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ANOMALOUS DEFORMATION OF CONSTRAINED AUXETIC SQUARE

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Abstract. A uniform compression is applied to opposite sides of a two-dimensional unit square with remaining sides fixed. Deformation of the square made of an isotropic material is studied numerically for different values of its Poisson's ratio, ν. The simulations indicate that in a range of negative values of the Poisson's ratio, the displacement vector near the vertices behaves in anomalous way - it can be even antiparallel to the direction of the loading force.

1. INTRODUCTION

Mechanical properties of materials are crucial for many practical applications. Modern technologies often require materials of unusual properties. These two facts have stimulated interest in various materials of anomalous mechanical properties. Recently, an increasing interest is observed in materials exhibiting anomalous (negative) Poisson's ratio [1-5]. Poisson's ratio (PR), ν, together with Young's modulus, (E), constitute a set of quantities describing mechanical properties of isotropic linear elastic (Hookean) bodies. PR is defined by the formula [6]

$$u_{yy} = -\nu u_{xx}. \quad (1)$$

where u_{xx} and u_{yy} are, respectively, the longitudinal and transverse strains accompanying infinitesimal change of the longitudinal stress σ_{xx}. According to (1), Poisson's ratio measures the negative ratio of transverse to longitudinal response due to longitudinal stress acting. The negative sign is used to make this quantity positive for common materials, as they usually shrink transversally when stretched.

It follows from the above definition that if PR is negative, the body will shrink transversally when compressed and expand when stretched. Such a behaviour, although counterintuitive, is admissible from the point of view of thermodynamic stability even for isotropic materials, see e.g. equation (6) in Ref. [7]. Mechanical [8] and thermodynamic [9] models of materials showing negative PR have been proposed in eighties. The true interest in the field of such counterintuitive systems has been started, however, when Lakes manufactured foams of negative PR [1]. Because of exhibiting uncommon way of elastic deformation such materials have been coined 'anti-rubber' [10], dilational materials [11] or auxetics [12]. In the following text we will use the last of those names.

Despite auxetics are already known for more than two decades, the mechanisms responsible for their anomalous properties are still under intensive studies [5]. Investigations of various models are particularly useful in this context [13-23]. Moreover, analysis of simple models can reveal various unusual phenomena [12,24]. A recent example was de-

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scribed by Stręk et al [25] who studied, a thick, three-dimensional elastic plate, made of isotropic material of a given Poisson's ratio. A horizontal loading was applied to two smaller opposite parallel faces of the cuboid plate. The two lateral smaller faces were fixed while the remaining (upper and lower) large faces were free. It has been observed that when the Poisson's ratio was close to its minimum value, which is equal to -1 for isotropic materials [6], the displacement vector in some regions near the corners showed a component directed opposite to the applied force. Such a deformation reminds those which occur in materials of negative stiffness, or negative rigidity, or negative compliance [26-29]. The latter materials are of interest and importance because they can be applied as inclusions in composites of extreme properties, like extreme stiffness, extreme hardness, extreme dumping, etc. [26-30].

In this paper we consider two-dimensional (2D) analogue of the three-dimensonal (3D) plate studied by Stręk et al. [25] - a square with two fixed horizontal sides under horizontal loading applied to vertical sides. The aim of the present study is to check if the counterintuitive behavior observed by Stręk et al [25] occurs also in 2D. Taking into account that 2D systems are usually simpler than 3D ones because the former have less degrees of freedom, it is natural to expect that studies of 2D systems require less computational effort and may give a better insight into the phenomenon discussed. Indeed, the numerical investigations described in this paper not only show that anti-parallel displacement observed earlier in 3D [25] occurs also in 2D but allow one to conclude that this phenomenon occurs at much higher values of the Poisson's ratio than observed in 3D.

The paper is organized as follows. In section 2 the model under study is defined and some theoretical background concerning elasticity of continuous media is reminded briefly. Numerical solution of the model is shown in section 3. The last section 4 presents the conclusions.

2. THE MODEL AND ITS SOLUTION

It is assumed that the considered two-dimensional unit square (Fig. 1) is made of a linear and isotropic 2D material of Young's modulus, E, and Poisson's ratio, ν.

