Investigation of the problem of motion of a heavy dynamically symmetric body on a perfectly rough plane by the Kovacic algorithm

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Abstract

The paper deals with a classical problem in mechanics of nonholonomic systems - the problem of motion of a dynamically symmetric body bounded by a surface of revolution on a fixed perfectly rough horizontal plane. Using the Kovacic algorithm [1] we found some new cases when it is possible to express the solution of the problem in quadratures.

Introduction

The problem of rolling without sliding of a heavy dynamically symmetric body bounded by a surface of revolution on a fixed horizontal plane is a classical problem of nonholonomic mechanics. In 1897, S.A. Chaplygin in his paper [2] proved the integrability of this problem and found out that its solution is reduced to integration of the second-order linear differential equation with respect to angular velocity component of the body in the projection on its axis of symmetry. However, the solution of this differential equation cannot always be found. In case when a moving body is an nonhomogeneous dynamically symmetric ball, the solution to the corresponding equation is expressed in terms of elementary functions [2]. In case of motion of a circular disk or a hoop on a horizontal plane, the solution to the mentioned equation is expressed in terms of a hypergeometric series [2]. In work [3], Kh.M. Mushtari continued the investigation of the problem of motion of a heavy body of revolution on a perfectly rough horizontal plane. Under additional condition imposing restrictions on the shape of surface of the body and mass distribution in it, two new particular cases were found, when the motion of the body can be investigated completely. In the first case the moving body is bounded by the surface of revolution formed by rotating a parabolic arc about an axis passing through its focus, and in the second case the moving rigid body is a paraboloid of revolution.

In 1986, American mathematician J. Kovacic proposed the algorithm [1] for finding a general solution of a second-order linear differential equation with variable coefficients for a case when this solution can be expressed in terms of so called liouvillian functions [1, 4]. Recall that liouvillian functions are functions that are built up from the rational functions by algebraic operations, taking exponentials and by integration. If a linear differential equation has no liouvillian solutions, the Kovacic algorithm also allows to ascertain that fact.

In this paper we discuss the application of Kovacic algorithm to the problem of motion of a rigid body on perfectly rough horizontal plane. The conclusions on existence of liouvillian solutions of equations of motion of a body are obtained in cases of motion of an infinitely thin disk, a disk of finite thickness, a paraboloid of revolution and a spindle-shaped body considered by Kh.M. Mushtari [3].

1 Problem formulation and equations of motion

Let us consider the problem of motion without sliding of a rigid body, bounded by a surface of revolution, on a fixed horizontal plane in a homogeneous gravity field. Let the center of mass G of the body be situated on the axis of symmetry $G\zeta$, and moments of inertia of the body about principal central axes of inertia $G\xi$ and $G\eta$, perpendicular to $G\zeta$, are equal to each other.

Let $Oxyz$ be a fixed coordinate system with the origin in at any point O of the plane of motion Oxy. Let us denote by θ the angle between the axis of symmetry of the body and the vertical. The height GQ of the center of mass G of the body over the plane Oxy is a function of angle θ (see [2]): $GQ = f(\theta)$.

Let us denote by M the contact point of the body with the supporting plane, by α – the angle between the horizontal tangent line MQ to the meridian $M\zeta$ of the body and the fixed axis Ox, while β is the angle between the meridian $M\zeta$ of the body and its arbitrary fixed meridian plane. The position of the body is completely defined by angles α, β, θ and coordinates x and y of the point M.

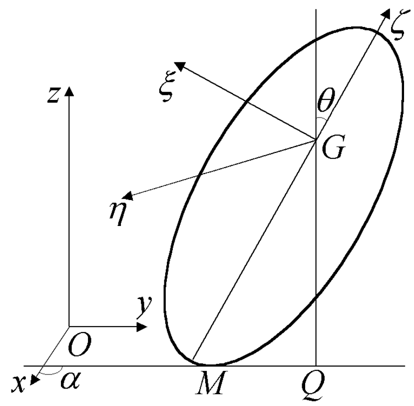

Figure 1: The rolling of the body of revolution: main frames.

