# The evolution of a satellite motion in the gravitational field of a viscoelastic planet with a core

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**Abstract**

We investigate the motion of a satellite in the gravitational field of a massive deformable planet. Planet is modeled as body, that consists of a solid core and a viscoelastic shell of a Kelvin-Voigt material. The satellite is modeled as a point mass. The system of integro-differential equations for a motion of a mechanical system is got out from the variational principle of the d'Alembert-Lagrange according to the linear theory of elasticity. Approximate equations of motion in vector are constructed with asymptotic method of motions separation. This system of equations describes the dynamics of the "planet-satellite" with regard to the perturbations caused by elasticity and dissipation. The solution of the quasi-static problem of the elasticity for a deformable shell of the planet obtained explicitly. To describe the evolution of the orbital parameters of a satellite, averaged differential equations was derived.

Phase trajectories were constructed for particular cases, their stationary solutions were found and investigated on stability. In the case of the existence of two stationary orbits stationary solution that corresponding to the motion along the orbit of larger radius is asymptotically stable, and the orbit of smaller radius is unstable. Some of the planets in the solar system and their satellites are considered as examples.

This problem is a model for the study of the tidal theory of planetary motion. Research the tidal evolution of "planet-satellite" was conducted by many authors [1, 2, 3]. We use the methods of analytical mechanics of systems with an infinite number of degrees of freedom [4]. Previously, this approach has been applied to a number of problems on the translational and rotational motion of a viscoelastic sphere [5, 6, 7].

## 1 Formulation of the problem. The equations of motion

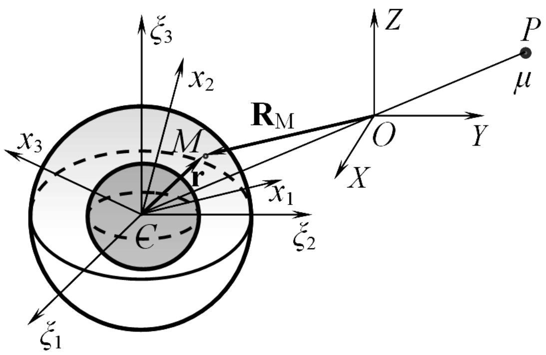


Figure 1: The problem's illustration

We consider the problem of translational and rotational motion of the "planet-satellite" in the gravitational field of the mutual attraction. The satellite is modeled as a point $P$ of mass $\mu$. The planet is modeled as a body, which consists of a solid core and a viscoelastic shell of a Kelvin-Voigt material and occupies a region $V = V_0 \cup V_1$ of three-dimensional Euclidean space in its natural undeformed state. Here $V_0 = \left\{ \mathbf{r} \in E^3 : |\mathbf{r}| \le r_0 \right\}$, $V_1 = \left\{ \mathbf{r} \in E^3 : r_0 < |\mathbf{r}| \le r_1 \right\}$. Then $\rho_0$, $\rho_1$ are the corresponding densities of the core and viscoelastic shell, and $m_0$, $m_1$ are their masses. It is assumed that the material of the planet's shell is homogeneous and isotropic.

We introduce an inertial coordinate system $OXYZ$ with the origin at the center of mass of the "planet-satellite". To describe the rotational motion of the planet, we introduce a moving coordinate system $Cx_1x_2x_3$ rigidly attached to the kernel and the system of Koenig axes $C\xi_1\xi_2\xi_3$, where $C$ is the center of mass of the planet in its natural undeformed state (Fig. 1).

