Rev.Adv.Mater.Sci. 23(2010) 141-146

STATISTICAL DESCRIPTION OF DOMAINS IN THE POTTS MODEL

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Received: December 10, 2009

Abstract. The Zipf power law and its connection with the inhomogeneity of the system is investigated. We describe the statistical distributions of domain masses in the Potts model near the temperature-induced phase transition. We found that the statistical distribution near the critical point is described by the power law form with a long tail, while beyond the critical point the power law tail is suppressed.

We use the Potts model [1] for a description of the phase transition. The Potts model is a generalization of the Ising model with more than two spin components and it has more experimental realizations than the Ising model. For a detailed review of the Potts model see [2].

The Hamiltonian for the q"state Potts model [2] is:

$$H = -\sum_{i,j} J_{ij} \delta_{\sigma_i,\sigma_j}, \tag{1}$$

where $\sigma_i \in \{1, 2, ..., q\}$, $\delta_{x,y}$ is the Kronecker delta

$$\delta_{x,y} = \begin{cases} 1 & \text{if} \quad x = y, \\ 0 & \text{otherwise}. \end{cases}$$

and

$$J_{ij} = \begin{cases} J & \text{if } i, j \text{ are neighbour pairs of spins,} \\ 0 & \text{in the opposite case,} \end{cases}$$

The case $q = 2$ describes the Ising model.

The results presented hereafter are obtained by applying the Monte Carlo techniques, based on the Metropolis algorithm [3], to the two-dimensional q"state Potts model with the periodic boundary conditions and with $q = 3$ and 6.

A number of exact results for the two-dimensional Potts models are known in the infinite volume limit. For example, the phase transition appears at the critical temperature $T = T_c (T_c = 2J/k_B \ln(1 + \sqrt{q}\,))$. It is the second order phase transition for $q \le 4$ and the first order one for $q \ge 5$, see [4] and [2].

The main goal of this paper is the statistical description of the domains in the Potts model when one approaches the critical point of the phase transition induced by the temperature. This problem has been investigated for the Ising model in [5]. Our considerations concerning the statistical description of domain masses have universal character and may be used to arbitrary fractal system of elements which are described by the random variables x. This variables can be listed in the decreasing order. For example: the rank of the city connected with its population, the frequency of the occurrence of any word in the text, the trading value of largest European's companies or the hyperbolic processes in finance [6,7].

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We consider the Zipf power law [8] in the Bouchaud notation [6]. We shall concentrate on the random variable X which we shall call [μ] - variable. We say that the random variable X is [μ] - variable if for some $x_0 > 0$ and $\mu > 0$, a tail of its distribution function decays as $(x_0/x)^{\mu}$.

The main property of [μ] - variable is that all its moments $m_q =< x^q >$ with $q \geq \mu$ are infinite. The distribution of the cluster size appears to be [μ]-distribution and the index m is a critical exponent.

Suppose that a set of N [μ]-variables is ordered and decreasing, then

$$x_k \propto x_0 \left(\frac{N}{k} \right)^{1/\mu}, \tag{2}$$

which means that the largest variable is is of the order $x_0 N^{1/\mu}$ while the smallest is of the order x_0, [6].

This law is used often in the description of self-organized critical phenomena. In our case x is the number of the Potts spins in the domain, called the domain mass, and k denotes the rank of the domain with mass x. The greatest cluster has the rank 1, smaller 2 and so on.

The geometrical clusters are considered in [9]. In the Potts model, the clusters form sets of nearest neighbour sites occupied by Potts spins with the same orientation. For a given configuration of spins the clusters are uniquely determined. There is the well established connection between the thermal and the geometric phase transitions. On an infinite lattice an infinite cluster appears exactly at the critical point. The q"state Potts model has the ground state degeneracy q and after a quench from the high-temperature phase, small domains start to grow, thus reducing the domain boundary curvature. One can notice that there is a difference between the domains of the Potts model with $q = 2$ (the Ising model case) and the Potts model with $q > 2$. For $q = 2$ the domain boundaries are represented by long filaments of one phase within the other one, whereas for $q > 2$, the domains form a structure characteristic for polycrystalline grains [10-12] - i.e. the boundaries are straight and meet at fixed angles.

We take into account, in our considerations, all domains and we concentrate on their size. Bouchaud [6] pointed out the strong correlation between the Zipf power law and the inhomogeneity of the system. We are going to test the log-log distribution of domain masses versus the rank index k when we come near the critical point in the temperature-induced phase transition.