Suppose the axis $G\xi$ is always situated in the plane of vertical meridian $M\zeta$, and axis $G\eta$ is normal to this plane (Fig. 1). Therefore the coordinate frame $G\xi\eta\zeta$ is moving both in space and in body. Let the vector $\mathbf{v}$ of velocity of the center of mass G, the vector $\boldsymbol{\omega}$ of the angular velocity of the body, the the angular velocity vector $\boldsymbol{\Omega}$ of trihedral $G\xi\eta\zeta$ and the reaction of supporting plane $\mathbf{R}$ be given with respect to coordinate frame $G\xi\eta\zeta$ by components v_ξ, v_η, v_ζ; p, q, r; Ω_ξ, Ω_η, Ω_ζ and R_ξ, R_η, R_ζ respectively.

Let m be the mass of the body, A_1 – its moment of inertia about the axes $G\xi$ and $G\eta$, A_3 – moment of inertia about the symmetry axis. Unit vector of axis Oz (upward vertical) is denoted by $\boldsymbol{\gamma}$.

Equations of motion of the body with respect to the $G\xi\eta\zeta$ can be written as follows:

$$m\dot{\mathbf{v}} + m\,[\boldsymbol{\Omega} \times \mathbf{v}] = -mg\boldsymbol{\gamma} + \mathbf{R}, \tag{1.1}$$

$$\dot{\mathbf{K}} + [\boldsymbol{\Omega} \times \mathbf{K}] = \left[\overrightarrow{GM}, \mathbf{R}\right], \tag{1.2}$$

$$\dot{\boldsymbol{\gamma}} + [\boldsymbol{\Omega} \times \boldsymbol{\gamma}] = 0, \tag{1.3}$$

$$\mathbf{v} + \left[\boldsymbol{\omega} \times \overrightarrow{GM}\right] = 0. \tag{1.4}$$

The equations (1.1) and (1.2) express respectively the laws of changing of momentum and angular momentum of the body, the equation (1.3) expresses the condition of constancy of vector $\boldsymbol{\gamma}$ in coordinate system $Oxyz$, and the equation (1.4) expresses the condition of absence of slipping in the point of tangency of the body and the plane. Herein g designates the value of free fall acceleration, $\mathbf{K}$ denotes angular momentum of the body with respect to the center of mass.

Let ξ, η, ζ be coordinates of the point M of the body and the plane in the moving frame $G\xi\eta\zeta$. Then $\eta = 0$ while ξ and ζ will be functions of angle θ, namely (see [2])

$$\xi = -f(\theta)\sin\theta - f'(\theta)\cos\theta, \quad \zeta = -f(\theta)\cos\theta + f'(\theta)\sin\theta. \tag{1.5}$$

Therefore the function $f(\theta)$ completely describes the shape of the moving body.

We represent the equations (1.1), (1.2) and (1.4) in scalar form:

$$\begin{gathered} \dot{v}_\xi + \Omega_\eta v_\zeta - \Omega_\zeta v_\eta = -g\sin\theta + \frac{R_\xi}{m}, \quad \dot{v}_\eta + \Omega_\zeta v_\xi - \Omega_\xi v_\zeta = \frac{R_\eta}{m}, \\ \dot{v}_\zeta + \Omega_\xi v_\eta - \Omega_\eta v_\xi = -g\cos\theta + \frac{R_\zeta}{m}; \end{gathered} \tag{1.6}$$

$$\begin{gathered} A_1\dot{p} + A_3 r\Omega_\eta - A_1 q\Omega_\zeta = -\zeta R_\eta, \quad A_1\dot{q} + A_1 p\Omega_\zeta - A_3 r\Omega_\xi = \zeta R_\xi - \xi R_\zeta, \\ A_3\dot{r} + A_1 q\Omega_\xi - A_1 p\Omega_\eta = \xi R_\eta; \end{gathered} \tag{1.7}$$

$$v_\xi + q\zeta = 0, \quad v_\eta + r\xi - p\zeta = 0, \quad v_\zeta - q\xi = 0. \tag{1.8}$$

Let us find out the relation between the components of angular velocity $\boldsymbol{\Omega}$ of trihedron $G\xi\eta\zeta$ and the angular velocity $\boldsymbol{\omega}$ of body. Since the axis $G\zeta$ is fixed in the body, then

$$\Omega_\xi = p, \quad \Omega_\eta = q. \tag{1.9}$$

The value Ω_ζ can be expressed via p. Indeed from (1.3) we have $(\boldsymbol{\Omega} \cdot \dot{\boldsymbol{\gamma}}) = 0$. Takin into account (1.9) we obtain that

$$\Omega_\zeta = \Omega_\xi \operatorname{ctg}\theta = p\operatorname{ctg}\theta. \tag{1.10}$$

