# On optimal anisotropic bodies in the heat conduction problems

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**Abstract**

Some problems of internal structural optimization are formulated for heat conduction bodies made of locally orthotropic material. State variable (inverse temperature) is determined with the help of solution of heat propagation boundary value problem. The orthogonal tensor of rotation defining the optimal orientation of orthotropic axes is considered as a control variable. Necessary conditions for total dissipation functional extremum are derived and some basic properties of optimal structures are investigated. Examples of solution of the problems on optimal orthotropic material distribution are presented.

## 1 Basic relations and formulation of optimization problem

Consider stationary process of the heat conduction in the rigid body occupied the domain $\Omega$ with the boundary $\Gamma$ where $\Gamma = \Gamma_g + \Gamma_i$, $\Gamma_i \bigcap \Gamma_g = 0$. The temperature is given on the part $\Gamma_g$ of the boundary. The part $\Gamma_i$ is assumed to be isolated. The material of the body is anisotropic with respect to the heat conduction process described by the known relations [1], [2]

$$\overrightarrow{q} = D \cdot \nabla\beta, \quad \nabla\beta = K \cdot \overrightarrow{q}, \quad \beta = \theta^{-1}, \quad K \cdot D = E, \tag{1}$$

where $\theta$ - temperature, $\overrightarrow{q}$ - vector of heat flux and $D = \{D_{ij}\}$, $K = \{K_{ij}\}$ - heat conduction tensors of the second rank, $E = \{\delta_{ij}\}$ - unit tensor, $\delta_{ij}$ - Kronecker symbol $(i, j = 1, 2, 3)$. In the case of absence of the heat source in the domain $\Omega$ we have the following governing equation

$$\nabla \cdot (D \cdot \nabla\beta) = \frac{\partial}{\partial x_i}(D_{ij}\frac{\partial}{\partial x_j}\beta) = 0 \tag{2}$$

and boundary conditions

$$(\beta)_{\Gamma_g} = \beta^0, \quad (\overrightarrow{q} \cdot \overrightarrow{n})_{\Gamma_i} = (n_i D_{ij} \cdot \frac{\partial\beta}{\partial x_j})_{\Gamma_i} = 0. \tag{3}$$

Here $\overrightarrow{n}$ - outward unit normal vector, $\beta^0$ - given function and in what follows we use as generally accepted index notation assuming summation for repeating indexes as symbolic form [3]. Vector $\nabla\beta$ - gradient operator applied to scalar value $\beta$, $\nabla \cdot \overrightarrow{q} = \partial q_i/\partial x_i$ - divergence operator applied to the vector $\overrightarrow{q}$. Here and in what follows point and double points mean respectively scalar product and double scalar product.

Total dissipation can be written as

$$J_D = \int_\Omega \nabla\beta \cdot D \cdot \nabla\beta d\Omega = \int_\Omega (\nabla\beta \otimes \nabla\beta) \cdot \cdot D d\Omega, \tag{4}$$

where $\otimes$ - tensor product. In accordance with the variational principle [2] the actual distribution of the function $\beta$ realizes a minimum for the functional $J_D$ on the set of admissible functions satisfying the first boundary condition (3), i.e.

$$J_D \to \min_\beta . \tag{5}$$

Note that the equation (2) is the Eulerian equation for the functional (4) expressing necessary extremum conditions in the problem (5) and the second boundary condition (3) plays the role of transversality condition and is satisfied "automatically" for extremum solution.

