscopic parameters that determine the Chapman-Enskog distribution function lead to the Euler equations' parameters. Formally we have values (density, linear moment, energy) with the first order errors. This fact was noted by Hilbert without further use and correction. Therefore

$$f(t,x,\xi) \equiv f_0(t,x,\xi) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m}{2kT}c^2\right\},$$

$$\left(c^2 = c_1^2 + c_2^2 + c_3^2\right) = (\xi - \boldsymbol{u})^2$$

and

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$$f = f_0\left[1+\frac{p_{ij}m}{2pkT}c_ic_j - \frac{q_im}{pkT}c_i\left(1-\frac{mc^2}{6kT}\right)\right]$$

has the same macroparameters in f_0. Here T is a temperature, $\boldsymbol{u}$ is the velocity, k is the Boltzmann constant. Besides, it is essential for the value of viscosity. So it is necessary to do iteration for all values (density, linear moment, energy). Tensor P enters as p in the Euler equations and as a symmetric tensor in the Navier-Stokes equations. Consequently the p is not equal to the p in the macroscopic Euler equations. Numerical solution of the Bolzmann equation does not contain such mistakes. Besides, we have the equation for an angular moment, if we take into account rotation of an elementary volume for great gradients of physical values. In classical case, the law of angular momentum is postulated in spite of the fact that in general case the movement of particles is non-inertial. In general, the nonstationary operator for the motion equations and energy is $(\partial/\partial x_j)x_j(\partial/\partial t)$. It follows from the fact that $\partial u/\partial t$ has the dimension of a force [8]. The main attention in our report is given to the analysis of the enumerated questions. Steady motion of conducting fluids in pipes under transverse magnetic fields is investigated in framework of the modified equations.

2. INFLUENCE OF DISPERSION IN FLUID MECHANICS

In the classical case the law of the angular moment can be formulated into the integral form

$$\frac{d}{dt}\int_r \vec{\mathbf{r}}\times\rho\vec{\mathbf{V}}\,d\tau = \int_r \frac{d\vec{\mathbf{r}}}{dt}\times\rho\vec{\mathbf{V}}\,d\tau +$$

$$\int_r \vec{\mathbf{r}}\times\rho\frac{d\vec{\mathbf{V}}}{dt}\,d\tau + \int_r \vec{\mathbf{r}}\times\rho\vec{\mathbf{V}}\frac{d}{dt}(\rho\,d\tau)$$

or

$$\int_\tau\left[\vec{\mathbf{r}}\times\left(\rho\frac{d\vec{\mathbf{V}}}{dt}+\dot{M}\vec{\mathbf{V}}-\rho\vec{\mathbf{F}}-\frac{\partial\vec{\mathbf{P}}_x}{\partial x}-\frac{\partial\vec{\mathbf{P}}_y}{\partial y}-\frac{\partial\vec{\mathbf{P}}_z}{\partial z}\right)-\right.$$

$$\left.\frac{\partial\vec{\mathbf{r}}}{\partial x}\times\vec{\mathbf{P}}_x-\frac{\partial\vec{\mathbf{r}}}{\partial y}\times\vec{\mathbf{P}}_y-\frac{\partial\vec{\mathbf{r}}}{\partial z}\times\vec{\mathbf{P}}_z\right]d\tau = 0.$$

If we get the equilibrium conditions of forces, the symmetrical stress tensor, received as the first term, is equal to zero. Another form of the equilibrium is angular moment equilibrium. In this case the first term is not zero, but it is divergence of the angular moment. The angular moment does not

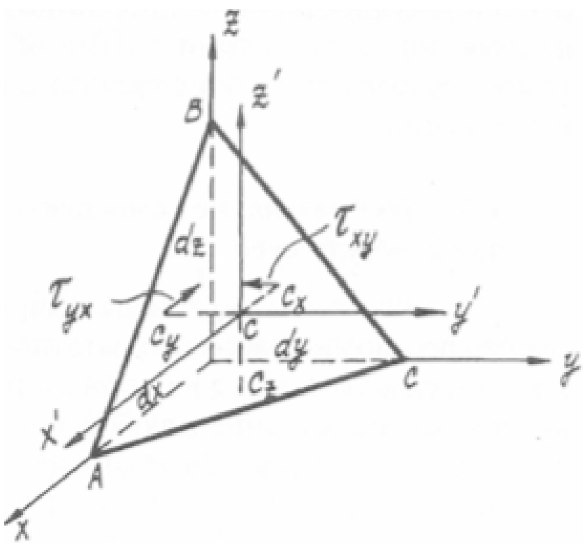

Fig. 1.

