

Motion of a Rigid Rod on a Convex Surface

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Abstract

Motion of the rigid infinitely thin rod on a fixed convex surface is investigated. The similar problem has been studied previously in [1]-[3] for the particular case when the supporting surface is an inclined circular cylinder. In the present paper equations of motion of the rod on an arbitrary convex surface under the action of gravity is derived. The condition when the obtained equations possess an invariant measure is found; the possible cases, when this condition is valid are described. All equilibria of the rod on the surface are found and discussed.

1 Necessary facts from Differential Geometry

To derive equations of motion of the rod on the surface we need some facts from Differential Geometry. Below we briefly discuss these facts.

We shall assume that the coordinate grid on the surface

$$\mathbf{r} = \mathbf{r}(q_1, q_2) \tag{1}$$

consists on the lines of curvature and directions of these lines are defined at every point of the surface by unit vectors

$$\mathbf{u}_1 = \frac{1}{h_1} \frac{\partial \mathbf{r}}{\partial q_1}, \quad \mathbf{u}_2 = \frac{1}{h_2} \frac{\partial \mathbf{r}}{\partial q_2}, \quad (\mathbf{u}_i \cdot \mathbf{u}_j) = \delta_{ij}. \tag{2}$$

Here h_1, h_2 are the Lamé's coefficients:

$$h_i(q_1, q_2) = \left| \frac{\partial \mathbf{r}}{\partial q_i} \right|, \quad i = 1, 2.$$

The vector

$$\mathbf{e}(q_1, q_2) = [\mathbf{u}_1 \times \mathbf{u}_2]$$

is the normal vector to the surface (1) at the point (q_1, q_2) .

Let us denote by $k_i(q_1, q_2)$, $i = 1, 2$ the principal curvatures of the surface (1), then we get:

$$\frac{\partial \mathbf{e}}{\partial q_1} = -h_1 k_1 \mathbf{u}_1, \quad \frac{\partial \mathbf{e}}{\partial q_2} = -h_2 k_2 \mathbf{u}_2. \tag{3}$$

Equations (3) follow from the Rodrigues's theorem [4]). Furthermore, using (2) and (3) it is possible to derive the following equations:

$$\begin{aligned} \frac{\partial \mathbf{u}_1}{\partial q_1} &= -\frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \mathbf{u}_2 + h_1 k_1 \mathbf{e}, & \frac{\partial \mathbf{u}_1}{\partial q_2} &= \frac{1}{h_1} \frac{\partial h_2}{\partial q_1} \mathbf{u}_2, \\ \frac{\partial \mathbf{u}_2}{\partial q_1} &= \frac{1}{h_2} \frac{\partial h_1}{\partial q_2} \mathbf{u}_1, & \frac{\partial \mathbf{u}_2}{\partial q_2} &= -\frac{1}{h_1} \frac{\partial h_2}{\partial q_1} \mathbf{u}_1 + h_2 k_2 \mathbf{e}. \end{aligned} \tag{4}$$

2 Kinematic Equations

Let the rigid infinitely thin rod moves without sliding on the convex surface defined by the equation (1). Suppose that the rod touches the surface at a single point P . Let us introduce the moving coordinate system $Px_1x_2x_3$ that is rigidly connected with the rod. We denote by \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e} the unit vectors of this system. The unit vector \mathbf{e}_1 is directed such that the radius-vector \overrightarrow{PG} of the center of mass G of the rod has a form: $\overrightarrow{PG} = s\mathbf{e}_1$. The unit vector \mathbf{e} is a normal vector to the supporting surface at the point P . The unit vector \mathbf{e}_2 is chosen such that the vectors \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e} form the right handed set of vectors.

We shall define the position of the rod on the surface by the previously introduced variable s and by the angle φ defined as follows:

$$\mathbf{e}_1 = \mathbf{u}_1 \cos \varphi + \mathbf{u}_2 \sin \varphi, \quad \mathbf{e}_2 = -\mathbf{u}_1 \sin \varphi + \mathbf{u}_2 \cos \varphi. \quad (5)$$

Since the rod moves on the surface without sliding the velocity of the point P of the rod equals to zero. This condition can be described by two nonholonomic constraints:

$$h_1 \dot{q}_1 + \dot{s} \cos \varphi = 0, \quad h_2 \dot{q}_2 + \dot{s} \sin \varphi = 0.$$

If we denote $\dot{s} = u$ the corresponding nonholonomic constraints can be rewritten as follows:

$$\dot{q}_1 = -\frac{u}{h_1} \cos \varphi, \quad \dot{q}_2 = -\frac{u}{h_2} \sin \varphi. \quad (6)$$