Let us fix global orthogonal coordinate system $x_1$, $x_2$, $x_3$ with the unit vectors $\overrightarrow{e_1^0}$, $\overrightarrow{e_2^0}$, $\overrightarrow{e_3^0}$. The unit vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, $\overrightarrow{e_3}$ of the principal directions of orthotropic material at the arbitrary point $(x_1, x_2, x_3) \in \Omega$ are related with the global coordinate vectors $\overrightarrow{e_1^0}$, $\overrightarrow{e_2^0}$, $\overrightarrow{e_3^0}$ by means of the rotation tensor $Q = Q(x)$:

$$\overrightarrow{e_i} = Q * \overrightarrow{e_i^0} = Q \cdot \overrightarrow{e_i^0}, \quad Q \cdot Q^T = E \quad (i = 1, 2, 3), \tag{6}$$

where symbol $T$ means operation of transposition and $*$ - rotation operation. In the axes of symmetry of orthotropic material the tensor of heat conductivity $D$ is written as

$$\begin{aligned} D &= D_{ij}^0 \overrightarrow{e_i} \otimes \overrightarrow{e_j} = D_{ij}^0 Q \cdot \overrightarrow{e_i^0} \otimes Q \cdot \overrightarrow{e_j^0} = \\ &= Q * (D_{ij}^0 \overrightarrow{e_i^0} \otimes \overrightarrow{e_j^0}) = Q * D^0, \quad D^0 = D_{ij}^0 \overrightarrow{e_i^0} \otimes \overrightarrow{e_j^0}. \end{aligned} \tag{7}$$

This expression for $D$ can be rewritten in the form

$$D = Q * D^0 = Q \cdot D^0 \cdot Q^T. \tag{8}$$

If $\kappa_i^0$ and $\overrightarrow{e_i^0}$ - eigenvalues and eigenvectors of the tensor $D^0$, i.e. $D^0 \cdot \overrightarrow{e_i^0} = \kappa_i^0 \overrightarrow{e_i^0}$, then $\kappa_i^0$ and $Q \cdot \overrightarrow{e_i^0} = \overrightarrow{e_i}$ - eigenvalues and eigenvectors of the tensor $Q * D^0 = Q \cdot D^0 \cdot Q^T$. Actually we have

$$D \cdot \overrightarrow{e_i} = (Q * D^0) \cdot (Q * \overrightarrow{e_i^0}) = Q * (D^0 \cdot \overrightarrow{e_i^0}) = \kappa_i^0 Q * \overrightarrow{e_i^0} = \kappa_i^0 \overrightarrow{e_i}. \tag{9}$$

Considered optimization problem consists in determination of the tensor $Q^*$ in each point $x \in \Omega$ for which the minimum of the quality functional $J_D$ is realized, i.e.

$$J_D^* = \min_Q J_D \tag{10}$$

under fulfilment of (2)-(7). Described optimization problem can be reformulated as the problem of successive determination of the functional $J_D$ extremums

$$J_D^* = J_D(Q^*, \beta^*) = \min_Q \min_\beta J_D(Q, \beta). \tag{11}$$

The external minimum (with respect to $Q$) is finding taking into account the orthogonality condition (6).

## 2 Necessary optimality conditions

To derive necessary condition for extremum determining the tensor of rotation $Q = Q(x)$ and characterizing the optimal orientation of orthotropy axes let us use the method of Lagrange multipliers and construct augmented functional

$$J^L = J_D + J_P, \quad J_P = \int_\Omega P \cdot\cdot (Q^T \cdot Q - E) d\Omega,$$
$$J_D = \int_\Omega \nabla\beta \cdot (Q * D^0) \cdot \nabla\beta d\Omega = \int_\Omega B \cdot\cdot (Q \cdot D^0 \cdot Q^T) d\Omega, \tag{12}$$

where symmetric tensor of the second rank $P = P(x)$ ($x \in \Omega$) - Lagrange multiplier corresponding to condition of orthogonality (6). By means of $B$ we denote in (12) the following symmetric second rank tensor

$$B = \nabla\beta \otimes \nabla\beta, \quad B = B^T. \tag{13}$$

Using known formulae $a \cdot\cdot (b \cdot c) = c \cdot\cdot (a \cdot b)$ and $a \cdot\cdot b = b \cdot\cdot a = a^T \cdot\cdot b^T$ for second rank tensors $a$, $b$, $c$ we derive the following expressions for the first variations $\delta J_D$ and $\delta J_P$ with respect to variation $\delta Q$ of rotation tensor