The position of the planet's point $M$ in the inertial coordinate system $OXYZ$ is defined by the vector field

$$\mathbf{R}_M(\mathbf{r}, t) = \mathbf{OC} + \Gamma(\mathbf{r} + \mathbf{u}(\mathbf{r}, t)) \tag{1}$$

where $\Gamma$ is the operator of convertion from the moving coordinate system $Cx_1x_2x_3$ to the system of Koenig axes $C\xi_1\xi_2\xi_3$, $\mathbf{u}(\mathbf{r}, t)$ is the vector of elastic displacement, identically zero for the points of the solid core $V_0$. Since $O$ is the center of mass of the considered mechanical system,

$$\int\limits_V \mathbf{R}_M\left(\mathbf{r}, t\right) \rho dv + \mu \mathbf{OP} = 0 \tag{2}$$

Here $p = \begin{cases} \rho_0, \; if \, \mathbf{r} \in V_0 \\ \rho_1, \; if \, \mathbf{r} \in V_1 \end{cases}$.

We introduce the vector $\mathbf{R} = \mathbf{CP}$. Then from (1) and (2) we get:

$$\begin{aligned} \mathbf{OC} &= -\frac{\mu}{m+\mu}\mathbf{R} - \frac{1}{m+\mu}\int\limits_{V_1} \Gamma \mathbf{u} \rho_1 dv_1, \\ \mathbf{OP} &= \frac{m}{m+\mu}\mathbf{R} - \frac{1}{m+\mu}\int\limits_{V_1} \Gamma \mathbf{u} \rho_1 dv_1 \end{aligned} \tag{3}$$

Here $m$ is the mass of the planet, $m = m_0 + m_1$.

The potential energy of the gravitational field is determined by the following functional:

$$\Pi = -\mu f \int\limits_V \frac{\rho dv}{\left| -\mathbf{R} + \Gamma\left(\mathbf{r} + \mathbf{u}\right) \right|}, \tag{4}$$

where $f$ is the universal gravitational constant.

The functional of potential energy of elastic deformations is given in accordance with the linear model of the theory of elasticity:

$$\mathcal{E} = \int\limits_{V_1} \mathcal{E}\left[\mathbf{u}\right] dv_1, \quad \mathcal{E}\left[\mathbf{u}\right] = \alpha_1 \left( I_{\mathrm{E}}^2 - \alpha_2 II_{\mathrm{E}} \right), \tag{5}$$

where

$$\alpha_1 = \frac{\mathrm{E}\left(1-\nu\right)}{2\left(1+\nu\right)\left(1-2\nu\right)}, \quad \alpha_2 = \frac{2\left(1-2\nu\right)}{1-\nu}, \quad \alpha_1 > 0, \; 0 < \alpha_2 < 3,$$

$$I_{\mathrm{E}} = \sum_{j=1}^{3} e_{jj},\ II_{\mathrm{E}} = \sum_{k<l} \left( e_{kk} e_{ll} - e_{kl}^2 \right),\ e_{kl} = \frac{1}{2} \left( \frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right), \mathbf{u} = (u_1,\ u_2,\ u_3)$$

where E is the Young's modulus, $\nu$ is the Poisson's coefficient of viscoelastic shell of the planet, $I_{\mathrm{E}}, II_{\mathrm{E}}$ are the invariants of the tensor of small deformations.

Dissipative properties of viscoelastic shell are described by dissipative functional

$$\mathcal{D} = \int\limits_{V_1} \mathcal{D}\left[\dot{u}\right] dv_1, \quad \mathcal{D}\left[\dot{u}\right] = \chi \mathcal{E}\left[\dot{\mathbf{u}}\right],$$

corresponding to the Kelvin-Voigt model. Here $\chi > 0$ is the coefficient of internal viscous friction.

We set $\mathbf{R}_P = \mathbf{OP}$. The equations of motion of the "planet-satellite" is obtained from the D'Alembert-Lagrange variational principle:

$$\int\limits_{V} \left( \ddot{\mathbf{R}}_M,\ \delta \mathbf{R}_M \right) \rho dv + \mu \left( \ddot{\mathbf{R}}_P,\ \delta \mathbf{R}_P \right) + \delta \Pi + \\ + \int\limits_{V_1} \left( \nabla_{\mathbf{u}} \mathcal{E}\left[\mathbf{u}\right] + \nabla_{\dot{\mathbf{u}}} \mathcal{D}\left[\dot{\mathbf{u}}\right], \delta \mathbf{u} \right) dv_1 = 0 \quad (6)$$