Elimination the values R_ξ, R_η and R_ζ from equations (1.6) and (1.7) and some simplifications, based on (1.5), (1.8)–(1.10), lead to three equations

$$\begin{aligned} &\left[A_1 + m(\xi^2 + \zeta^2)\right]\dot{q} = mgf'(\theta) + (A_3 r - A_1 p\operatorname{ctg}\theta)p- \\ &-mp(\zeta\operatorname{ctg}\theta + \xi)(p\zeta - r\xi) - mq(\xi\dot{\xi} + \zeta\dot{\zeta}), \\ &A_1\dot{p} + A_3\frac{\zeta}{\xi}\dot{r} = (A_1 p\operatorname{ctg}\theta - A_3 r)q, \\ &\frac{d}{dt}(p\zeta - r\xi) - \frac{A_3}{m\xi}\dot{r} = (\zeta\operatorname{ctg}\theta + \xi)pq. \end{aligned} \tag{1.11}$$

Herein ξ and ζ are functions of angle θ, determined by formulae (1.5). Adding to (1.11) the evident equation

$$q = -\dot{\theta}, \tag{1.12}$$

we obtain complete system of four differential equations with respect to unknown variables p, q, r, θ.

Let us suppose that $\theta \neq \text{const}$. Then, basing on (1.12), pass in the second and third of equations (1.11) to a new independent variable – the angle θ. Then obtain:

$$\begin{aligned} &A_1 \frac{dp}{d\theta} + A_3 \frac{\zeta}{\xi} \frac{dr}{d\theta} = -A_1 p \operatorname{ctg}\theta + A_3 r, \\ &\zeta \frac{dp}{d\theta} - \frac{A_3 + m\xi^2}{m\xi} \frac{dr}{d\theta} = -(\zeta \operatorname{ctg}\theta + \xi + \zeta')p + \xi' r. \end{aligned} \tag{1.13}$$

Two first order equations (1.13) linear with respect to p and r can be reduced to one second order linear differential equation

$$\begin{aligned} &\frac{d^2 r}{d\theta^2} + \left[\frac{\cos\theta}{\sin\theta} + \frac{3m(A_1 \xi\xi' + A_3 \zeta\zeta')}{\Delta} - \frac{\frac{d}{d\theta}(\xi(\xi + \zeta'))}{\xi(\xi + \zeta')} \right] \frac{dr}{d\theta} + \\ &+ \frac{m\xi(\xi + \zeta')}{\Delta \sin\theta} \left[\frac{d}{d\theta} \left(\frac{(A_1 \xi' - A_3 \zeta)\sin\theta}{\xi + \zeta'} \right) - A_3 \sin\theta \right] r = 0, \end{aligned} \tag{1.14}$$

where $\Delta = A_1 A_3 + A_1 m\xi^2 + A_3 m\zeta^2$.

The further solving of the problem is reduced to integration of second order linear differential equation (1.14). We will consider the motion of different-shaped bodies on horizontal plane, for each one we will present a corresponding equation of the form (1.14) and through the use of Kovacic algorithm find out whether the obtained second order linear equation admits a solution expressed by liouvillian functions.

2 The specificity of application of the Kovacic algorithm to differential equations

In this section we discuss peculiarities of application of the Kovacic algorithm to second order linear differential equations. The algorithm under consideration provides obtention of general solution of second order linear differential equation in explicit form, expressed by liouvillian functions [1, 4], or show that the equation has no such solution. The step-by-step execution of the algorithm is concerned with cumbersome though uncomplicated calculations. Therefore we are not going to deepen into details of the algorithm: they are particularized in the original work of J. Kovacic [1]. We will provide with the initial concepts.

The second order linear differential equation which can be investigated by the Kovacic algorithm is supposed to have the following form:

$$\frac{d^2 y}{dx^2} + a(x) \frac{dy}{dx} + b(x) y = 0, \tag{2.1}$$

where $a(x)$ and $b(x)$ are rational functions of variable x. Through the variable change

$$z = \exp\left(\frac{1}{2} \int a(x)\, dx \right) y \tag{2.2}$$

the equation (2.1) can be rewritten as following:

$$\frac{d^2 z}{dx^2} = R(x) z, \quad R(x) = \frac{1}{2} \frac{da(x)}{dx} + \frac{1}{4} \big(a(x)\big)^2 - b(x). \tag{2.3}$$

Hereafter the rational function $R(x)$ is expanded in partial fractions and its finite poles are analyzed as well as the infinite pole. On the basis of the obtained data we conclude about the existence of liouvillian solutions of the equation (2.3).