The slope α of the regression line describing the relation between the logarithm of the cluster mass and the logarithm of its rank is given by μ ($-1/\mu = \mathrm{tg}\alpha$), and it characterizes the inhomogeneity of the physical structure of the system. The inhomogeneity of the system means that its structure has become fractal and more hierarchical. We expect that μ value will depend on whether we are far or close to the critical point. In particular one should have $\mu \approx 1$ in the critical point and $\mu > 1$ beyond the criticality. This follows from the following considerations. It is shown in [13] that there is a conjecture between Zipf exponent and the Hurst exponent H [14] of the form

$$1/\mu = |2H - 1|. \tag{3}$$

It is well known that any complex system exhibits long-range (infinite-long) correlations at the critical point. This corresponds to the biggest value of Hurst exponent, i.e. $H = 1$. In such a case one obtains from Eq. (3) $\mu \approx 1$. The criticality in the rank statistics was also considered experimentally in [15,16] investigating distributions of island areas of discontinuous metal films near the percolation threshold.

The simplest technique which helps to test a heavy tail hypothesis and estimates the tail index μ is based on the following reasoning, for more details see [17].

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with the distribution function F. We may re-arrange the X_i in the decreasing order. Denoting the largest value, with rank 1, by $X_{(1)}$, the next largest, with rank 2, by $X_{(2)}$ etc., we arrive at the sequence of order statistics

$$X_{(1)} \geq X_{(2)} \geq ... \geq X_{(n)}.$$

Assume that X is a [μ] - variable with tail of its distribution function satisfying for some $x_0 > 0$ and m > 0

$$P(X > x) = \left(\frac{x_0}{x} \right)^{\mu}, \quad x > x_0. \tag{4}$$

The assumption (4) means that the distribution function of X/x_0 is the Pareto distribution with the left endpoint 1 and the parameter μ, and for $x > 0$

$$P\left(\mu \ln \frac{X}{x_0} > x \right) = \exp(-x),$$

which means that the distribution function of the random variable $\mu \ln(X/x_0)$ is exponential with the parameter equal to 1. Therefore

$$P(\ln X > x) = \exp\left(-\frac{x - \ln x_0}{\mu^{-1}} \right),$$

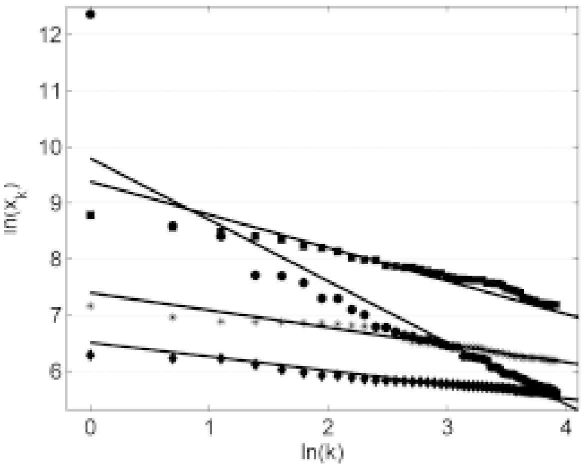

Fig. 1. The log-log distribution of the domain masses x_k versus the rank order index k for the 3-state Potts model for $\beta = 0.40$ (◊), 0.45 (⁎), 0.49 (□) and critical β(○), for one configuration. The straight line are with the slopes -1/μ given in Table 1.

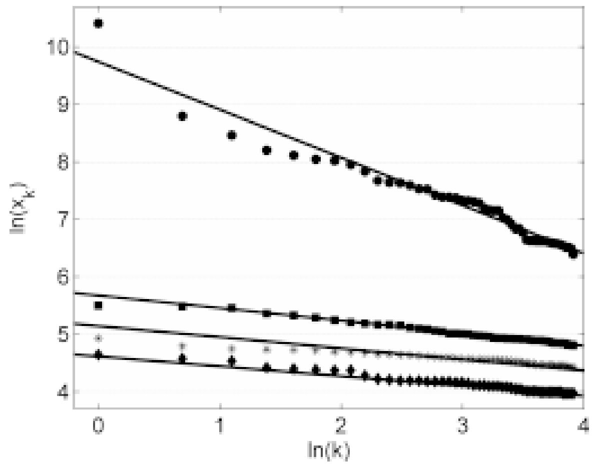

Fig. 2. The log-log distribution of the domain masses x_k versus the rank order index k for the 6-state Potts model for $\beta = 0.50$ (◊), 0.55 (⁎), 0.59 (□) and critical β(○), for one configuration. The straight line are with the slopes -1/μ given in Table 2.

which is a tail of shifted by ln x_0 and rescaled by μ^{-1} exponential distribution function with the parameter equal to 1.