contain new dimension constants, so we have the modified Navier-Stokes equations [6,7] and the conservation equations

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u_i}{\partial x_i}+\frac{\partial}{\partial x_i}\left(x_i\frac{\partial\rho u_i}{\partial x_i}\right)=0.$$

$$\frac{\partial\rho u_i}{\partial x_i}+\frac{\partial}{\partial x_i}\left(\rho u_iu_j+P_{ij}+x_i\frac{\partial P_{ij}}{\partial x_i}\right)-\frac{X_i}{m}\rho=0.$$

$$\frac{\partial}{\partial t}\rho\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+\frac{\partial}{\partial x_i}\left[\rho u_j\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+\right.$$

$$\left.u_kP_{kj}+q_j\right]+\frac{\partial}{\partial x_i}x_i\frac{\partial}{\partial x_j}\left[\rho u_j\left(\frac{3}{2}RT+\frac{1}{2}u^2\right)+\right.$$

$$\left.u_kP_{kj}+q_j\right]=0.$$

We obtain the equation for angular moment from the modified Boltzmann equation.

$$\frac{\partial\vec{r}}{\partial x}\times\vec{p}_x+\frac{\partial\vec{r}}{\partial y}\times\vec{p}_y+\frac{\partial\vec{r}}{\partial z}\times\vec{p}_z+x_j\frac{\partial}{\partial x_j}\left(\vec{P}_j\right)=M_1$$

For small gradients the angular moment $M_1 \approx 0$. The last equation for usual Boltzmann equation degenerates into the equality $P_{ij} = P_{ji}$. The number of molecules is defined by the one-particle distribution function $f(\mathrm{r} + \vec{\xi}\cdot\Delta t)$. Here Δt is the spacing time for molecules, $\vec{\xi}$ is the phase velocity. The total number of collisions for ξ molecules in an element $dxd\xi$ is

$$\Delta^{-} = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\xi f(t,x,\xi)\int\Big[f(t,x,\xi) +$$

$$O\left(\Delta t\xi\frac{\partial f_1}{\partial x}\right)\Big] gb\,\mathrm{d}b\,\mathrm{d}\varepsilon\,\mathrm{d}\xi_1,$$

$$\Delta^{+} = \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\xi' f(t,x,\xi')\int\Big[f(t,x,\xi')f(t,x,\xi'_1) +$$

$$O\left(\Delta t\xi\frac{\partial f_1}{\partial x}\right)\Big] g'b'\mathrm{d}b'\mathrm{d}\varepsilon'\mathrm{d}\xi_1,$$

$$I = \Delta^{-} - \Delta^{+}.$$

Here ε is an angle, b is the sighting distance, $\boldsymbol{g} = \xi_1$, ξ, are parameters of a molecules; after collisions they are noted by a prime. The usual Boltzmann equation follows from it, if the macrofunction is varied slowly at the mean free path. Then tensor P is symmetric. Formally by this way we have values (density, linear moment, energy) with error of the first order. This fact was noted by Hilbert for the problem with small perturbations without further use and correction, $\left(c^2 = c_1^2 + c_2^2 + c_3^2\right) = (\xi - \boldsymbol{u})^2$. As the movement of a body is investigated in the coordinates connected with a body, the distribution function is

$$f = f_0\left[1 + \frac{p_{ij}m}{2pkT}c_ic_j - \frac{q_im}{pkT}c_i\left(1 - \frac{mc^2}{5kT}\right)\right]$$

with density, linear moment, energy in f_0 from Euler equation.

Consequently the usual theory gives uncoordinated actions at approximation density, linear moment, and temperature calculated for the distribution function from Euler and Navier-Stokes equations. The way out of this situation is to do new iteration for determination of macroparameters for the local distribution function, i.e. the derivatives of macroparameters are determined from Navier-Stokes equations. So we have in classical case

$$\left.\frac{\mathrm{d}f^{(0)}}{\mathrm{d}t}\right|_{t=0} = f^{(0)}\left\{\frac{m}{kT}\left(c_ic_j - \frac{1}{3}c^2\delta_{ij}\right)\frac{\partial u_i}{\partial x_j} + \frac{1}{2T}\frac{\partial T}{\partial x_i}c_i\left[\left(\frac{m}{kT}\right)c^2 - 5\right]\right\}.$$