The equilibrium equation of static linear elasticity theory without any internal forces is [6]

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{2}$$

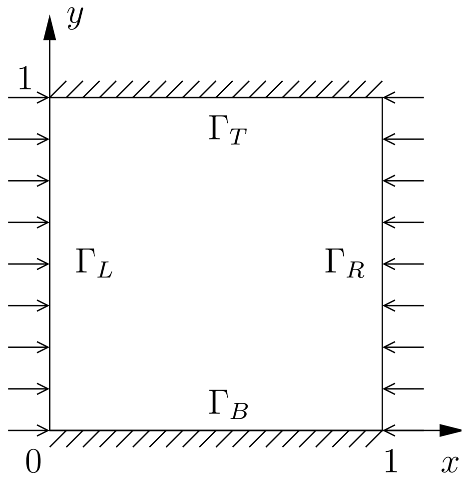

Fig. 1. Geometry of the system studied. Arrows indicate the (uniform compression) force field applied. Oblique lines indicate the fixed sides, i.e. those with Dirichlet boundary conditions.

where the Einstein summation convention is used, stress tensor, σ_{ij}, is related to the linear strain, ε_{ij}, by the formula [6]:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{ll}\delta_{ij}, \tag{3}$$

and the latter is expressed through the displacement vector, u_i, by the relation

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

In terms of the displacement field the equilibrium, Eq. (2) can be written as

$$\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_j \partial x_i} = 0. \tag{5}$$

The considered boundary conditions (see Fig. 1) are as follows:

$$\begin{aligned} &\Gamma_R:\ \sigma_{xx} = -1.0 \ \text{and}\ \sigma_{xy} = \sigma_{yy} = 0,\\ &\Gamma_B \cup \Gamma_T:\ u_x = u_y = 0,\\ &\Gamma_L:\ \sigma_{xx} = 1.0 \ \text{and}\ \sigma_{xy} = \sigma_{yy} = 0. \end{aligned} \tag{6}$$

The quantities λ and μ, called Lamé constants, are in 2D space connected with Young's modulus and Poisson's ratio by the equalities

$$\lambda = \frac{E\nu}{(1-\nu)(1+\nu)},\quad \mu = \frac{E}{2(1+\nu)}. \tag{7}$$

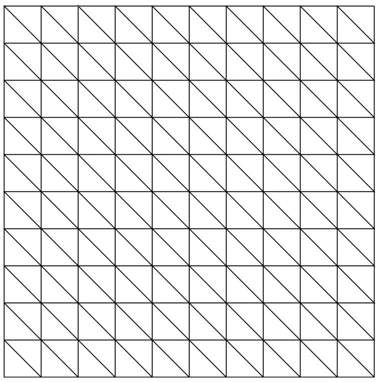

Fig. 2. Uniform mesh of 200 Lagrange triangle elements what corresponds to 11x11 points on the edges and to $N = 10$ intervals on each edge. In the simulations, typical values were $N = 250, 500, 1000, 2000$.

Finite Element Method (FEM) based on Galerkin approach has been used to solve the system of elliptic differential Eqs. (5) with the boundary conditions (6). The idea of the method is to transform Eqs. (5) to a discrete variational problem by multiplying that equation by components of a (vector) test function v_i in a discrete test space $\hat{V}_h$ and integrating it over a mesh on the square. For a chosen mesh and chosen set of test functions solution is searched for, which minimizes the residuum of the operator defined by the Eqs. (5) with the boundary conditions (6) [31].

The numerical simulations were run on own software written using FEniCS [32,33] - the finite element suite. FEniCS is an opensource package, which provides complete set of FEM routines, namely

- generating the mesh,
- generating test (and trial) function spaces,
- assembly of the matrices,
- backends to the most advaned algebra packages.

FEniCS consists of several sub-projects. Each is responsible for a different step of FEM procedure. The results obtained by FEniCS have been verified by using ABAQUS [34] and GET-FEM++ [35]. Excellent agreement was observed between all the three packages.

3. NUMERICAL RESULTS

The domain was a unit square *AxA*, divided into Lagrange (triangle) elements, see Fig. 2, so the mesh is uniform and no refinement was done.

