High-Performance Computations in Numerical Modeling of Wave Propagation in Block Media with Thin Interlayers

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Abstract

Parallel computational algorithms for the numerical modeling of dynamic interactions of elastic blocks through thin elastic and viscoelastic interlayers in structurally inhomogeneous medium such as rock are developed. Monotonous grid-characteristic ENO-scheme with the balanced number of time steps in blocks and interlayers is used for the numerical solution. The algorithms are implemented in 1d and 2d cases with the CUDA technology (Compute Unified Device Architecture) for supercomputers with graphics processing units. The results of numerical analysis demonstrate peculiar quality of planar wave propagation in materials with layered microstructure.

Introduction

Several nature materials such as rock have distinct structurally inhomogeneous block-hierarchical structure [1]. Block structure appears on different scale levels from the size of crystal grains to the blocks of rock. Blocks are connected to each other with thin interlayers of rock with significantly weaker mechanical properties.

One of the most important technological problems of coal mining is the prognosis of sudden collapse of the coal mining roof. This process is preceded by a weakening of the mechanical contact between the blocks: the rock gains weakened microstructure. Such state of the media can be detected with inducing elastic waves of small amplitude and recording the response to these disturbances. This method can be used to develop special technical devices for the forehanded prediction and prevention of the emergency situations.

The purpose of this paper is to apply reliable computational algorithms for the calculation of elastic wave propagation in rock media based on mathematical models that take into account complex rheological properties of layered materials [2]. Parallel computation algorithms for the numerical modeling of dynamic interactions of elastic blocks through thin viscoelastic interlayers in structurally inhomogeneous medium such as rock are developed as a program complex for the multiprocessor computers with graphics processing units. The computations performed on supercomputer for the large number of layers allowed to analyze specific "pendulum" waves related to the structural inhomogeneity. Experimental and theoretical research of pendulum waves related to the microstructure of materials is described in papers [3, 4, 5].

1 Mathematical Model

Hierarchical structure of rock media can be represented as a nested layered structure with invariant ratio of the characteristic sizes of blocks and interlayers. Let us consider at first a fragment of the structure in one dimension, i.e. the interleaved system of n elastic layers of thickness h and elastic interlayers of thickness δ.

Let ρ and ρ_0, c and c_0, $a = 1/(\rho c^2)$ and $a_0 = 1/(\rho_0 c_0^2)$ be the densities, velocities and elastic compliances of materials in the layer and the interlayer respectively. One-dimensional equations of elasticity theory inside the k^{th} layer have the form:

$$\rho \frac{\partial v^k}{\partial t} = \frac{\partial \sigma^k}{\partial x}, \quad a \frac{\partial \sigma^k}{\partial t} = \frac{\partial v^k}{\partial x}. \tag{1}$$

Here v^k is the longitudinal velocity in x-direction (x varies from 0 to h in each layer) and σ^k is the normal stress.

The behavior of the interlayer material is described by the system of ordinary differential equations:

$$\rho_0 \frac{d}{dt} \frac{v^{k+1} + v^k}{2} = \frac{\sigma^{k+1} - \sigma^k}{\delta}, \quad a_0 \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} = \frac{v^{k+1} - v^k}{\delta}. \tag{2}$$

This system contains boundary conditions for the mentioned above velocities and stresses, i.e. the left boundary condition is for the $(k+1)^{th}$ interlayer and the right is for the k^{th} interlayer.

Systems of equations (1), (2) are complemented with the initial and boundary conditions

$$v^k = \sigma^k = 0, \quad (k = 1, ..., n), \quad \sigma^1(0,t) = -p(t), \quad v^n(h,t) = 0.$$

Here $p(t)$ is given external pressure. Statement correctness of the initial-boundary value problem can be proved by the methods based on integral estimates derived from the energy conservation law [6]. The thermodynamic consistency of the mathematical model is fulfilled.