It is necessary to sum the term $(1/\rho)\cdot(\partial P_{ij}/\partial x_j)$ into the braces multiplied by the first bracket. Uniting the new term with exponent, we have the local distribution function concerted with the macroparameters of the Navier-Stokes equations. For the modified case, we have

$$f_0 = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left[-\frac{n\left(\vec{c} + \vec{v} + \mathrm{rot}\vec{v}\times\vec{r}\right)^2}{kRT}\right].$$

As in the classical case the united new term with exponent has distribution function with macroparamerters from Navier-Stokes equations. This function has parameters which concerted in order approximation. At that time, we shall solve the equation in order to describe viscosity (as in the classical case)

$$\int f^{(0)}f_1^{(0)}\left(\varphi_1^{(1)'} + \varphi^{(1)'} - \varphi_1^{(1)} - \varphi^{(1)}\right)gb\,\mathrm{d}b\,\mathrm{d}\varepsilon\,\mathrm{d}\xi_1 = \frac{1}{2}f^0\left\{\frac{1}{2T}c_i\left[\left(\frac{m}{kT}\right)c^2 - 5\right]\frac{\partial T}{\partial x_i} + \frac{m}{kT}\left(c_ic_j - \frac{1}{3}c^2\delta_{ij}\right)\frac{\partial u_i}{\partial x_j}\right\}.$$

Formally we have old values.

The degree of the asymmetry for the stress tensor, we can receive from the moment equation (in projections)

$$y\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right) - z\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z}\right) + \tau_{zy} - \tau_{yz} = 0,$$

$$x\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right) - z\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right) + \tau_{zx} - \tau_{xz} = 0,$$

$$x\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z}\right) - y\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right) + \tau_{yx} - \tau_{xy} = 0.$$

Thus we have (Fig. 1)

$$\frac{\partial}{\partial y}\left[x\left(\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{zy}}{\partial z}\right) - y\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)\right] = \Delta_{yx},$$

$$\frac{\partial}{\partial z}\left[x\left(\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right) - z\left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{yx}}{\partial y} + \frac{\partial\sigma_{zx}}{\partial z}\right)\right] = \Delta_{zx},$$

For the same indices (*ii, kk, jj*) it is necessary to calculate the gradient of physical values, but it gives the classical stress tensor. To define connection of the stress tensor with the velocity (rheology), it should be found the point into an elementary volume with forces equal zero. Then we have to con-

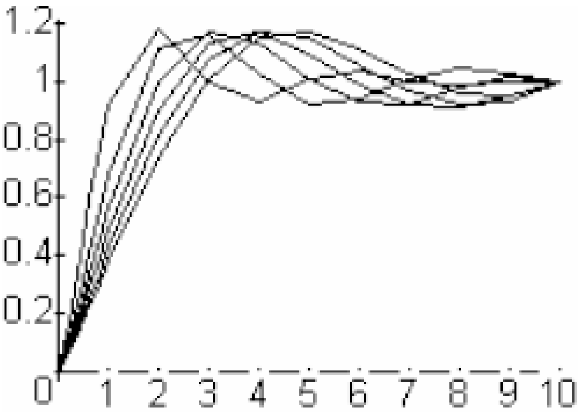

Fig. 2. Dimensionless function *u*.

sider the Taylor series for coordinates. Another way is to employ the classical processes; to keep uniformity of designation for gas and for solid body, it should be $S \approx s$.

$$\dot{S}\begin{pmatrix} \dot{S}_{xx} & \dot{S}_{xy} & \dot{S}_{xz} \\ \dot{S}_{yx} & \dot{S}_{yy} & \dot{S}_{yz} \\ \dot{S}_{zx} & \dot{S}_{zy} & \dot{S}_{zz} \end{pmatrix} =$$

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}+\dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}.$$