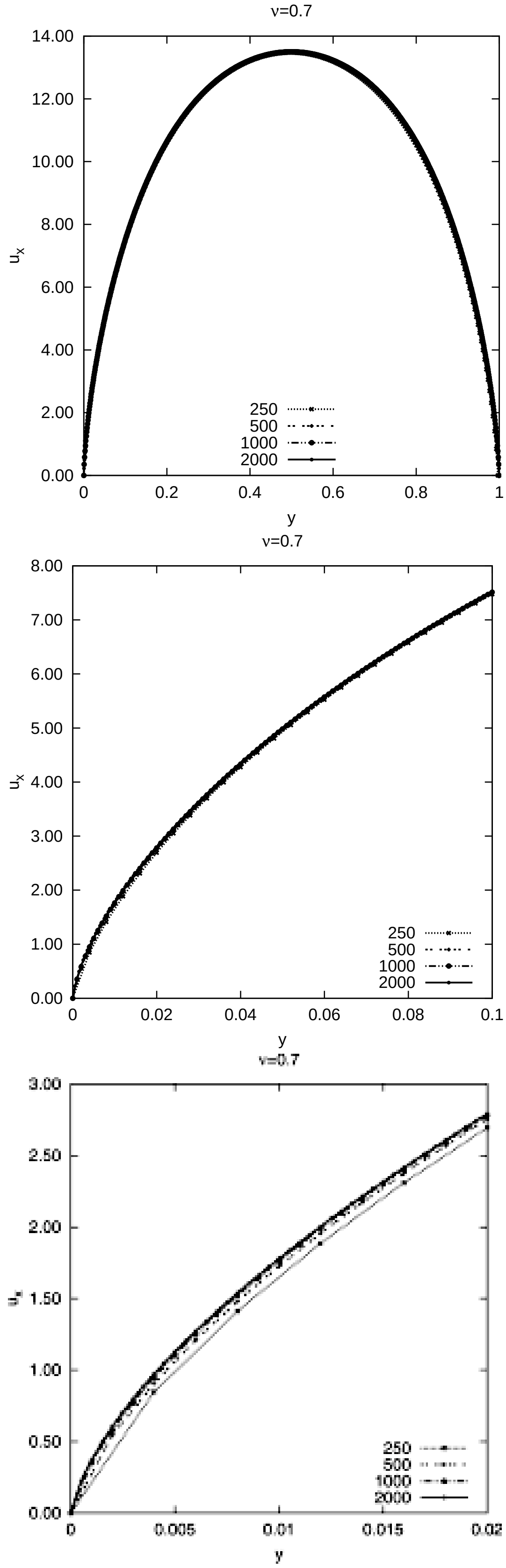

Fig. 3. u_x, being the x-component of the displacement vector multiplied by 10^9, as a function of y for the Poisson's ratio $\nu = 0.7$; the figures (a), (b), and (c) show, respectively, the whole dependence and its details near the bottom-left corner. The numbers in the legend describe the values of N for the meshes used.