For each body rolling on a fixed perfectly rough plane we will obtain the corresponding equation (1.14) and transform it into the form (2.3). After that we will briefly inform about the results of application of the Kovacic algorithm for finding liouvillian solutions.

3 The rolling of infinitely thin disk

Let us consider the problem of motion of dynamically symmetric infinitely thin round disk on a perfectly rough plane. Suppose that center of mass of the disk coincides with its geometric center; disk radius is equal to R. Then the distance between the center of mass of the disk and the supporting plane is expressed by formula:

$$f(\theta) = R\sin\theta.$$

According to (1.5) we obtain $\xi = -R$, $\zeta = 0$, and as a result the equation (1.14) for infinitely thin disk takes the form:

$$\frac{d^2r}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dr}{d\theta} - \frac{A_3mR^2}{A_1(A_3+mR^2)}r = 0, \quad \theta \in (0,\pi). \tag{3.1}$$

In the equation (3.1) we change the independent variable by formula $\sin\theta = x$ and introduce the following notations:

$$B = \frac{A_3mR^2}{A_1(A_3+mR^2)}, \quad y(x) = r(\theta(x)).$$

As a result the equation (3.1) is rewritten as follows:

$$\frac{d^2y}{dx^2} + \frac{1-2x^2}{x(1-x^2)}\frac{dy}{dx} - \frac{B}{1-x^2}y = 0. \tag{3.2}$$

The obtained equation is initial for application of the Kovacic algorithm. If we denote

$$a(x) = \frac{1-2x^2}{x(1-x^2)}, \quad b(x) = -\frac{B}{1-x^2},$$

then the equation (3.2) will have the form of the equation (2.1). By the variable change (2.2) the equation (3.2) is reduced to the equation

$$\frac{d^2z}{dx^2} = D(x)z, \tag{3.3}$$

$$D(x) = \frac{8B-5}{16(x+1)} - \frac{8B-5}{16(x-1)} - \frac{1}{4x^2} - \frac{3}{16(x-1)^2} - \frac{3}{16(x+1)^2}.$$

The Laurent series expansion of $D(x)$ at $x = \infty$ is given by

$$D(x)|_{x=\infty} \approx -\frac{B}{x^2} + O\left(\frac{1}{x^3}\right).$$

All initial operations necessary for application of the Kovacic algorithm are fulfilled. The direct application of the Kovacic algorithm to the equation (3.3) leads to the following result:

Theorem 3.1. *For all physically admissible values of parameters the equation (3.3) has no solutions expressed by liouvillian functions.*□

Remark 3.1. Hereinafter we consider only physically meaningful values of parameters of the problem. However, the Kovacic algorithm makes possible finding of liouvillian solutions of second order linear differential equations for all values of parameters. For example, if $B = 0$ (disk mass is concentrated on its symmetry axis), then the equation (3.1) has a solution expressed by elementary functions:

$$r(\theta) = C_1 \ln\left(\operatorname{tg}\frac{\theta}{2}\right) + C_2.$$

Remark 3.2. As it discussed previously, in works of S.A. Chaplygin [2] the general solution of the equation (3.1) was stated to be expressed in terms of hypergeometric series. Therefore we have proved that these hypergeometric series cannot be reduced to any simpler functions for all physically admissible values of parameters.

4 The rolling of disk of finite thickness

Let us consider again a moving dynamically symmetric disk on a fixed perfectly rough horizontal plane. Now we suppose that its center of mass does not coincide with its geometric center but is at a distance h of it along the symmetry axis. Such disk is accepted to be called a disk of "finite thickness" [5]. Let us suppose the moving disk to be inclined with respect to vertical and hence it contacts with the plane in a single point. We will keep the notation R for the disk radius. The height of the disk's center of mass over the supporting plane is expressed by formula

$$f(\theta) = R\sin\theta + h\cos\theta.$$

Then $\xi = -R$, $\zeta = -h$ and the equation (1.14) takes the form

$$\frac{d^2r}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dr}{d\theta} - \frac{A_3mR(R\sin\theta + h\cos\theta)}{(A_1A_3 + A_1mR^2 + A_3mh^2)\sin\theta}r = 0, \quad \theta \in (0, \frac{\pi}{2}). \tag{4.1}$$