2 Numerical Algorithm

Numerical solution of the problem is based on the collapse of the gap Godunov scheme on a uniform grid with a time step $\tau = \triangle x/c$ admissible by the Courant-Friedrichs-Levy condition. In this case the scheme in the layer does not possess an artificial energy dissipation. Piece-wise linear ENO-reconstruction of the second-order accuracy is used with the lower values of time step [2].

This scheme significantly reduces the effect of smoothing the numerical solution peaks with corresponding refinement of the obtained results. Consistency conditions of the form (2) on the boundaries between layers and interlayers are calculated with the Godunov scheme as well. For this purpose in each artificially introduced cell simulating a single interlayer the collapse of the gap scheme is implemented. The independent time step in the interlayer in this scheme $\tau_0 = \delta/c_0 << \tau$ is admissible by the Courant-Friedrichs-Levy condition. In the interlayer the number of time steps necessary to achieve the next time step $t + \tau$ of the main scheme is calculated.

Grid-characteristic interpretation of the method is shown schematically on fig. 1. At the stage of solving system (2) with time step τ_0 the equations of collapse of the gap are used on the boundaries of interlayers (stage “predictor” in the interlayer):

$$z_0 v_+ - \sigma_+ = z_0 v - \sigma, \quad z v_+ + \sigma_+ = z v^{k+1} + \sigma^{k+1},$$
$$z_0 v_- + \sigma_- = z_0 v + \sigma, \quad z v_- - \sigma_+ = z v^k - \sigma^k.$$

Here $z = \rho c$ and $z_0 = \rho_0 c_0$ are the acoustic impedances of materials in layer and interlayer respectively. The values with upper indexes correspond to the boundary cells of interacting layers. The values with lower indexes "+" and "–" correspond to the right and the left boundaries of the interlayer respectively.

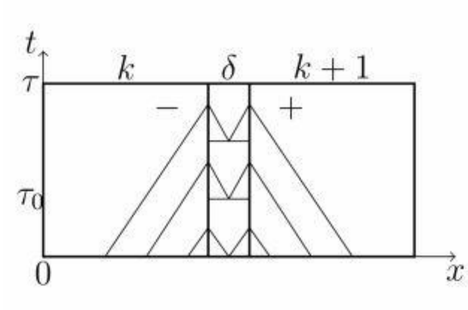


Figure 1: Scheme of the grid-characteristic method in the boundary between the layers.

The subsequent recalculation of the solution (stage "corrector") is done according to the formulas:

$$\hat{v} = v + (\sigma_+ - \sigma_-)\,\frac{\tau_0}{\rho_0\,\delta}, \qquad \hat{\sigma} = \sigma + (v_+ - v_-)\,\frac{\tau_0}{a_0\,\delta}. \tag{3}$$

Here $\hat{v}$ and $\hat{\sigma}$ correspond to the next time step. Predictor values on the boundary between layers in the main scheme with time step τ are calculated by averaging values related to the cell boundaries on small mesh steps. Stage "corrector" of the main scheme in layers is performed in the usual way based on the integral analogues of differential equations (1). The scheme possess at least the first-order approximation relative to the introduced parameters of time step and mesh step.

Numerical algorithm to solve planar and spatial problems is based on the space-variable two-cycling decomposition method. At first stage one-dimensional problem in x_1 direction in the time interval $(t; t + \tau/2)$ is solved. Second and third stages are similar to the first in directions x_2 and x_3 respectively. Fourth, fifth and sixth stages are the stages of recalculation of the problem in directions x_3, x_2 and x_1 respectively in the time interval $(t{+}\tau/2; t{+}\tau)$. In planar case third and fourth stages are absent. Two-cycling decomposition method preserves the second-order accuracy if schemes of second-order accuracy are used in its stages. It also ensures the stability of the numerical solution in dynamic problems provided the Courant-Friedrichs-Levy stability condition for one-dimensional systems is fulfilled.

3 Viscoelastic Interlayers

Analysis of experimental data of wave propagation in layered media shows that the interlayers behave non-elastically even under small wave amplitudes. The rheological method allows to construct more complex models taking into account natural dissipation processes in interlayers. These are the viscoelastic Maxwell and Kelvin–Voigt models.