Using (5) and taking into account (3), (4) and (6) we get:

$$\begin{aligned} \dot{\mathbf{e}}_1 &= \left(\dot{\varphi} + \left(\frac{\partial h_1}{\partial q_2} \cos \varphi - \frac{\partial h_2}{\partial q_1} \sin \varphi \right) \frac{u}{h_1 h_2} \right) \mathbf{e}_2 - \left(k_1 \cos^2 \varphi + k_2 \sin^2 \varphi \right) u \mathbf{e}, \\ \dot{\mathbf{e}} &= \left(k_1 \cos^2 \varphi + k_2 \sin^2 \varphi \right) u \mathbf{e}_1 + (k_2 - k_1) u \sin \varphi \cos \varphi \mathbf{e}_2. \end{aligned}$$

Let us define the vector $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega \mathbf{e}$ such that $\dot{\mathbf{e}}_1 = [\boldsymbol{\omega} \times \mathbf{e}_1]$, $\dot{\mathbf{e}} = [\boldsymbol{\omega} \times \mathbf{e}]$. Then the vector $\boldsymbol{\omega}$ is the absolute angular velocity vector of the system $Px_1x_2x_3$. Its components on the axes of the system $Px_1x_2x_3$ have the following form:

$$\begin{aligned} \omega_1 &= (k_1 - k_2) u \sin \varphi \cos \varphi, \quad \omega_2 = (k_1 \cos^2 \varphi + k_2 \sin^2 \varphi) u, \\ \omega &= \dot{\varphi} - \left(\frac{\partial h_2}{\partial q_1} \sin \varphi - \frac{\partial h_1}{\partial q_2} \cos \varphi \right) \frac{u}{h_1 h_2}. \end{aligned} \quad (7)$$

3 Dynamical Equations

In order to derive equations of motion of the rod on the surface we shall use the Gibbs – Appell method. First of all let us find the expression for the Gibbs function (energy of accelerations) of the system. We shall use the variables u and ω as quasivelocities in this problem. The Gibbs function can be found using the formula:

$$S = \frac{M}{2} \mathbf{a}_G^2 + \frac{1}{2} (\boldsymbol{\varepsilon} \cdot \boldsymbol{\Theta}_G \boldsymbol{\varepsilon}) + ([\boldsymbol{\omega} \times \boldsymbol{\Theta}_G \boldsymbol{\omega}] \cdot \boldsymbol{\varepsilon}). \quad (8)$$

Here M is the mass of the rod, \mathbf{a}_G is the acceleration of the rod's center of mass, $\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}}$ is the angular acceleration of the rod, $\boldsymbol{\Theta}_G$ is the central inertia tensor of the rod. We shall

assume that the vectors \mathbf{e}_1 , \mathbf{e}_2 and \mathbf{e} are directed along the principal axes of inertia of the rod. Then Θ_G has the following form:

$$\Theta_G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}.$$

In the explicit form expression (8) can be written as follows:

$$\begin{aligned} S = & J_P (k_1 - k_2) u k_\varphi \sin \varphi \cos \varphi \left(u \dot{\omega} - 3\omega \dot{u} - 2 \left(\frac{\partial h_2}{\partial q_1} \sin \varphi - \frac{\partial h_1}{\partial q_2} \cos \varphi \right) \frac{u \dot{u}}{h_1 h_2} \right) - \\ & - J_P k_\varphi \left(\left(\frac{\partial k_1}{\partial q_1} \cos^2 \varphi + \frac{\partial k_2}{\partial q_1} \sin^2 \varphi \right) \frac{\cos \varphi}{h_1} + \left(\frac{\partial k_1}{\partial q_2} \cos^2 \varphi + \frac{\partial k_2}{\partial q_2} \sin^2 \varphi \right) \frac{\sin \varphi}{h_2} \right) u^2 \dot{u} + \\ & + \frac{J_P k_\varphi^2}{2} \dot{u}^2 + \frac{J_P}{2} \dot{\omega}^2 + M s u \omega \dot{\omega} + M k_\varphi^2 s u^2 \dot{u}. \end{aligned}$$

Here $J_P = J + M s^2$ are the moments of inertia of the rod about the axes Px_2 and Px_3 and $k_\varphi = k_1 \cos^2 \varphi + k_2 \sin^2 \varphi$ is the normal section curvature in the direction \mathbf{e}_1 .