In the equation (4.1) we change independent variable by formula $\operatorname{ctg}\theta = x$ and introduce the following notations:

$$B_1 = \frac{A_3mR^2}{A_1A_3 + A_1mR^2 + A_3mh^2}, \quad B_2 = \frac{h}{R}, \quad y(x) = r(\theta(x)).$$

As a result we get the equation (4.1) in form:

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{B_1(B_2x+1)}{(x^2+1)^2}y = 0. \tag{4.2}$$

By the variable change (2.2) the equation (4.2) is rewritten as follows:

$$\frac{d^2z}{dx^2} = D_1(x)z, \tag{4.3}$$

$$D_1(x) = \frac{(4B_1+1)i}{16(x+i)} - \frac{3+4B_1-4B_1B_2i}{16(x+i)^2} - \frac{(4B_1+1)i}{16(x-i)} - \frac{3+4B_1+4B_1B_2i}{16(x-i)^2}.$$

The Laurent series expansion of $D(x)$ at $x = \infty$ is given by

$$D_1(x)|_{x=\infty} \approx -\frac{1}{4x^2} + O\left(\frac{1}{x^3}\right).$$

The direct application of the Kovacic algorithm to the equation (4.3) leads to the following result:

Theorem 4.1. *For all physically admissible values of parameters the equation (4.3) has no solutions expressed by liouvillian functions.*□

Remark 4.1. In paper of M. Batista [5] it was stated that the general solution of the equation (4.1) is expressed by hypergeometric series. As we can see, in this case, as well as in case of infinitely thin disk, the resultant hypergeometric series cannot be reduced to liouvillian functions for all admissible values of parameters.

5 The rolling of paraboloid of revolution

Suppose the meridian of the rolling body to be parabola with parameter 2λ. Let the center of mass of the body be situated in the focus of this parabola. Then the height of center of mass of the paraboloid above the supporting plane is equal

$$f(\theta) = \frac{\lambda}{\cos\theta}$$

and by formulae (1.5)

$$\xi = -\frac{2\lambda\sin\theta}{\cos\theta}, \quad \zeta = \frac{\lambda\sin^2\theta}{\cos^2\theta} - \lambda, \quad \xi^2 = 4\lambda\zeta + 4\lambda^2.$$

The equation (1.14) in the case of rolling of dynamically symmetric paraboloid of revolution on the plane takes the form

$$\frac{d^2r}{d\theta^2} + b_1\frac{dr}{d\theta} + b_2 r = 0, \tag{6.1}$$

$$\begin{aligned} b_1 &= \frac{\cos^2\theta - 4}{\sin\theta\cos\theta} + \\ &+ \frac{6(A_3 - 2(A_3 - A_1)\cos^2\theta)m\lambda^2\sin\theta}{((A_1A_3 + 4(A_3 - A_1)m\lambda^2)\cos^4\theta - 4(A_3 - A_1)m\lambda^2\cos^2\theta + A_3m\lambda^2)\cos\theta}, \\ b_2 &= \frac{2m\lambda^2(A_3 - 2A_1)(1 + \cos^2\theta)}{(A_1A_3 + 4(A_3 - A_1)m\lambda^2)\cos^4\theta - 4(A_3 - A_1)m\lambda^2\cos^2\theta + A_3m\lambda^2}. \end{aligned}$$

Obviously, when $A_3 = 2A_1$ the equation (5.1) has a simple partial solution $r = r_0 =$ const. For the first time this fact was mentioned in the paper by Kh.M. Mushtari [3].

In the equation (5.1) we change the independent variable by formula $\cos^2\theta = x$ and introduce the following notations:

$$B = m\lambda^2, \quad y(x) = r(\theta(x)).$$

As a result we get the equation (5.1) in the following form:

$$\frac{d^2y}{dx^2} + d_1(x)\frac{dy}{dx} + d_2(x)y = 0, \tag{5.2}$$

$$d_1(x) = \frac{5 - 3x}{2x(1 - x)} - \frac{3(A_3 - 2(A_3 - A_1)x)B}{((A_1A_3 + 4(A_3 - A_1)B)x^2 - 4(A_3 - A_1)Bx + A_3B)x},$$

$$d_2(x) = \frac{(A_3 - 2A_1)B(x + 1)}{2x(1 - x)((A_1A_3 + 4(A_3 - A_1)B)x^2 - 4(A_3 - A_1)Bx + A_3B)}.$$