The steady motion of conducting fluids in pipes under transverse magnetic fields was investigated for classical case in [9]. Taking into account the problem for the classical case, we have

$$j_z = \sigma E_z = 0;\;\; j_z = 0,\;\; E_z = 0,$$
$$B_x = 0,\;\; B_y = B_0 = \text{const},\;\; B_z = B(x,y)$$

or, in projections,

$$\mu_0 j_x = \frac{\partial B}{\partial y},\;\; \mu_0 j_y = -\frac{\partial B}{\partial x},$$
$$\frac{\partial P}{\partial x} = j_y B_z = -\frac{1}{\mu_0} B\frac{\partial B}{\partial x},\;\; \frac{\partial P}{\partial y} = -j_x B_z = -\frac{1}{\mu_0} B\frac{\partial B}{\partial y},$$
$$\frac{\partial P}{\partial z} = \rho\nu\nabla^2 w + j_x B_y - j_y B_x = \rho\nu\nabla^2 w + \frac{B_0}{\mu_0}\frac{\partial B}{\partial y}.$$

The first two equations lead to

$$\frac{\partial}{\partial x}\left(P+\frac{B^2}{2\mu_0}\right)=0,\;\; \frac{\partial}{\partial y}\left(P+\frac{B^2}{2\mu_0}\right)=0,$$

For the modified theory, which takes into account the angular moment and the second type of rheology, we have

$$F_{Mx} = \frac{\partial}{\partial y}\left[x\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right)-y\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\right]-\frac{\partial}{\partial z}\cdot\left[x\left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)-z\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\right]$$

$$\frac{\partial}{\partial x}\left(P+\frac{B^2}{2\mu_0}\right)+\frac{\partial^2}{\partial y\partial x}(Py)-\frac{\partial^2}{\partial y^2}(xP)=0,$$
$$\frac{\partial}{\partial y}\left(P+\frac{B^2}{2\mu_0}\right)+\frac{\partial^2}{\partial x^2}(yP)-\frac{\partial^2}{\partial y\partial x}(xP)=0.$$

We shoud add equation for *w*

$$\rho\nu\nabla^2 w+\frac{B_0}{\mu_0}\frac{\partial B}{\partial y}=-\frac{\Delta p}{l}$$

the term

$$\Delta_F = \frac{\partial}{\partial x}\left[x\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right]+\frac{\partial}{\partial y}\left[y\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\right].$$

Other equation is

$$\nabla^2 B+\sigma\mu_0 B_0\frac{\partial w}{\partial y}=0.$$

We suggested the new equations but we do not change the force (without angular moment). The angular moments of the forces are

$$M_x = yj_xB_y + zj_xB_z,\;\; M_y = zj_yB_z - xj_xB_y,$$
$$M_z = -xj_xB_z - yj_yB_z.$$

The asymmetry degree of the force moment is

$$F_{Mx}=\frac{\partial M_z}{\partial y}-\frac{\partial M_y}{\partial z},\;\; F_{My}=\frac{\partial M_z}{\partial x}-\frac{\partial M_x}{\partial z},$$
$$F_{Mz}=\frac{\partial M_y}{\partial x}-\frac{\partial M_x}{\partial y},$$

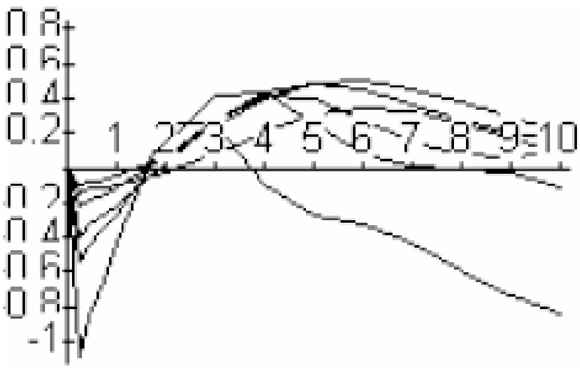

Fig. 3. Dimensionless function *v*.

3. FOKNER-SKEN PROBLEM

The new equations for uniform translational movement of a cylinder are

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}=U_e\frac{\partial U_e}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+$$

$$\frac{\partial}{\partial y}\left(\mu y\frac{\partial^2 u}{\partial y^2}\right),\ \ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}y\frac{\partial v}{\partial y}=0$$

the boundary conditions can be written as follows

$$u=0,\ \ v=0,\ \ \mu\frac{\partial u}{\partial y}=\tau_w,\ \ y=0;$$

$$u=U_e,\ \ y\to\infty,\ \ x>0;\ \ u=U_e,\ \ x=0.$$

Here U_e is the velocity at upper boundary, τω is friction.

$$u=cx^m\phi(\eta),\ \ \eta=\sqrt{\frac{c}{\mu}}\,yx^{(m-1)/2},$$

$$v=\sqrt{\mu c}\,x^{(m-1)/2}V(\eta),\ \ v=\mu y.$$

The profiles of velocity *u* and *v* for the boundary condition at *m*=0.05 are shown in Figs. 2 and 3. Here *d* = 0.9 is multiplied by the Fokner-Sken function *í* at infinity. It is essential that the basic equation is an equation for the velocity.