If the rod moves under the action of the forces with the potential $V(q_1, q_2, s, \varphi)$ then its equations of motion written in the form of the Gibbs – Appell equations have the form:

$$\begin{aligned} \frac{\partial S}{\partial \dot{u}} &= -\frac{\partial V}{\partial s} + \frac{\cos \varphi}{h_1} \frac{\partial V}{\partial q_1} + \frac{\sin \varphi}{h_2} \frac{\partial V}{\partial q_2} - \left(\frac{\partial h_2}{\partial q_1} \sin \varphi - \frac{\partial h_1}{\partial q_2} \cos \varphi \right) \frac{1}{h_1 h_2} \frac{\partial V}{\partial \varphi}, \\ \frac{\partial S}{\partial \dot{\omega}} &= -\frac{\partial V}{\partial \varphi}. \end{aligned} \quad (9)$$

Equations (9) together with the equations (6) and with the equations

$$\dot{s} = u, \quad \dot{\varphi} = \omega + \left(\frac{\partial h_2}{\partial q_1} \sin \varphi - \frac{\partial h_1}{\partial q_2} \cos \varphi \right) \frac{u}{h_1 h_2}, \quad (10)$$

form the complete system of equations of motion of the rod on the surface.

4 Motion of the Rod on the Surface of Revolution

Suppose that the supporting surface is the surface of revolution. It is defined with respect to the fixed cartesian frame by the equation

$$\mathbf{r} = \begin{pmatrix} \rho(q_1) \cos q_2 \\ \rho(q_1) \sin q_2 \\ \zeta(q_1) \end{pmatrix}. \quad (11)$$

In this case the Lamé's coefficients h_1 and h_2 take the form:

$$h_1 = h_1(q_1) = \sqrt{\left(\frac{d\rho}{dq_1} \right)^2 + \left(\frac{d\zeta}{dq_1} \right)^2}, \quad h_2 = h_2(q_1) = \rho(q_1), \quad (12)$$

and the principal curvatures k_1 and k_2 may be written as follows:

$$k_1 = k_1(q_1) = \frac{\left(\frac{d^2\zeta}{dq_1^2} \frac{d\rho}{dq_1} - \frac{d\zeta}{dq_1} \frac{d^2\rho}{dq_1^2} \right)}{\left(\left(\frac{d\rho}{dq_1} \right)^2 + \left(\frac{d\zeta}{dq_1} \right)^2 \right)^{\frac{3}{2}}}, \quad k_2 = k_2(q_1) = \frac{\frac{d\zeta}{dq_1}}{\rho \sqrt{\left(\frac{d\rho}{dq_1} \right)^2 + \left(\frac{d\zeta}{dq_1} \right)^2}}. \quad (13)$$

We shall assume that the rod moves on the surface under the action of gravity. Suppose that the direction of gravity is defined with respect to the fixed cartesian frame by the following formula:

$$\mathbf{F} = -Mg(\cos \alpha \mathbf{e}_x + \sin \alpha \mathbf{e}_z),$$

i.e. the direction of gravity form the angle $\pi/2 - \alpha$ with the axis of symmetry of the surface. Then the potential energy of the rod has the following form:

$$V = Mg \left[\left(\rho \cos q_2 + \frac{s \frac{d\rho}{dq_1} \cos \varphi \cos q_2}{h_1} - s \sin \varphi \sin q_2 \right) \cos \alpha + \left(\zeta + \frac{s \frac{d\zeta}{dq_1} \cos \varphi}{h_1} \right) \sin \alpha \right]. \quad (14)$$

Equations of motion of the rod on the surface of revolution have the form (6), (9) and (10) where we should substitute h_1 and h_2 by the corresponding expressions (12) and k_1 and k_2 by the corresponding expressions (13). We should use also the formula (14) for the potential energy of the system.

5 Equilibria of the rod on the supporting surface

Equations of motion of the rod on the supporting surface have a particular solution $s = s_0$, $\varphi = \varphi_0$, $q_1 = q_{10}$, $q_2 = q_{20}$, where s_0 , φ_0 , q_{10} , q_{20} are constants. This solution corresponds to the equilibrium of the rod on the supporting surface. Conditions of existence of equilibria have the form:

$$\frac{\partial V}{\partial s} = \frac{\cos \varphi}{\sqrt{\left(\frac{d\rho}{dq_1} \right)^2 + \left(\frac{d\zeta}{dq_1} \right)^2}} \frac{\partial V}{\partial q_1} + \frac{\sin \varphi}{\rho} \frac{\partial V}{\partial q_2}, \quad \frac{\partial V}{\partial \varphi} = 0. \quad (15)$$

In order to clarify the physical meaning of these conditions we represent the potential energy of the rod in the form:

$$V(q_1, q_2, s, \varphi) = (M\mathbf{g} \cdot (\mathbf{r} + s\mathbf{e}_1)).$$

Substituting this expression to the conditions (15) and taking into account (2)-(5) we can rewrite the conditions (15) as follows:

$$(M\mathbf{g} \cdot s\mathbf{e}_2) = 0, \quad (M\mathbf{g} \cdot (-k_\varphi s\mathbf{e})) = 0. \quad (16)$$

Let us discuss the possible solutions of the system (16). Obviously the system has a solution $s = 0$. In this case the rod touches the surface by its center of mass and it will be in equilibrium independently on the values of the other coordinates φ , q_1 and q_2 . Therefore equilibria of this type form the manifold $\{(q_1, q_2, s, \varphi) | s = 0\}$ and they are not isolated. The investigation of stability of these equilibria is a sufficiently difficult problem. We shall not study this problem here.