According to (1) and (3) we have

$$\begin{aligned}
&\ddot{\mathbf{R}}_M = -\frac{\mu}{m+\mu} \ddot{\mathbf{R}} - \frac{1}{m+\mu} \Gamma \int\limits_{V_1} \left\{ \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \mathbf{u}\right] + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \dot{\boldsymbol{\omega}} \times \mathbf{u} + \ddot{\mathbf{u}} \right\} \rho_1 dv_1 \\
&+\Gamma \left\{ \boldsymbol{\omega} \times \left[ \boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) \right] + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \dot{\boldsymbol{\omega}} \times (\mathbf{r} + \mathbf{u}) + \ddot{\mathbf{u}} \right\}, \\
&\delta \mathbf{R}_M = -\frac{\mu}{m+\mu} \delta \mathbf{R} - \frac{1}{m+\mu} \Gamma \int\limits_{V_1} \left\{ \delta \boldsymbol{\alpha} \times \mathbf{u} + \delta \mathbf{u} \right\} \rho_1 dv_1 + \\
&+\Gamma \left\{ \delta \boldsymbol{\alpha} \times (\mathbf{r} + \mathbf{u}) + \delta \mathbf{u} \right\}, \qquad (7) \\
&\ddot{\mathbf{R}}_P = \frac{m}{m+\mu} \ddot{\mathbf{R}} - \frac{1}{m+\mu} \Gamma \int\limits_{V_1} \left\{ \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times \mathbf{u}\right] + 2\boldsymbol{\omega} \times \dot{\mathbf{u}} + \dot{\boldsymbol{\omega}} \times \mathbf{u} + \ddot{\mathbf{u}} \right\} \rho_1 dv_1, \\
&\delta \mathbf{R}_P = \frac{m}{m+\mu} \delta \mathbf{R} - \frac{1}{m+\mu} \Gamma \int\limits_{V_1} \left\{ \delta \boldsymbol{\alpha} \times \mathbf{u} + \delta \mathbf{u} \right\} \rho_1 dv_1.
\end{aligned}$$

Here $\boldsymbol{\omega}$ is the angular velocity vector of the planet, $\delta \boldsymbol{\alpha}$ is the vector appearing in the variation of the orthogonal operator $\Gamma$:

$$\boldsymbol{\omega} \times (\cdot) = \Gamma^{-1} \dot{\Gamma} (\cdot),\ \delta \Gamma (\cdot) = \Gamma \left[ \delta \boldsymbol{\alpha} \times (\cdot) \right].$$

Substituting into (6) expressions for $\ddot{\mathbf{R}}_M$, $\delta \mathbf{R}_M$, $\ddot{\mathbf{R}}_P$, $\delta \mathbf{R}_P$ from (7), and equating the coefficients of the independent variations $\delta \mathbf{R}$, $\delta \boldsymbol{\alpha}$, $\delta \mathbf{u}$, we obtain the equations of motion of the "planet-satellite" in the following form:

$$-\frac{\mu}{m+\mu} \int\limits_{V} \ddot{\mathbf{R}}_M \rho dv + \frac{\mu m}{m+\mu} \ddot{\mathbf{R}}_P + f\mu \int\limits_{V} \frac{\mathbf{R} - \Gamma (\mathbf{r} + \mathbf{u})}{\left| \mathbf{R} - \Gamma (\mathbf{r} + \mathbf{u}) \right|^3} \rho dv = \mathbf{0}, \qquad (8)$$