4. GENERAL CLASS OF THE NONSTATIONARY SOLUTIONS OF NAVIER-STOKES EQUATIONS

Consider system of the Navier-Stokes equations of for the plate case

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}+\nu\frac{\partial}{\partial y}y\frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial P}{\partial x},\ \ \frac{\partial v}{\partial y}=0,$$

$$\frac{\partial u}{\partial t}+v_0\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+\nu\frac{\partial^2 u}{\partial y^2}+\nu\frac{\partial}{\partial y}y\frac{\partial^2 u}{\partial y^2},$$

$$U(t)=U_0\left[1+f(t)\right],\ \ u(y,t)=V_0\left[\xi(y)+g(y,t)\right],$$

$$-\frac{\partial\xi}{\partial\eta}=\frac{\partial^2\xi}{\partial\eta^2}-\frac{\partial}{\partial\eta}\eta\frac{\partial^2\xi}{\partial\eta^2},\ \ \eta=\frac{y(-v_0)}{\nu},\ \ T=\frac{tv_0^2}{4\nu},$$

$$\frac{\partial g}{\partial T}-4\frac{\partial g}{\partial\eta}=f'(t)+4\frac{\partial^2 g}{\partial\eta^2}-4\frac{\partial}{\partial\eta}\eta\frac{\partial^2 g}{\partial\eta^2}.$$

Here we also have a monotonic profile for the first term.

5. PRANDTL PROBLEM (THE SECOND FORMULATION)

Usually the equilibrium conditions are postulated as the conditions of equilibrium of forces. Then the angular moment law is fulfilled if stress tensor is symmetric, but the second type of equilibrium is

$$y\left(\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}\right)-z\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}\right)+$$

$$\tau_{yz}-\tau_{zy}=0.$$

We consider the Prandtl problem of layer compression by two rough plates [6]. We make use the equations of equilibrium. This formulation of problem of moment theory is not well. The theory contradicts the Navier equations as in equilibrium the movement is absent. We suggest that the velocity

$$\frac{\partial}{\partial y}\left[x\left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right)-\right.$$

$$\left.z\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\right]=\Delta_{yx},\ \ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0.$$

From this it follows

$$\frac{\partial^2\sigma_{yy}}{\partial y^2}-\frac{\partial^2\sigma_{xx}}{\partial x^2}=0,\ \ \frac{\partial\sigma_{yy}}{\partial y}-\frac{\partial\sigma_{yx}}{\partial x}=k,$$

$$\sigma_{yx}=k_1y,\ \ \frac{\partial\sigma_{xx}}{\partial x}=k-k_1=k_2,\ \ \sigma_{xx}=k_2x+\varphi(y),$$

$$\frac{\partial\sigma_{xy}}{\partial x}=0.\ \ \frac{\partial}{\partial x}\left\{\left(x\frac{\partial\sigma_{yy}}{\partial y}\right)-yk\right\}+\frac{\partial\sigma_{yy}}{\partial y}=0,$$

$$\frac{\partial\sigma_{yy}}{\partial y}+x\frac{\partial^2\sigma_{yy}}{\partial x\partial y}+\frac{\partial\sigma_{yy}}{\partial y}=0,$$

$$\partial\sigma_{yy}=f(x),\ \ k_2+k_1-k=0.$$

One of the solutions

$$u=-ax\pm ah\sqrt{1-\left(\frac{y}{h}\right)^2}+c_1,\ \ v=ay,\ \ c_1=\text{const}$$

satisfies to Navier equations, and the angular moment compensates the difference between the ideal plasticity and Navier theories.

In our case we can write the second solution in the form

$$\frac{\partial^2\sigma_{yy}}{\partial y^2}-\frac{\partial^2\sigma_{xx}}{\partial x^2}=0,\ \ \frac{\partial\sigma_{xx}}{\partial x}-\frac{\partial\sigma_{yx}}{\partial y}=k,$$

$$\sigma_{yx}=k_1y,\ \ \frac{\partial\sigma_{xx}}{\partial x}=k-k_1=k_2.$$

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