Let us denote x_1, x_2 complex conjugate roots of the quadratic expression

$$(A_1A_3 + 4(A_3 - A_1)B)x^2 - 4(A_3 - A_1)Bx + A_3B.$$

The variable change (2.2) turns the equation (5.2) to the equation:

$$\frac{d^2z}{dx^2} = \Pi(x)z, \tag{5.3}$$

$$\Pi(x) = \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x} + \frac{\alpha_2}{x^2} + \frac{\beta_3}{x-x_1} + \frac{\alpha_3}{(x-x_1)^2} + \frac{\beta_4}{x-x_2} + \frac{\alpha_4}{(x-x_2)^2},$$

$$\alpha_1 = \frac{3}{4}, \quad \alpha_2 = \frac{5}{16}, \quad \alpha_3 = \alpha_4 = -\frac{3}{16},$$

$$\beta_1 = \frac{4x_1 + 4x_2 - 3x_1x_2 - 5}{4(x_1-1)(x_2-1)}, \quad \beta_2 = \frac{x_1 + x_2 + 2x_1x_2}{8x_1x_2},$$

$$\beta_3 = -\frac{4x_1 + x_2 - 7x_1x_2 - 2x_1^2 + 4x_1^2x_2}{8x_1(x_1-x_2)(x_1-1)}, \quad \beta_4 = \frac{x_1 + 4x_2 - 7x_1x_2 - 2x_2^2 + 4x_1x_2^2}{8x_2(x_1-x_2)(x_2-1)}.$$

The Laurent series expansion of function $\Pi(x)$ at $x = \infty$ has the form

$$\Pi(x)|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

The direct application of the Kovacic algorithm to the equation (5.3) leads to the following result:

Theorem 5.1. For all physically admissible values of parameters of the problem all solutions of the equation (5.3) can be expressed in terms of liouvillian functions.□

However the obtained general solution has rather complicated structure. Actually, it can be expressed as

$$z(x) = \sqrt{\frac{(1 - 2x_1 - 2x_2 + 4x_1x_2)x^2 - 2x_1x_2x + x_1x_2}{(x-x_1)(x-x_2)}}\left(C_1\cos\Phi(x) + C_2\sin\Phi(x)\right),$$

$$\Phi(x) = \sqrt{(2x_1-1)(2x_2-1)(2x_1x_2 - x_1 - x_2)x_1x_2}\times$$

$$\times\int\limits_{x_0}^{x}\sqrt{\frac{u}{(u-x_1)(u-x_2)}}\frac{(u-1)du}{(1 - 2x_1 - 2x_2 + 4x_1x_2)u^2 - 2x_1x_2u + x_1x_2}$$

For the first time the fact that the solution of the problem of rolling of paraboloid of revolution on a fixed perfectly rough horizontal plane can be reduced to quadratures was mentioned by A.S. Kuleshov [6].

6 The rolling of spindle-shaped body

Consider a motion on a perfectly rough horizontal plane of the rigid body of a following shape: its surface be formed by rotation of parabolic arc about the axis, parallel to the directrix of the parabola and passing throw its focus. The shape of the body reminds a spindle, hence it was called in such way (Fig. 2). The problem of motion of spindle-shaped body was considered by Kh.M. Mushtari [3], who gave the full solution of the corresponding problem in case of additional restriction to moments of inertia of the body:

$$A_3 = \frac{2}{3}A_1. \tag{6.1}$$

For the considered body the height of its center of mass over the supporting plane is expressed by formula

$$f(\theta) = \frac{\lambda}{\sin\theta}, \quad \lambda = \text{const.}$$

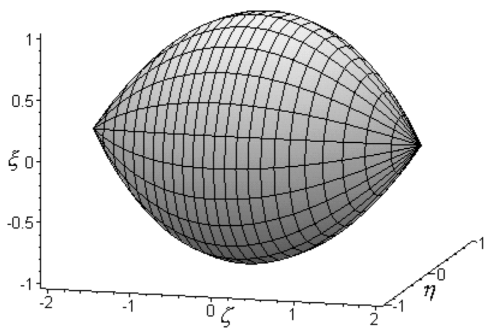

Figure 2: The rolling of spindle-shaped body on horizontal plane.