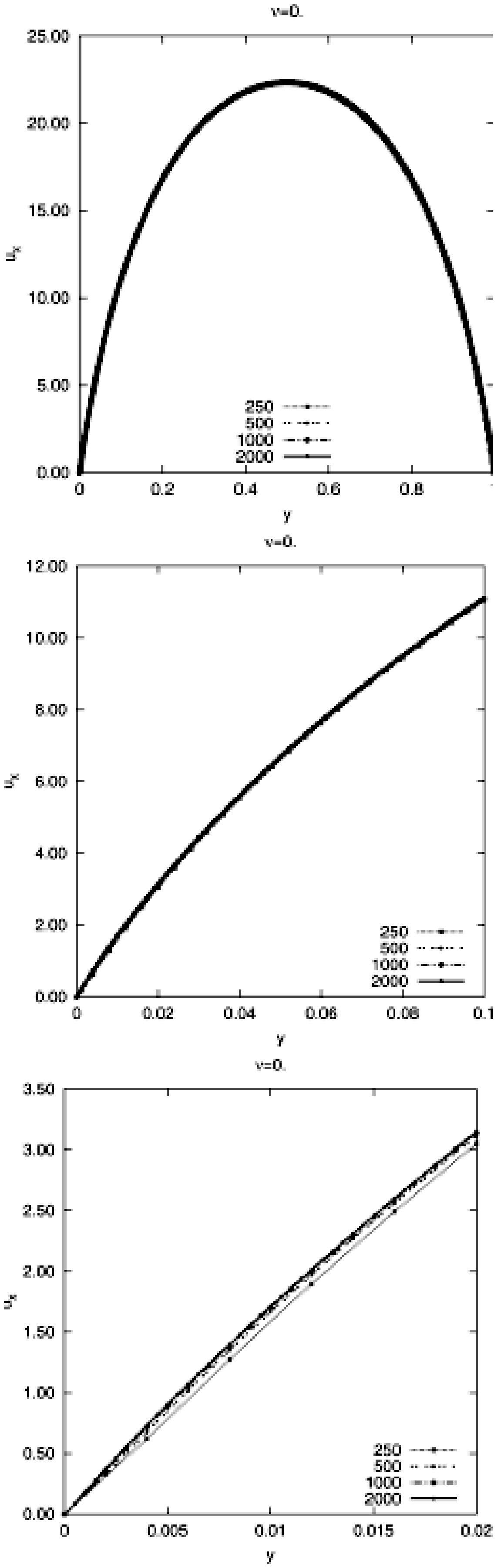

Fig. 4. The same as in Fig. 3 but for the Poisson's ratio ν = 0.

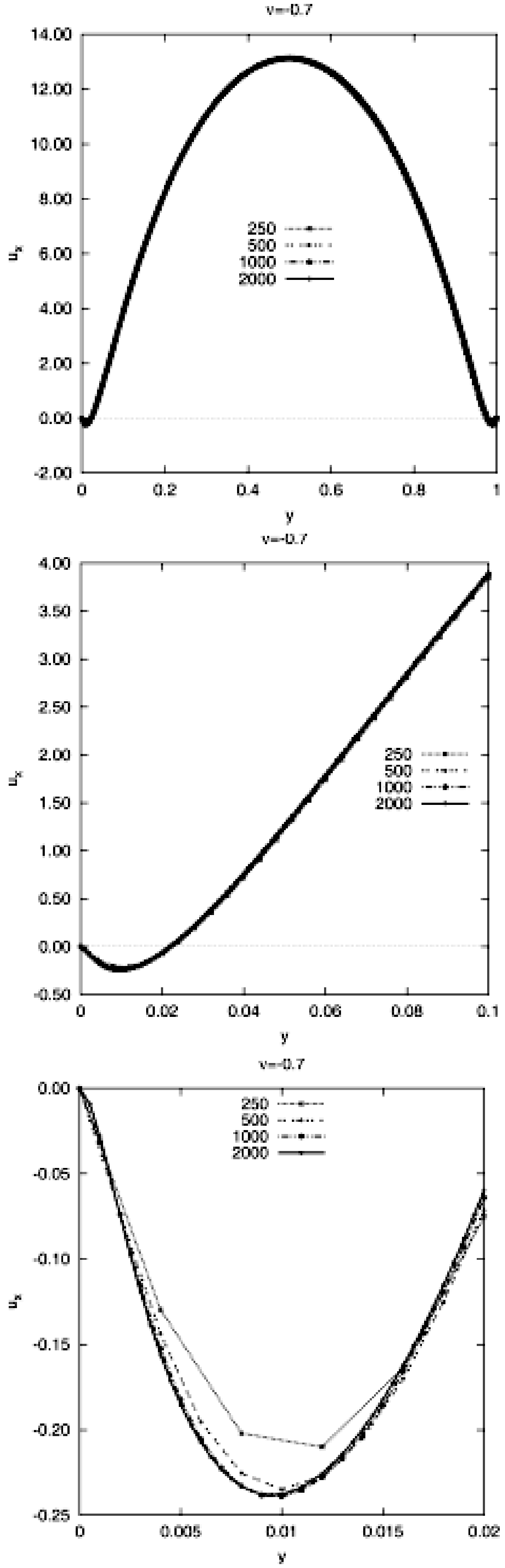

Fig. 5. The same as in Fig. 3 but for the Poisson's ratio ν = -0.7.