$$\int\limits_V \left[\mathbf{r}+\mathbf{u}-\frac{1}{m+\mu}\int\limits_{V_1}\mathbf{u}\rho_1 dv_1\right]\times\Gamma^{-1}\ddot{\mathbf{R}}_M\rho dv-\frac{\mu}{m+\mu}\int\limits_{V_1}\mathbf{u}\rho_1 dv_1\times\Gamma^{-1}\ddot{\mathbf{R}}_P -$$
$$- f\mu\int\limits_V \frac{(\mathbf{r}+\mathbf{u})\times\left(\Gamma^{-1}\mathbf{R}-(\mathbf{r}+\mathbf{u})\right)}{\left|\mathbf{R}-\Gamma(\mathbf{r}+\mathbf{u})\right|^3}\rho dv = \mathbf{0}, \quad (9)$$

$$\rho_1\left\{\Gamma^{-1}\ddot{\mathbf{R}}_M-\frac{1}{m+\mu}\int\limits_V\Gamma^{-1}\ddot{\mathbf{R}}_M\rho dv-\frac{\mu}{m+\mu}\Gamma^{-1}\ddot{\mathbf{R}}_P-f\mu\frac{\Gamma^{-1}\mathbf{R}-(\mathbf{r}+\mathbf{u})}{\left|\mathbf{R}-\Gamma(\mathbf{r}+\mathbf{u})\right|^3}\right\}+$$
$$+\nabla_{\mathbf{u}}\mathcal{E}\left[\mathbf{u}+\chi\dot{\mathbf{u}}\right]=\mathbf{0}. \quad (10)$$

## 2 The disturbed system of equations of motion. Deformations of viscoelastic shell of the planet

We assume that the stiffness of a viscoelastic shell of the planet is large, therefore we can introduce a small parameter $\varepsilon = \mathrm{E}^{-1}$. When $\varepsilon = 0$ the vector of elastic displacement $\mathbf{u}$ is assumed to be zero. In this case the original problem is reduced to the problem of describing the motion of a mechanical system consisting of a rigid spherical body and a particle in the force field of mutual attraction. The unperturbed motion is described by the following system of equations:

$$\ddot{\mathbf{R}}+\frac{f(m+\mu)}{R^3}\mathbf{R}=\mathbf{0},\qquad A\dot{\boldsymbol{\omega}}=\mathbf{0}, \quad (11)$$

where $A=\dfrac{8\pi}{15}\left[\rho_0 r_0^5+\rho_1\left(r_1^5-r_0^5\right)\right]$ is the moment of inertia of the planet in its undeformed state relative to the diameter.

Using the method of separation of motions [4] we construct the approximate system of vector ordinary differential equations, that describes translational and rotational motion of planet-satellite with the disturbances caused by elasticity and dissipation. We search for the vector of elastic displacement $\mathbf{u}$ as the solution of (10) in the form $\mathbf{u}=\varepsilon\mathbf{u}_1+\varepsilon^2\mathbf{u}_2+\cdots$. The first approximation $\mathbf{u}=\varepsilon\mathbf{u}_1$ has been found [8].

The vector ODE system after calculations and transformations takes the following form:

$$\ddot{\mathbf{R}}+\frac{f(m+\mu)}{R^3}\mathbf{R}+\frac{3f(m+\mu)\rho_1^2\varepsilon D}{mR^4}\Gamma\left\{-5\,\boldsymbol{\xi}\,(\boldsymbol{\xi},\boldsymbol{\omega})^2+\boldsymbol{\xi}\boldsymbol{\omega}^2+2\boldsymbol{\omega}(\boldsymbol{\xi},\boldsymbol{\omega})+\right.$$
$$\left.+\frac{6f\mu}{R^3}\left(1+\frac{3\chi\dot{R}}{R}\right)\boldsymbol{\xi}+\frac{6\chi f\mu}{R^3}\dot{\boldsymbol{\xi}}\right\}=\mathbf{0}, \quad (12)$$

$$\dot{\mathbf{L}}+\frac{6f\mu\rho_1^2\varepsilon D}{R^3}\Gamma\left\{\frac{3\chi f\mu}{R^3}\left[\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi}\right]+(\boldsymbol{\xi}\,,\boldsymbol{\omega})\left[\boldsymbol{\omega}\times\boldsymbol{\xi}\right]\right\}=\mathbf{0}, \quad (13)$$

$$D=\frac{4\pi r_1^7}{105}\varphi(x,\nu),$$
$$\varphi(x,\nu)=\frac{(1+\nu)}{\Delta_0}\left\{-16(9k+14)x^{17}-200(3k+8)x^{14}+672(4k+9)x^{12}-\right.$$

$$-\left(210k^2+3044k+5824\right)x^{10}+\left(525k^2+1256k+1576\right)x^7+84\,(17k+12)\,x^5-$$
$$-25\left(21k^2+92k+56\right)x^3+210k^2+716k+416\Big\}.$$