$$\delta J_P = \int_\Omega P \cdot\cdot (\delta Q^T \cdot Q + Q^T \cdot \delta Q) d\Omega = 2 \int_\Omega \delta Q \cdot\cdot (P \cdot Q^T) d\Omega,$$
$$\delta J_D = \int_\Omega B \cdot\cdot \delta D d\Omega = \int_\Omega B \cdot\cdot (\delta Q \cdot D^0 \cdot Q^T + Q \cdot D^0 \cdot \delta Q^T) d\Omega = \tag{14}$$
$$= 2 \int_\Omega \delta Q \cdot\cdot (D^0 \cdot Q^T \cdot B) d\Omega.$$

Taking into account the expressions (14) we will find the total variation $\delta J^L$ with respect to variation $\delta Q$

$$\delta J^L = \delta J_D + \delta J_P = 2 \int_\Omega \delta Q \cdot\cdot (D^0 \cdot Q^T \cdot B + P \cdot Q^T) d\Omega. \tag{15}$$

Using extremum condition $\delta J^L = 0$ and arbitrariness of $\delta Q$ we will have $D^0 \cdot Q^T \cdot B + P \cdot Q^T = 0$ in $\Omega$. Multiplying this relation by $Q$ and using formulae (3), (8) we find

$$D \cdot \nabla\beta \otimes \nabla\beta = -Q \cdot P \cdot Q^T. \tag{16}$$

This relation means the symmetry of the second rank tensor written in the left side of (16), i.e.

$$D \cdot \nabla\beta \otimes \nabla\beta = \nabla\beta \otimes D \cdot \nabla\beta. \tag{17}$$

In another form we will have

$$\vec{q} \otimes \nabla\beta = \nabla\beta \otimes \vec{q}. \tag{18}$$

The equality (18) is satisfied if the vectors of heat flux $\vec{q}$ and gradient of inverse temperature are parallel, i.e. $\vec{q} = \lambda \nabla\beta$, where $\lambda$ - some scalar value. Presented relation we write in the following form

$$D \cdot \nabla\beta = \lambda \nabla\beta. \tag{19}$$

## 3 Analysis of optimal solutions and some examples of extremum material orientation

Presented necessary optimality condition(19), equalities (6), (8) and equation (2) with boundary conditions (3) determine the extremum orientations of locally orthotropic material and also variables $\beta$, $\overrightarrow{q}$. Taking into account that the eigenvalues $\lambda_i$ $(i = 1, 2, 3)$ of the tensors $D$ and $D^0$ are given we assume

$$\lambda_1 = \lambda_{\min} < \lambda_2 < \lambda_3 = \lambda_{\max}. \tag{20}$$

If we assume that the same way of extremum orientation of the principal axes of orthotropy is realized for all domains $\Omega$ then the heat conduction process is described by the equation

$$\nabla \overrightarrow{q} = \nabla \cdot (D \cdot \nabla\beta) = \lambda\Delta\beta = 0, \quad (\lambda = \lambda_i, \quad i = 1, 2, 3), \tag{21}$$

where $\Delta$ - Laplace operator acting in 3-dimensional space. The last equality in (21) means that in the case of the body with extremum orthotropy the heat conductivity process is described by the harmonic equation $\Delta\beta = 0$, the same as in isotropic case.

The existence and uniqueness of the extremum distribution of the variable $\beta$ which is independent on $\lambda$ follows from the existence and uniqueness of solution of the "isotropic" boundary value problem

$$\Delta\beta = 0, \quad (\beta)_{\Gamma_g} = \beta^0, \quad (n \cdot \nabla\beta)_{\Gamma_i} = 0. \tag{22}$$

If the domain $\Omega$ consists of several sub-domains $\Omega_i$ $(\Omega = \bigcup \Omega_i, \ \Omega_i \bigcap \Omega_j = 0 \ (i \neq j))$ and for each separate sub-domain $\Omega_i$ the same extremum way of material orientation is taken, then the isotropic distribution of the heat flux is realized for all considered sub-domains.