Drawing the quantile to quantile plot for the exponential distribution function we conclude that

$$\left\{\left(-\ln\left(\frac{i}{n+1}\right), \ln X_{(i)}\right), 1 \le i \le n\right\},$$

should be linear with slope μ^{-1} and intercept lnx_0. In this way we obtain the Zipf law, if $\mu = 1$. We may estimate μ in a different way. The most popular estimator of μ is the Hill estimator, [18]. For the convenience of the reader we recall its definition. The Hill's estimator of 1/μ based on m upper-ordered statistics $X_{(1)}$ ³ $X_{(2)}$ ³ ... ³ $X_{(m)}$ is defined as

$$H_{m,n} = \frac{1}{m}\sum_{i=1}^{m} \log \frac{X_{(i)}}{X_{(m+1)}}. \tag{5}$$

Our simulations were performed on two dimensional lattices 600 × 600 with periodic boundary conditions. We used the Metropolis algorithm [3]. All simulations were initiated with the random initial orientation of spins. We formalized the system to equilibrate the orientations of spins. The number of Monte Carlo steps which are necessary to reach the equilibrium state was chosen by measuring the average energy for one spin.

We performed 1 000 000 Monte Carlo steps to equilibrate the system. Once the equilibrium of the system is reached we took a sample of the cluster configuration. Then we studied the distribution of the cluster masses. The total mass of the cluster was defined as the number of spins in the cluster.

Our aim is to examine the statistics of domain masses for the Potts model at the critical temperature, and at the temperature higher than the critical temperature. When, starting from the paramagnetic phase, we decrease temperature of the system then parameters of domain masses statistic change (as it is seen in Fig. 1, for $q = 3$ and Fig. 2 for $q = 6$). At the critical point ($T = T_c$) the distribution function, which describes the distribution of the domain masses, changes from the distribution function with a light tail to the Pareto distribution, which has a heavy tail.

The estimations of indexes 1/μ, for the 3-state Potts model with the different inverse temperature $\beta = 1/T$, based on 1000 realizations for 600 × 600 lattice are presented in Table 1 and Fig. 3.

The same estimations of indexes 1/μ and its standard deviations as in Table 1 and Fig. 3, but for the 6-state Potts model, are presented in Table 2 and Fig. 4.

When β increases, temperature decreases, we observe that the size of domains is growing. Their structure is fractal (loss of an oval) and more hierarchical, the inhomogeneity of the system increases. In the case when ($T \approx T_c$) we see in Fig. 1 and Fig. 2 the straight lines with slope approximately -1, rep-

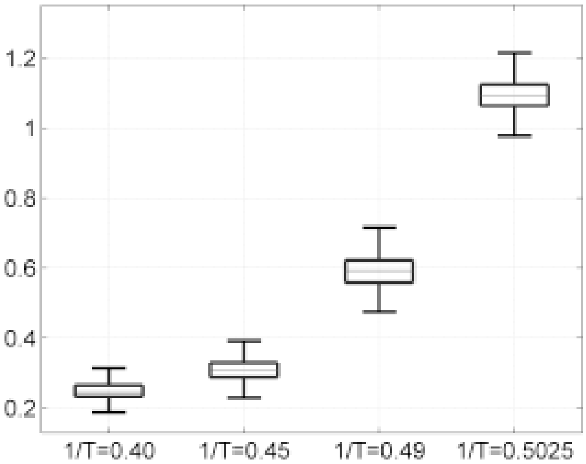

Fig. 3. Box plot: The mean value of μ^{-1} (–) with standard deviation and 95% confidence interval, for inverse temperatures $\beta = 0.40, 0.45, 0.49$,and close to critical 0.50253, obtained by averaging of 1000 Monte Carlo realizations of the 3-state Potts model on 600 × 600 lattice.