Here $\mathbf{L}$ is the angular momentum vector of the planet relative to the center of mass:

$$\mathbf{L}=\int\limits_V \Gamma(\mathbf{r}+\mathbf{u})\times\frac{d}{dt}\left[\Gamma(\mathbf{r}+\mathbf{u})\right]\rho dv. \tag{14}$$

The system of equations (12)–(13) has a first integral — the law of conservation of angular momentum of "planet-satellite" relative to the common center of mass,

$$m_r\mathbf{R}\times\dot{\mathbf{R}}+\mathbf{L}=\mathbf{G}_0, \tag{15}$$

where $m_r=\frac{\mu m}{m+\mu}$, $\mathbf{G}_0$ is a constant vector.

## 3 The evolution of the orbital motion of the satellite

Since the angular momentum vector of the planet up to terms of order $\varepsilon$ is given by $\mathbf{L}=A\Gamma\boldsymbol{\omega}$, then $\Gamma\boldsymbol{\omega}=A^{-1}\mathbf{L}$. Then, taking into account (12) and (14) the vector differential equation of the orbital motion of the satellite is

$$\mu\ddot{\mathbf{R}}=\mathbf{F}_0+\varepsilon\mathbf{F}_1+\varepsilon\chi\mathbf{F}_2,$$

where

$$\mathbf{F}_0=-\frac{f_0\mu}{R^3}\mathbf{R},$$
$$\mathbf{F}_1=-C_1\left\{\frac{\left(\mathbf{G}_0-m_r\mathbf{R}\times\dot{\mathbf{R}}\right)^2}{A^2R^5}\mathbf{R}+\frac{2\,(\mathbf{R},\mathbf{G}_0\,)\left(\mathbf{G}_0-m_r\mathbf{R}\times\dot{\mathbf{R}}\right)}{A^2R^5}-\right.$$
$$\left.-\frac{5\mathbf{R}\,(\mathbf{R},\mathbf{G}_0\,)^2}{A^2R^7}+\frac{6f\mu}{R^8}\mathbf{R}\right\},$$
$$\mathbf{F}_2=-C_2\left\{\frac{\dot{\mathbf{R}}}{R^8}+\frac{2\dot{R}}{R^9}\mathbf{R}-\frac{\left[\mathbf{G}_0-m_r\mathbf{R}\times\dot{\mathbf{R}}\right]\times\mathbf{R}}{\mathrm{AR}^8}\right\},$$
$$C_1=3f_0\mu\rho_1^2Dm^{-1}\,,\quad C_2=6f\mu C_1\,,\quad f_0=f\,(m+\mu)\,.$$

We use the canonical variables Delaunay $L$, $G$, $H$, $g$, $h$ [9, 11] to describe the evolution of motion of the satellite. After averaging over the "fast" angular variable $l$ — the mean anomaly — we obtain a closed system of ordinary differential equations. This system describes the evolution of the "action" variables $L$, $G$, $H$ and slow angular variables $g$, $h$. The evolutionary system of equations of the orbital motion of the satellite is written in terms of dimensionless variables $n_0$, $e$, $i$, $g$, $h$, where $n_0=nAG_0^{-1}$, the axis $OZ$ of the inertial coordinate system $OXYZ$ is directed along the vector $\mathbf{G}_0=(0,0,G_0)$ ($n$ is the mean orbital motion, $e$ is the eccentricity, $i$ is the inclination and $g$ is the longitude of the perihelion from the ascending node):