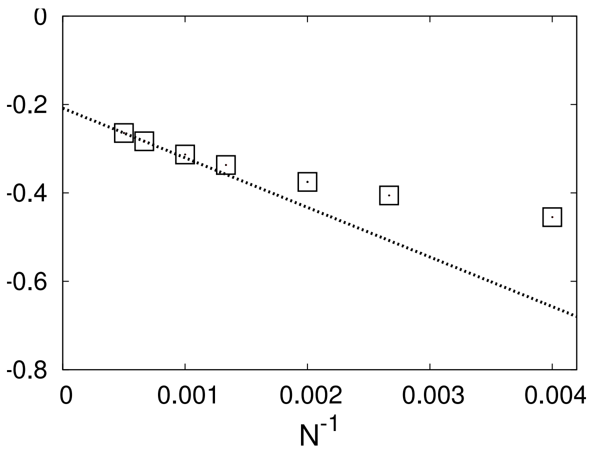

Fig. 6. The inverse *N* dependence of $\nu_c(N)$. The dotted line goes through the points corresponding to the two largest values of *N* = 1400; 2000.

Functions of first order were taken as the test (so also basis) functions. The accuracy of calculations grows with increasing density of the mesh reaching exact result for infinitely dense mesh. Obviously, only finite meshes can be simulated by present computers. The upper limit of mesh density was 15 680 000 triangle elements of first order with 7 845 601 points, what is equal to *N* = 2800 intervals at a single side of the unit square. The (material) constants defining the sample and the experiment were chosen to be the same as in Ref. [25], i.e. $E = 2{:}1\text{x}10^{11}$ N/m, $|P| \equiv |\sigma_{xx}| = 10^4$ N/m, and $A = 1\ m$; one should note that in 2D the units of *E; P* are different than in 3D.

Figs. 3-5 show details of the dependence *x*-component of the displacement field, u_x, on the *y*-component of the boundary Γ_L. It can be seen in those figures that for positive and zero values of Poisson's ratio ν the sign of u_x is positive, i.e. the same as the sign of the acting force. However, when ν = -0.7 parts of the surface of the square behave in anomalous way having *negative* u_x which is *opposite* to the force direction!

The critical values of Poisson's ratio $\nu_c(N)$, below which anomalous behaviour of u_x was observed for chosen *N*, were determined for some *N* and are plotted in Fig. 6. It can be seen there that $\nu_c(N)$ is an increasing function of *N* and is convex as a function of 1/*N*. Thus, $-0.25 < \nu_c \equiv \lim_{N\to\infty} \nu_c(N)$.

To ensure that the counterintuitive results are reliable some of crucial cases were checked using another FEM libraries - *GETFEM++* [35] and ABAQUS [34] which showed exactly the same unusual behavior of the sample.

4. SUMMARY AND CONCLUSIONS

Deformation of a two-dimensional isotropic material forming a square sample with two sides fixed and the other two remaining under uniform compression load was studied for a positive Young's modulus and Poisson's ratios in the range $\nu \in (-1.0, 1.0)$, i.e. in the whole range of mechanical stability of the material. It has been shown that for negative Poisson's ratios in the range $(-1, n_c)$ certain domains of the material near the corners of the sample behave in a counterintuitive way - the material in those domains moves in the direction *opposite* to the pressure applied. As such a behavior can be thought of as a locally negative compliance, the following comments are worth to be drawn.

First, the problem of existence of a material of negative compliance and without internal instability is very interesting and important. One of the reasons is that combining two materials of the same absolute value of compliance but of opposite signs, gives a material of infinite modulus. When the absolute values are not the same but just close to each other, one may obtain very large modulus, e.g. very hard material as has been recently demonstrated by Lakes and co-workers [30].

Second, the observed unusual phenomenon has its source in the constrains (fixed sides) applied to the auxetic material. This constitutes a further evidence supporting the thesis presented in Ref. [29] that negative compliance can be obtained without internal instability when constraints are present in the system.

Third, the present calculations prove that in 2D the critical value of Poisson's ratio, ν_c, below which the negative compliance effect occurs is not less than -0.25. Taking into account that, in the finite mesh calculations performed, the finer mesh the lower amplitude of Poisson's ratio for which the local negative compliance effect was observed, it is attractive to expect that the true critical value in 2D is zero, $\nu_c = 0$.

Extensive simulations are in progress to verify the latter hypothesis both in 2D and 3D. If it is true, then *any* auxetic material will lead to negative compliance!