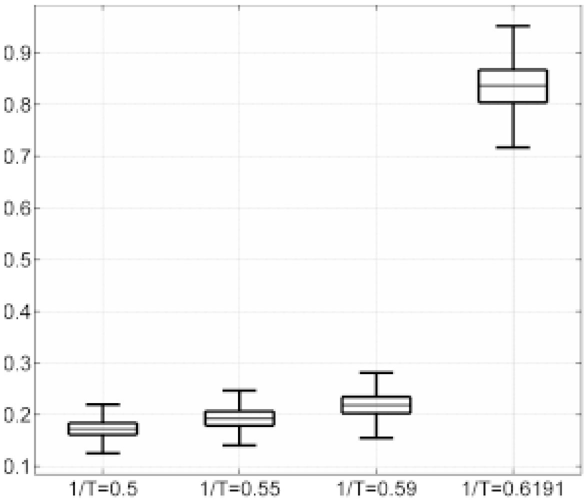

Fig. 4. Box plot: The mean value of μ^{-1} (–) with standard deviation and 95% confidence interval, for inverse temperatures $\beta = 0.50, 0.55, 0.59$, and close to critical 0.6191, obtained by averaging of 1000 Monte Carlo realizations of the 6-state Potts model on 600 × 600 lattice.

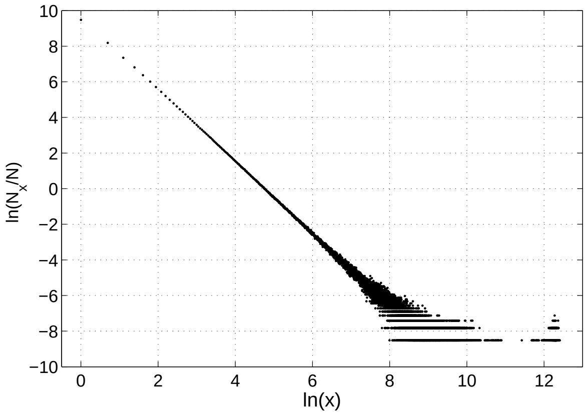

Fig. 5. The dependence between the number of domains with mass x normalized by number of configurations (N_x/N) and the mass (x) in the log-log scale for the 3-state Potts model near the critical point T_c for 5000 configurations and $L = 600$.

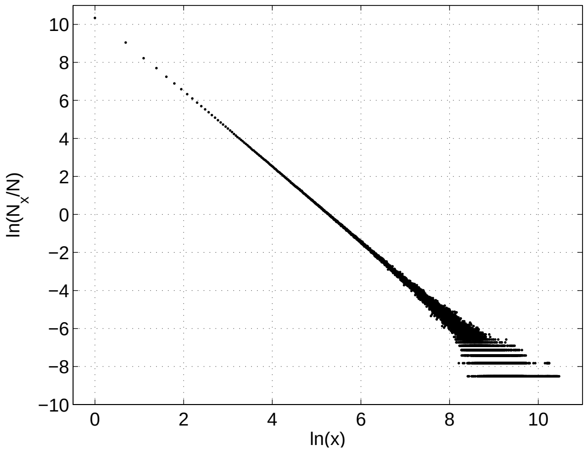

Fig. 6. The dependence between the number of domains with mass x normalized by number of configurations (N_x/N) and the mass (x) in the log-log scale for the 6-state Potts model near the critical point T_c for 5000 configuration and $L = 600$.

resenting the Zipf law. In this case estimation based on the Hill estimator gives us $\mu \approx 1.031$ for the 3-state Potts model and $\mu \approx 0.990$ for the 6-state Potts model. We can notice that in Fig. 1 and Fig. 2 we have straight lines for temperatures different than critical, but their slopes are less then one which is required for Zipf's law.

Fig. 5 and Fig. 6 represent the dependence, in the log-log scale, between the mass (x) and the number of domains with the mass x, normalized by the number of averaged configurations, (N_x/N) and the mass(x) for the 3-state and the 6-state Potts models respectively, near the critical point. The distribution of domain mass is [μ]-distribution with $\mu \approx 1$, which means that it satisfies power law with a long tail: $x_0^{\mu}/x^{1+\mu}$, where x_0 denotes a typical scale and $\mu \approx 1$.