$$\dot{n}_0 = -\frac{3\Delta_1 n_0^{16/3}}{\left(1-e^2\right)^{15/2}} \left\{ \left[ \cos i - \frac{p}{n_0^{1/3}} \left(1-e^2\right)^{1/2} \right] \cdot F_2(e) \cdot \left(1-e^2\right)^{3/2} - n_0 \cdot F_3(e) \right\},$$

$$\dot{e} = \frac{\Delta_1 n_0^{13/3} e}{\left(1-e^2\right)^{13/2}} \left\{ \left[ \cos i - \frac{p}{n_0^{1/3}} \left(1-e^2\right)^{1/2} \right] \cdot F_5(e) \cdot \left(1-e^2\right)^{3/2} - n_0 \cdot F_4(e) \right\},$$

$$\frac{\mathrm{di}}{\mathrm{dt}} = -\frac{\Delta_1 n_0^{13/3} \sin i}{\left(1-e^2\right)^5} \left\{ \frac{1}{2} + \left( \frac{9}{4} - \frac{3}{2} \sin^2 g \right) e^2 + \left( \frac{5}{16} - \frac{1}{4} \sin^2 g \right) e^4 \right\},$$

$$\dot{g} = \frac{\Delta_1 n_0^{13/3}}{\left(1-e^2\right)^5} \cdot \cos i \cdot \sin 2g \cdot \left\{ \frac{3e^2}{4} + \frac{e^4}{8} \right\} - \frac{\Delta_2 n_0^2}{\left(1-e^2\right)^{3/2}} \left\{ 3\cos i - \frac{p}{n_0^{1/3}} \left(1-e^2\right)^{1/2} \right\} +$$

$$+ \frac{\Delta_3 n_0^{7/3} \sin i}{\left(1-e^2\right)^2} \left\{ \frac{5}{2} \cos^2 i - \frac{1}{2} + \frac{15 \mu n_0^2}{(m+\mu)\left(1-e^2\right)^3} \left( 1 + \frac{3e^2}{2} + \frac{e^4}{8} \right) \right\},$$

$$\dot{h} = -\frac{\Delta_1 n_0^{13/3}}{\left(1-e^2\right)^5} \cdot \sin 2g \cdot \left\{ \frac{3e^2}{4} + \frac{e^4}{8} \right\} + \frac{\Delta_2 n_0^2}{\left(1-e^2\right)^{3/2}} - \frac{\Delta_3 n_0^{7/3} \cos i}{\left(1-e^2\right)^2}.$$

where $\Delta_1 = \frac{18\varepsilon\chi\mu\rho_1^2 D}{m(m+\mu) f_0^{2/3}} \left(\frac{G_0}{A}\right)^{16/3}$, $\Delta_2 = \frac{3\varepsilon\mu\rho_1^2 \mathrm{D} \mathrm{G}_0^3}{(m+\mu)A^4}$, $\Delta_3 = \frac{3\varepsilon\rho_1^2 D}{\mathrm{m} \mathrm{f}_0^{2/3}} \left(\frac{G_0}{A}\right)^{13/3}$, $p = \frac{A^{1/3} f_0^{2/3} m_r}{G_0^{4/3}}$,
$F_2(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6$, $F_3(e) = 1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8$,
$F_4(e) = 9 + \frac{135}{4}e^2 + \frac{135}{8}e^4 + \frac{45}{64}e^6$, $F_5(e) = \frac{11}{2} + \frac{33}{4}e^2 + \frac{11}{16}e^4$.

We consider two specific cases: a) $i \equiv 0$, b) $e \equiv 0$. In both cases, $n_0$ is defined as a root of equation

$$n_0 + \frac{p}{n_0^{1/3}} = 1 \tag{16}$$

which, depending on the parameter $p$, can have two, one or none solutions. If there are two solutions $n_{01}$ and $n_{02}$, corresponding to the motion along the orbits of a larger and smaller radii respectively, the first of them is asymptotically stable, and the second one is unstable. Phase portraits of the system for $p = 0.375$ are constructed for a) and b) cases (Fig.2).