Let us assume that the orthotropic material is distributed in accordance with the same extremum orientation rule. Then we will have the following minimal and maximal values of the optimized quality functional $J_D$

$$\min_Q J_D = \lambda_{\min} I, \quad \max_Q J_D = \lambda_{\max} I, \quad I = \int_\Omega (\nabla\beta)^2 d\Omega. \tag{23}$$

Thus we obtain the double-side estimates of the total dissipation

$$\lambda_{\min} \leq J_D I^{-1} \leq \lambda_{\max}. \tag{24}$$

The rotational tensors $Q = Q(x)$ corresponding to the extremal material orientation are determined with the help of the equation

$$(Q \cdot D^0 \cdot Q^T) \cdot \nabla\beta = \lambda\nabla\beta. \tag{25}$$

As it is seen from the necessary optimality condition (19) the heat flux is given by the formula

$$\overrightarrow{q} = \lambda\nabla\beta \quad (\lambda = \lambda_i, \quad i = 1, 2, 3). \tag{26}$$

This formula means that the vector $\overrightarrow{q}$ is normal to the level surface (in the plane case to the level lines) of the function $\beta$, i.e. orthogonal to the heat front, where $\theta(x) = const$. In addition the relation (26) indicates the absence of the heat fluxes which are tangential to the level surfaces. Moreover, the minimum and maximum of dissipation are realized when the axes of orthotropy with the minimal and maximal heat conductivity are respectively oriented in normal and tangential directions to the level surface of the function $\beta$.

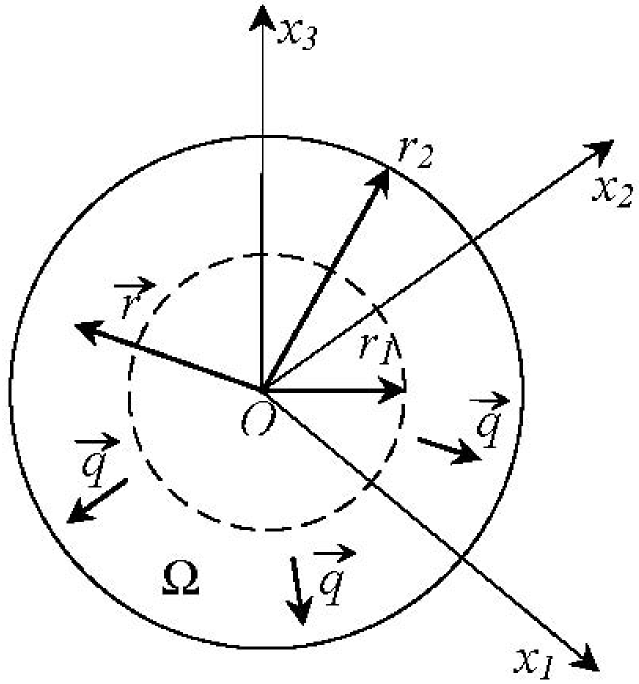


Figure 1: Heat transfer between spherical surfaces

Using derived necessary optimality conditions and investigated global properties of optimal solutions let us consider some particular problems of search for orthotropic material orientation in the heat conducting bodies.

Suppose at first that the orthotropic material occupies the 3-d domain $\Omega$ (see Fig.1) situated between the internal sphere of radius $r_1$ and the external sphere of radius $r_2$, where $r_1,\ r_2\ (r_1 < r_2)$ - given values. The temperature $\theta = \theta_1$ is defined at the internal boundary and the temperature $\theta = \theta_2\ (\theta_1 > \theta_2)$ is given at the external boundary. Thus we consider the following boundary conditions

$$\beta = \beta_1 = \theta_1^{-1}, \quad r = r_1; \quad \beta = \beta_2 = \theta_2^{-1}, \quad r = r_2. \tag{27}$$