In Maxwell's model the strain of the interlayers is formed from elastic and viscous components. Viscoelastic interaction according to this model is described by system (2) after replacing the second equation by the more general equation:

$$a_o \frac{d}{dt}\frac{\sigma^{k+1} + \sigma^k}{2} + \frac{\sigma^{k+1} + \sigma^k}{2\eta} = \frac{v^{k+1} + v^k}{\delta},$$

where η is the viscosity coefficient of interlayer material. In this case at stage "corrector" of the numerical algorithm an approximation of equations of viscoelastic media with Crank-Nicholson scheme is used instead of (3):

$$\rho_0 \frac{\hat{v} - v}{\tau_0} = \frac{\sigma_+ - \sigma_-}{\delta}, \quad a_0 \frac{\hat{\sigma} - \sigma}{\tau_0} + \frac{\hat{\sigma} + \sigma}{\eta} = \frac{v_+ - v_-}{\delta}, \tag{4}$$

which is also implemented with an explicit computational algorithm.

In Kelvin-Voigt model the stress in interlayer is formed of elastic and viscous components. The constitutive equation has the following form:

$$a_0 \frac{d}{dt} \frac{\sigma^{k+1} + \sigma^k}{2} = a_0 \eta \frac{d}{dt} \frac{v^{k+1} - v^k}{\delta} + \frac{v^{k+1} - v^k}{\delta}$$

Finite-difference scheme of "predictor-corrector" type is used for the approximation of the equations in interlayers. "Predictor" is calculated according to the equations of elastic model (3). "Corrector" is based on the system of equations

$$\rho_0 \frac{\hat{v} - v}{\tau_0} = \frac{\sigma_+ - \sigma_-}{\delta}, \quad a_0 \frac{\hat{s} - s}{\tau_0} = \frac{v_+ - v_-}{\delta}, \quad \sigma = s + \eta \frac{v_+ - v_-}{\delta}, \tag{5}$$

where s is the elastic stress.

4 Parallel Computational Algorithm

The modeling of a contact interaction between layers and interlayers is based on the system of equations (2) and its generalization taking into account viscous properties of the material in interlayer. Systems are written apart for the normal and tangent stresses and velocities relatively to the interfaces. To obtain numerical solution the computational algorithms with independent time steps for the longitudinal and lateral waves were applied.

The algorithms for calculation thin viscoelastic interlayers are implemented as complex of programs for the supercomputers with graphics processing units with CUDA technology (Compute Unified Device Architecture). The basic unit of executable program code in CUDA is the so-called kernel. The application consists of several kernels written in C / Fortran language extensions with the additional variables and functions for parallelization. The computational load is distributed between the blocks of the grid of graphics device. The blocks are composed from threads. Each thread corresponds to a cell of the computational domain.

To solve one-dimensional problem three kernels are used. At first the kernel for the boundary conditions and the stage "predictor" of the finite-difference scheme for each layer and the kernel for the interlayer interaction are executed independently. After that the computations in blocks in a graphics device are synchronized. The synchronization provides the correct calculation of the predictor variables in the whole computational domain. Finally the kernel for the stage "corrector" of the scheme is calculated.

To analyze the numerical solution additional kernels implement the calculation of seismograms and the Fourier transform of the velocities and the displacements. Fourier analysis of the displacement of layers seismograms allows to identify the characteristic frequency of the pendulum wave due to the compliances of interlayers and their thickness.

To solve planar problems 2D decomposition of the domain is used. The two-cyclic space-variable decomposition method leads to the one-dimensional equations that are solved with the proposed algorithm.

If the volume of calculations exceeds the memory of graphics device it is possible to use several devices with the data exchange in one of the interlayer.

5 Numerical Results

The computations of planar waves propagation induced by short and long $\Lambda-$ and $\Pi-$ impulses on the boundary of layered medium were performed. The layered medium consists of 512 layers of rock with microfractured elastic interlayers. Calculations were performed with the following parameters: $\rho_0/\rho = 0.73$, $c_0/c = 0.57$, $\delta/h = 0.034$. A uniform finite difference mesh in the layer consists of 16 cells and one cell is used in each interlayer.