Table 6 shows the numerical values of the parameter $p$, the stationary values $n_{01}$, $n_{02}$ and the current value of $n_0 = n_0(0)$ for different systems "planet-satellite".

For all examples, except the Mars-Phobos system, there is a double inequality $n_{01} < n_0(0) < n_{02}$. The value of the variable $n_0$ decreases during the motion. This means that the semi-major axes of the orbits of satellites increase, tending to their asymptotically stable stationary values. In this case, the satellites of Jupiter and Mars satellite Deimos current value $n_0(0)$ is closer to the unstable stationary value $n_{02}$, and for the Earth-Moon system $n_0(0)$ is closer to the asymptotically stable value $n_{01}$. For the Mars-Phobos system $n_{01} < n_{02} < n_0(0)$. The value of the variable $n_0$ increases. This means that Phobos is currently approaching Mars.

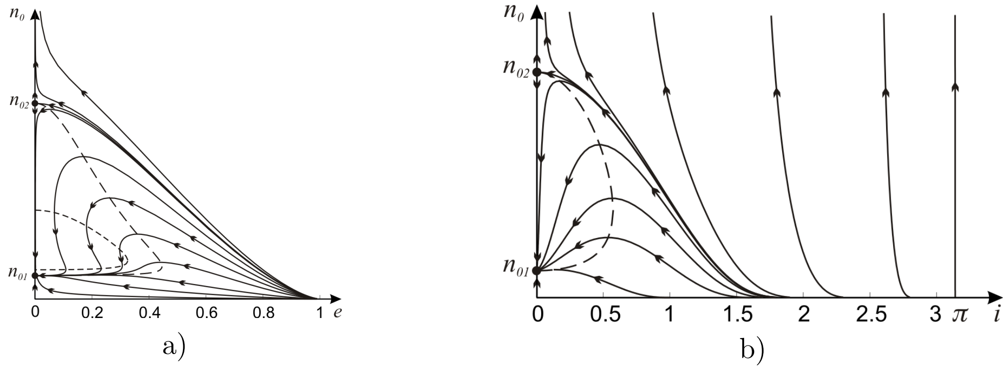


Figure 2: Phase portraits for the cases: a) $i \equiv 0$, b) $e \equiv 0$

Table 6: Parameters of planet-satellite systems

| Planet-satellite | $p$ | $n_0(0)$ | $n_{01}$ | $n_{02}$ |
|---|---|---|---|---|
| Earth-Moon | 0.15535 | $7.2589 \cdot 10^{-3}$ | $3.7922 \cdot 10^{-3}$ | 0.83503 |
| Mars-Phobos | $1.5235 \cdot 10^{-6}$ | 3.2171 | $3.5358 \cdot 10^{-18}$ | $1 - 1.5235 \cdot 10^{-6}$ |
| Mars-Deimos | $2.5391 \cdot 10^{-7}$ | $8.1268 \cdot 10^{-1}$ | $1.6369 \cdot 10^{-20}$ | $1 - 2.5391 \cdot 10^{-7}$ |
| Jupiter-Io | $5.8980 \cdot 10^{-4}$ | 0.2335 | $2.0517 \cdot 10^{-10}$ | 0.9994 |
| Jupiter-Europe | $3.1685 \cdot 10^{-4}$ | 0.1163 | $3.1811 \cdot 10^{-11}$ | 0.9997 |
| Jupiter and Ganymede | $9.7643 \cdot 10^{-4}$ | $5.7650 \cdot 10^{-2}$ | $9.3093 \cdot 10^{-10}$ | 0.9990 |
| Jupiter and Callisto | $7.0902 \cdot 10^{-4}$ | $2.4717 \cdot 10^{-2}$ | $3.5644 \cdot 10^{-10}$ | 0.9993 |

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