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REFERENCES

[1] R. S. Lakes // *Science* 235 **(**1987) 1038.

[2] K. E. Evans, M. A. Nkansah, I. J. Hutchinson and S. C. Rogers // *Nature* **353** (1991) 124.

[3] R. S. Lakes // *Advanced Materials* **5** (1993) 293.

[4] R. H. Baughman, S. O. Dantas, S. Stafstrom, A. A. Zakhidov, T. B. Mitchell and D. H. E. Dubin // *Science* **288** (2000) 2018.

[5] C. Remillat, F. Scarpa and K. W. Wojciechowski // *Physica Status Solidi (b)* **246** (2009) 2007; see also references therein.

[6] L. D. Landau and E.D Lifshits, *Theory of Elasticity* (Pergamon Press, London, 1986).

[7] K. W. Wojciechowski // *Journal of the Physical Society of Japan*, **72** (2003) 1819.

[8] R. F. Almgren //*Journal of Elasticity* **15** (1985) 427.

[9] K. W. Wojciechowski // *Molecular Physics* **61** (1987) 1247.

[10] R. S. Lakes, *http://silver.neep.wisc.edu/~lakes/*.

[11] G. W. Milton // *Journal of the Mechanics and Physics of Solids* **40** (1992) 1105.

[12] K. E. Evans // *Endeavour* **15** (1991) 170.

[13] R. J. Bathurst and L. Rothenburg // *International Journal of Engineering Science* **26** (1988) 373.

[14] K. W. Wojciechowski // *Physics Letters A* **137** (1989) 60.

[15] D. H. Boal, U. Seifert and J. C. Shillcock // *Phys. Rev. E* **48** (1993) 4274.

[16] D. Prall and R. S. Lakes // *Int. J. Mech. Sci.* **39** (1887) 305.

[17] J. N. Grima and K. E. Evans // *Journal of Materials Science Letters*, **19** (2000) 1563.

[18] Y. Ishibashi and M. Iwata // *Journal of the Physical Society of Japan*, **69** (2000) 2702.

[19] M. Bowick, A. Cacciuto and G. Thorleifsson // *Physical Review Letters* **87** (2001) 14.

[20] A. A. Vasiliev, S. V. Dmitriev, Y. Ishibashi and T. Shigenari // *Phys. Rev. E* **65** (2002) 094101.

[21] K. W. Wojciechowski // *Journal of Physics A - Mathematical and General* **36** (2003) 11765.

[22] J. N. Grima, R. Gatt and P. S. Farrugia // *Physica Status Solidi (b)* **245** (2008) 511.

[23] D. Attard, E. Manicaro and J. N. Grima // *Physica Status Solidi (b)* **246** (2009) 2033.

[24] A. W. Lipsett and A. I. Beltzer // *Journal of the Acoustical Society of America* **84** (1988) 2179.

[25] T. Strek, B. Maruszewski, J.W. Narojczyk and K.W. Wojciechowski // *Journal of Non-Crystalline Solids* **354** (2008) 4475.

[26] R. S. Lakes // *Physical Review Letters* **86** (2001) 2897.

[27] R. S. Lakes // *Philosophical Magazine Letters* **81** (2001) 95.

[28] R. S. Lakes, T. Lee, A. Bersie and Y. C. Wang // *Nature* **410** (2001) 565.

[29] R. S. Lakes and K. W. Wojciechowski // *Physica Status Solidi B* **245** (2008) 545.

[30] T. Jaglinski, D. Kochmann, D. Stone and R. S. Lakes // *Science* **315** (2007) 620.

[31] G.R. Liu and S.S. Quek, *Finite Element Method: A Practical Course* (Elsevier-Science, 2003).

[32] *FEniCS, FEniCS project.* URL: urlhttp//www.fenics.org/.

[33] T. Dupont, J. Hoffman, C. Johnson, R. C. Kirby, M. G. Larson, A. Logg and L. R. Scott, *The FEniCS project. Technical report* (Chalmers Finite Element Center Preprint Series, 2003).

[34] *Simulia. Abaqus/standard v6.9.1*, 2009.

[35] Y. Renard and J. Pommier. *The getfem++ project*. URL:http://download.gna.org/getfem/doc/getfem_project.pdf, 2009.