On fig. 2 and fig. 3 the dependencies of velocities of particles v of space coordinate x in a problem of the load of $\Lambda-$impulse of pressure are shown. The impulse with a unit amplitude was induced on the left boundary of computational domain, the right boundary was fixed. Figure 2 corresponds to the impulse duration equal to the time that elastic wave passes through one layer, fig. 3 corresponds to the duration two and a half times longer. Figure 2.a and fig. 3.a show the dependencies of velocities at the time the incident wave passes approximately 400 layers (6500^{th} time step of the main scheme). On fig. 2.b and fig. 3.b the reflected wave passes approximately 200 layers (12000^{th} time step of the main scheme).

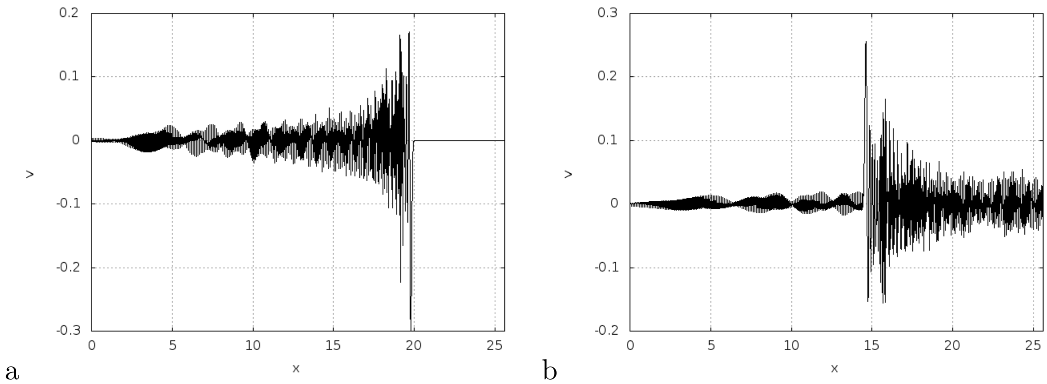


Figure 2: Velocity behind front wave of incident (a) and reflected (b) waves induced by the short impulse in layered medium.