Fig. 7 and Fig. 8 represent the dependence between the number of domains with the mass x normalized by the number of configurations (N_x/N) and the mass (x) of domain masses for the 3-state and the 6-state Potts models beyond the critical region

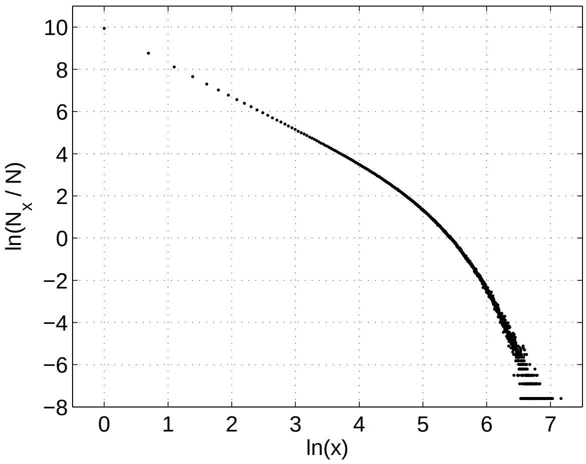

Fig. 7. The dependence between the number of domains with mass *x* normalized by number of configurations (N_x/N) and the mass (x) in the log-log scale for the 3-state Potts model for β = 0.4 for 2000 configurations and L = 600.

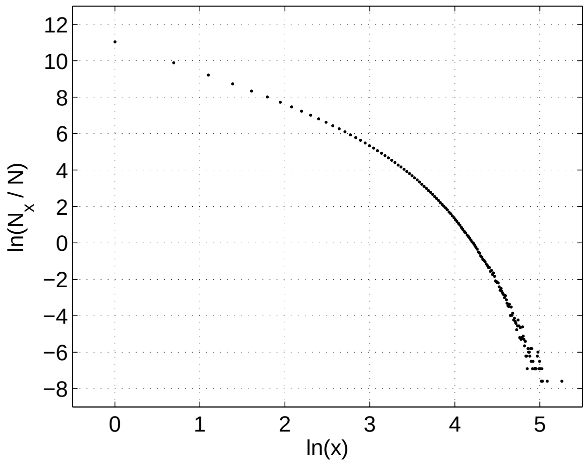

Fig. 8. The dependence between the number of domains with mass *x* normalized by number of configurations (N_x/N)) and the mass (x) in the log-log scale for the 6-state Potts model for β = 0.5 for 2000 configuration and L = 600.

Table 1. The estimation of indexes $1/\mu$ with standard deviations *s*.

β	μ^{-1}	s
0.40	0.249	0.024
0.45	0.310	0.030
0.49	0.591	0.047
$\beta_{kr} \cong 0.5025$	1.095	0.048

Table 2. The estimation of indexes $1/\mu$ with standard deviations *s*.

β	μ^{-1}	s
0.50	0.172	0.0174
0.55	0.193	0.020
0.59	0.219	0.022
$\beta_{kr} \cong 0.6191$	0.834	0.044

when $\mu > 1$, respectively. In this case the distribution of domain masses is without the heavy power-law tail. This results are in the agreement with the standard percolation theory [9].

The numerical results presented in Figs. 5-8, usually can be deduced from the standard percolation theory [9] and are in the agreement with the previous results of [19,20]. The origin of the power-law is well understood and explained as a consequence of the scale invariance at the criticality regime due to the divergence of the correlation length. Beyond the criticality range, where the system is a fractal its correlation length is finite, so the power-law tail in the distribution is suppressed. In our paper we describe size of the cluster by the number of spin in this cluster. We may consider other measures of the cluster size, for example area of the smallest disk or square covering cluster. Such measures are less dependent on the details of the cluster geometry. Numerical and analytical results for such measures of the cluster size have been obtained by Cardy and Ziff [21,22]. Very closed to the subject of domains size distribution closed to criticality but concerning their dynamics are papers by Arenzon et al. [23,24] where the authors have been focusing on geometric domains as well on the hull enclosed areas for which exact results can be obtained. Also a recent experimental verification was performed in [25]. In the physical phenomena different than considered above, but which properties are similar one of the possible experimental confirmations of the Zipf power law is the distribution of island areas of discontinuous metal films, obtained by evaporation of dielectric substances [15,16]. Another one is the distribution of local fields intensities [26].

In our paper we verify inverse power law for cluster size distribution in the Potts model in criticality region and beyond it. We have shown, by the simulation results, that the exponent in the Zipf power law describes the long tail behaviour of the distribution of domains masses at the critical point when $\mu \approx 1$.

ACKNOWLEDGEMENTS

One of the authors (K. L-W) would like to express her gratitude to Dr. Dariusz Grech for the interest in this paper and valuable comments.

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