Here $\beta_1 < \beta_2$. We use spherical coordinate system with the origin at $r = 0$. From the properties of symmetry it follows that the extremum orientations of the axes of orthotropy with $\lambda_1 = \lambda_{\min}$ and $\lambda_3 = \lambda_{\max}$ corresponding, respectively, to the cases $J_D \to \min_Q$ and $J_D \to \max_Q$ are realized in radial direction. Besides, the gradient of $\beta$, i.e. vector $\nabla\beta$, and also the heat flux vector $\overrightarrow{q}$ are directed along the radius vector at each point of the domain $\Omega$. Note that the heat flux is absent in circumferential directions. The following values characterize the extremum solutions:

$$\begin{gathered}
\nabla\beta = N\overrightarrow{r_0}, \quad \int_\Omega (\nabla\beta)^2 d\Omega = \frac{4}{3}\pi(\beta_2-\beta_1)^2\frac{r_1^2+r_1r_2+r_2^2}{r_2-r_1} = I, \\
\min_Q J_D = \lambda_{\min} I, \quad \max_Q J_D = \lambda_{\max} I, \quad \overrightarrow{r_0} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|}, \\
(\overrightarrow{q})_{\min} = \lambda_{\min} N \overrightarrow{r_0}, \quad (\overrightarrow{q})_{\max} = \lambda_{\max} N \overrightarrow{r_0}, \quad N = \frac{\beta_2-\beta_1}{r_2-r_1},
\end{gathered} \tag{28}$$

where $\overrightarrow{r_0}$ is a unit vector.

Next let us consider the problem of finding the optimal solution, when a simply-connected domain $\Omega$ occupied by the orthotropic material is a rectangular parallelepiped with the upper and lower faces at $x_3 = -c$, $x_3 = c$ and side faces at $x_1 = \pm a$ and $x_2 = \pm b$, where $a, b, c$ - given positive parameters. We use here Cartesian coordinate system $(x_1, x_2, x_3)$ and assume that the temperature $\theta$ is given at the lower and upper faces and the side faces are thermally insulated, i.e. the boundary conditions have the form

$$\begin{aligned} &\beta = \beta_1 = \theta_1^{-1}, \quad x_3 = -c; \quad \beta = \beta_2 = \theta_2^{-1}, \quad x_3 = c, \\ &\overrightarrow{q} \cdot \overrightarrow{n} = \overrightarrow{n} \cdot D \cdot \nabla\beta = 0, \quad x_1 = \pm a, \quad x_2 = \pm b, \end{aligned} \tag{29}$$

where $\theta_1 > 0,\ \theta_2 > 0\ \ (\theta_1 > \theta_2)$ - given temperature values and $a, b, c$ - given positive geometric parameters. Optimal solution of considered problem complying with necessary optimality condition derived in section 2 is characterized by the existence of level surfaces $x_3 = const$ $(-c < x_3 < c$ in $\Omega)$ with constant distribution of variable $\beta$ (constant temperature $\theta$). The gradient of $\beta$ is parallel to $x_3$-axis. Therefore the axes of orthotropy with minimal $\lambda = \lambda_{min}$ (in the case $J_D \to \min_Q$) and with maximal $\lambda = \lambda_{max}$ (in the case $J_D \to \max_Q$) are oriented in a parallel way with respect to the axis $x_3$. Such orientation provides, respectively, either the minimum or the maximum of dissipation. For considered problem we will have

$$\begin{aligned} &\nabla\beta = N\overrightarrow{x_3^0}, \quad \int_\Omega (\nabla\beta)^2 d\Omega = \frac{2ab}{c}(\beta_2 - \beta_1)^2 = I, \\ &\min_Q J_D = \lambda_{\min} I, \quad \max_Q J_D = \lambda_{\max} I, \quad \overrightarrow{x_3^0} = \frac{\overrightarrow{x_3}}{|\overrightarrow{x_3}|}, \\ &(\overrightarrow{q})_{\min} = \lambda_{\min} N\overrightarrow{x_3^0}, \quad (\overrightarrow{q})_{\max} = \lambda_{\max} N\overrightarrow{x_3^0}, \quad N = \frac{\beta_2 - \beta_1}{2c}, \end{aligned} \tag{30}$$

where $\overrightarrow{x_3^0}$ is a unit vector of the $x_3$-axis.