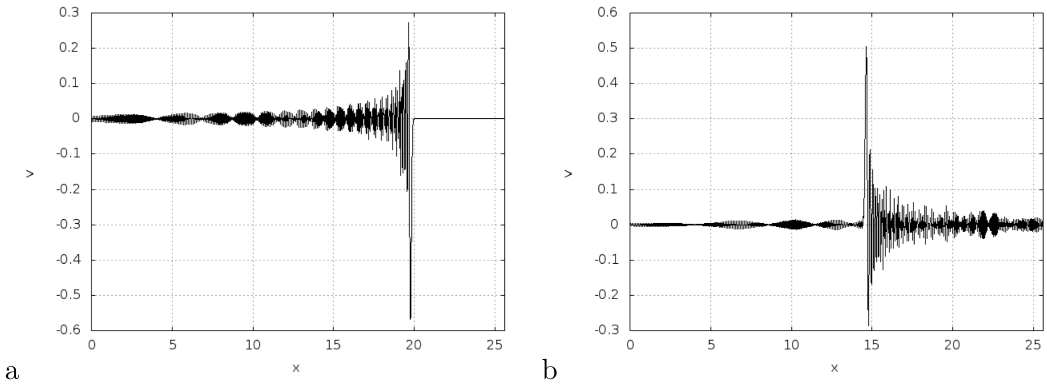


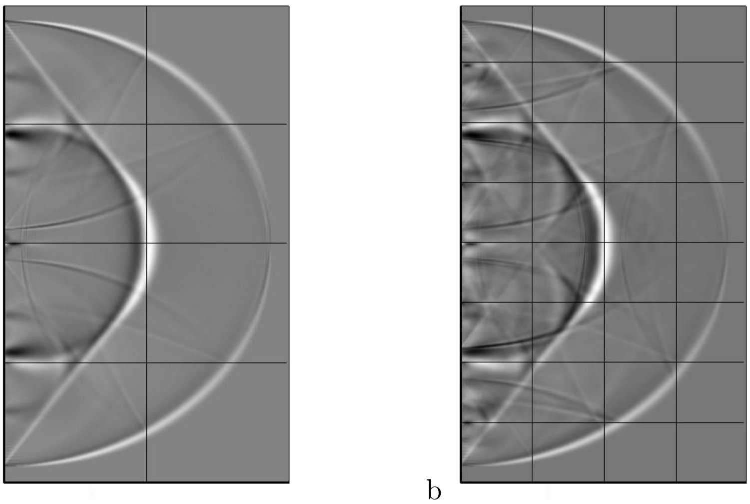
Figure 3: Velocity behind front wave of incident (a) and reflected (b) waves induced by the long impulse in layered medium.

These results demonstrate a qualitative difference between the wave pattern in layered media as compared to a homogeneous medium. This difference at the initial stage is revealed in the appearance of waves reflected from the interlayers, i.e. the characteristic oscillations behind the loading wave front as it passes through the interface. Eventually stationary wave pattern appears after multiple reflections behind the head wave front, i.e. the so-called pendulum wave which existence was predicted in papers [3, 4, 5].

Comparing of fig. 2 and fig. 3 shows that the head wave amplitude increases with the impulse duration increasing, and the amplitude of the oscillations behind the wavefront decreases and tends to zero. It is related to the fact that waves with wavelength considerably greater than the thickness of the layer are nearly not reflected from the interlayers. Thus it is possible to detect the weakened microstructure of layered or block medium with only sufficiently short wavelengths.

To verify the algorithm similar calculations were performed for the elastic interlayer, impedance of which coincides with the impedance of layers, and the density of which is less than the layers density. In this case the wave induced by the impulse load on the boundary passes through the interlayer as in homogeneous medium in form of solitary impulse. Its velocity is naturally corrected due to the difference between the velocities of impulse propagation in the layer and in the interlayer. The wave amplitude doesn't distort for the thousands of time steps of the main scheme provided steps τ and τ_0 are admissible by Courant-Friedrichs-Levy condition. With theirs decreasing the wave amplitude in layered medium attenuates very quickly.

The numerical results of Lamb problem on the action of instant concentrated load in tangent direction on a central point of a half-space for plane strain are presented on fig. 4. These results are obtained on a grid consisting of 512×1024 cells. Figure 4.a corresponds to the domain with 2×4 layers, fig. 4.b is related to the domain with 4×8 layers. The black lines mark the positions of interlayers.



a b

Figure 4: Lamb problem: contour levels of σ_{12}

All characteristic waves for the solution of Lamb's problem in the framework of the classical elasticity theory (incident longitudinal and transverse waves with circular fronts, two conical transverse waves in the form of symmetric straight-line segments tangent to the semicircle of smaller radius, the Rayleigh surface waves rapidly damped with depth shown by bright points on the boundary following by an incident transverse wave) are clearly distinguished on level curves. The distinction is that for the domain with thin interlayers multiple re-reflections from the interlayers arised.

The calculations were performed on the supercomputer "Flagman" of ICM SB RAS with 8 graphics processing units NVIDIA Tesla C2050.

The series of computations shows that if mechanical parameters of materials and thickness of layer and interlayer don't allow to perform balanced calculation of the problem with the maximum admissible time steps then it is necessary to set the time step in the interlayer as maximum allowed and apply the scheme of higher accuracy in the layer with the reconstruction of the solution making the calculations with lower time step.

It should be noted in conclusion that the choice of numerical scheme in the solution of problems of wave propagation in layered media is significant. In such problems non-monotonous finite-difference schemes with inadequate oscillations on a head wave front and monotonous schemes with artificial viscosity where amplitude attenuated incorrectly can't be applied.

Acknowledgements

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