When $s \neq 0$ equations (16) take the form:

$$(M\mathbf{g} \cdot \mathbf{e}_2) = 0, \quad -k_\varphi (M\mathbf{g} \cdot \mathbf{e}) = 0.$$

These conditions are valid when the rod is directed along the gravitational vertical. The center of mass G of the rod will be below (stable equilibrium) or above (unstable equilibrium) of the point P . These conditions are valid also when the normal section curvature $k_\varphi = 0$. In this case the normal section of the supporting surface through the direction of the rod is a straight line. In equilibrium position the rod touches the surface along this straight line. In this case we have the multipoint contact of the rod with the surface.

6 Existence of an invariant measure

Equations of motion of the rod on the surface of revolution resolved with respect to the derivative have a form:

$$\begin{aligned} \dot{u} &= -\frac{Ms u^2}{J_P} + \left(3\omega + 2\frac{d\rho}{dq_1} \frac{u \sin \varphi}{h_1 \rho}\right) \frac{(k_1 - k_2) u \sin \varphi \cos \varphi}{k_\varphi} + \\ &+ \left(\frac{dk_1}{dq_1} \cos^2 \varphi + \frac{dk_2}{dq_1} \sin^2 \varphi\right) \frac{u^2 \cos \varphi}{k_\varphi h_1} + \frac{Mgs}{J_P k_\varphi h_1} \left(\frac{d\rho}{dq_1} \sin \alpha - \frac{d\zeta}{dq_1} \cos q_2 \cos \alpha\right), \\ \dot{\omega} &= -\frac{Ms u \omega}{J_P} - (k_1 - k_2) k_\varphi \sin \varphi \cos \varphi u^2 + \frac{Mgs \cos \varphi \sin q_2 \cos \alpha}{J_P} + \\ &+ \frac{Mgs}{J_P h_1} \left(\frac{d\rho}{dq_1} \cos q_2 \cos \alpha + \frac{d\zeta}{dq_1} \sin \alpha\right) \sin \varphi, \\ \dot{\varphi} &= \omega + \frac{d\rho}{dq_1} \frac{u \sin \varphi}{h_1 \rho}, \quad \dot{s} = u, \quad \dot{q}_1 = -\frac{u \cos \varphi}{h_1}, \quad \dot{q}_2 = -\frac{u \sin \varphi}{\rho}. \end{aligned}$$

It is known (see e.g. [5]) that the smooth function $\mu(\mathbf{x})$ is a density of the invariant measure for the equation $\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$ if and only if $\operatorname{div}(\mu\mathbf{v}) = 0$, where

$$\operatorname{div}(\mu\mathbf{v}) = \sum_k \frac{\partial(\mu v_k)}{\partial x_k}. \quad (17)$$

Equation (17) can be represented in the form:

$$-\frac{d}{dt}(\ln \mu) = \sum_k \frac{\partial v_k}{\partial x_k},$$

Applying the corresponding formula to the equations of motion of the rod we found that these equations possess the invariant measure when the condition

$$\rho k_2 \frac{dk_1}{dq_1} - \rho k_1 \frac{dk_2}{dq_1} + 2k_1 k_2 \frac{d\rho}{dq_1} - 2k_1^2 \frac{d\rho}{dq_1} = 0 \quad (18)$$

is valid. The corresponding density of the invariant measure has the form

$$\mu = (J_P k_\varphi)^{\frac{3}{2}} \rho h_1 \sqrt{k_1}$$

(for $k_1 \neq 0$) or the form

$$\mu = \left(J_P \sin^2 \varphi \right)^{\frac{3}{2}} k_2^2 h_1$$

(for $k_1 = 0$). The existence of the invariant measure facilitates the process of integration of equations of motion of the rod on the surface of revolution. Moreover the existence of an integral invariant with a positive density is interesting also for the possible application of ergodic theory to this problem.

Note that the condition (18) is valid for many convex surfaces. Indeed it is valid in the case when the supporting surface is a cylinder, a cone, a paraboloid, a sphere, an ellipsoid etc.

Equations of motion of the rod on the surface of revolution forms the system of six first-order autonomous differential equations. We shall continue the study of this problem by various methods.

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