As it is shown above the function $\beta$ satisfies the equation (21) in the case of extremal placement of orthotropic material. Note that this equation describes also the deflection of elastic membrane fixed at the part $\Gamma_g$ of the boundary $\Gamma$ and free at the another part $\Gamma_i$. Therefore it is possible to use this analogy in the case of corresponding boundary conditions for qualitative description of optimal orthotropic material distribution and also for preliminary analysis of correctness of formulated optimization problems. As an example consider particular heat conduction problem for cylindrical tube, which is infinitely long in direction of $x_3$-axis. Solving this problem we take into account the independence of solution on the $x_3$-coordinate and consider the cross-section of the tube at some arbitrary value of $x_3$-coordinate. It is assumed that the tube cross-section is a plane double-connected domain $\Omega$ bounded by the internal and external confocal ellipses. The boundary conditions have the form: $\beta_1 = \theta_1^{-1},\ \beta_2 = \theta_2^{-1}$ $(\theta_1 < \theta_2,\ \beta_1 > \beta_2)$, where the lower indexes "1" and "2" correspond, respectively, to the internal and external ellipses. In accordance with the considered analogy it is suppose that the membrane deflections $w$ are given at the internal contour $w_1 = \beta_1$ and at the external contour $w_2 = \beta_2$, where $w_1 > w_2$ (see Fig.2). The level lines of the deflection function $w(x_1, x_2) = const$ characterize the behavior of the level lines of $\beta$. Normal directions to the level lines of deflections describe the heat fluxes. The gradients of the deflection function determine of the orientations of orthotropic material.

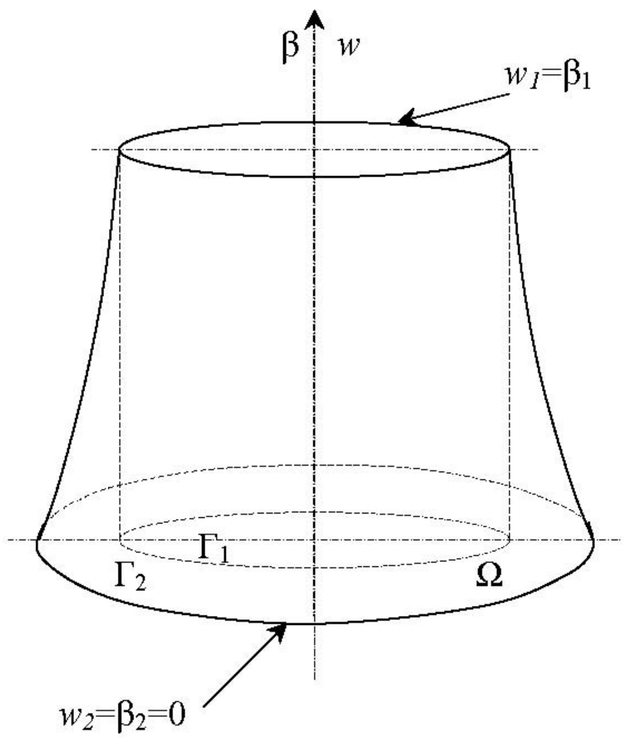


Figure 2: Membrane analogy

## 4 Some notes and conclusions

In this paper the theory of the heat conduction in the anisotropic rigid bodies having the locally orthotropic internal structure was used for investigation of the best orientation of the material. Optimality condition have been derived and analyzed. It was shown that for the bodies with the optimal structure the governing heat conduction equation is the same as for isotropic bodies. The two-side estimations have been found for the total dissipation functional. To simplify the search of the optimal orientation of the locally orthotropic material the mathematical analogy with the elastic membrane model has been proposed.

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