Using formulae (1.5), we find coordinates ξ and ζ of the contact point of the body with the supporting plane

$$\xi = \frac{\lambda\cos^2\theta}{\sin^2\theta} - \lambda, \quad \zeta = -\frac{2\lambda\cos\theta}{\sin\theta}, \quad \zeta^2 = 4\lambda(\xi+\lambda).$$

As a result the equation (1.14) in case of rolling of the spindle-shaped body takes the form

$$\frac{d^2r}{d\theta^2} + b_1\frac{dr}{d\theta} + b_2 r = 0, \quad \theta \in (0, \frac{\pi}{2}), \tag{6.2}$$

$$b_1 = \frac{(4\sin^4\theta - 24\sin^2\theta + 15)\cos\theta}{(1-2\sin^2\theta)(3-2\sin^2\theta)\sin\theta} -$$

$$-\frac{6(A_1 - 2(A_1-A_3)\sin^2\theta)m\lambda^2\cos\theta}{((A_1A_3 + 4(A_1-A_3)m\lambda^2)\sin^4\theta - 4(A_1-A_3)m\lambda^2\sin^2\theta + A_1m\lambda^2)\sin\theta},$$

$$b_2 = \frac{(3A_3-2A_1)m\lambda^2(1-2\sin^2\theta)^2}{(3-2\sin^2\theta)((A_1A_3+4(A_1-A_3)m\lambda^2)\sin^4\theta-4(A_1-A_3)m\lambda^2\sin^2\theta+A_1m\lambda^2)}.$$

If $3A_3 = 2A_1$, then the equation (6.2) admits the solution (see [3])

$$r = r_0 = \text{const.}$$

In the equation (6.2) we change the independent variable by formula $\sin^2\theta = x$ and introduce the following notations:

$$B = m\lambda^2, \quad y(x) = r(\theta(x)).$$

As a result we get the equation (6.2) in the following form:

$$\frac{d^2y}{dx^2} + d_1(x)\frac{dy}{dx} + d_2(x)y = 0, \tag{6.3}$$

$$d_1(x) = \frac{18 - 53x + 48x^2 - 12x^3}{2x(1-x)(1-2x)(3-2x)} -$$

$$- \frac{3(A_1 - 2(A_1 - A_3)x)B}{((A_1A_3 + 4B(A_1 - A_3))x^2 - 4B(A_1 - A_3)x + A_1B)x},$$

$$d_2(x) = \frac{(3A_3 - 2A_1)(1-2x)^2 B}{4x(1-x)(3-2x)((A_1A_3 + 4B(A_1 - A_3))x^2 - 4B(A_1 - A_3)x + A_1B)}.$$

Denote via x_1, x_2 complex conjugate roots of the quadratic expression

$$(A_1A_3 + 4B(A_1 - A_3))x^2 - 4B(A_1 - A_3)x + A_1B.$$

The variable change (2.2) turns the equation (6.3) to the equation:

$$\frac{d^2z}{dx^2} = S(x)z, \tag{6.4}$$

$$S(x) = \frac{\beta_0}{x} + \frac{\beta_1}{x-1} + \frac{\alpha_1}{(x-1)^2} + \frac{\beta_2}{x-\frac{1}{2}} + \frac{\alpha_2}{(x-\frac{1}{2})^2} + \frac{\beta_3}{x-\frac{3}{2}} + \frac{\alpha_3}{(x-\frac{3}{2})^2} +$$

$$+ \frac{\beta_4}{x-x_1} + \frac{\alpha_4}{(x-x_1)^2} + \frac{\beta_5}{x-x_2} + \frac{\alpha_5}{(x-x_2)^2},$$

$$\alpha_1 = \alpha_4 = \alpha_5 = -\frac{3}{16}, \quad \alpha_2 = \alpha_3 - \frac{3}{4},$$

$$\beta_0 = \frac{3(x_1+x_2) - 4x_1x_2}{48x_1x_2}, \quad \beta_1 = \frac{4x_1x_2 - 9(x_1+x_2) + 12}{16(x_1-1)(x_2-1)},$$

$$\beta_2 = \frac{3(x_1+x_2-1)}{(2x_1-1)(2x_2-1)}, \quad \beta_3 = \frac{15(x_1+x_2) - 8x_1x_2 - 27}{3(2x_1-3)(2x_2-3)},$$

$$\beta_4 = -\frac{\left(8x_1^3 - 36x_1^2 + 51x_1 - 25\right)(4x_2-3)\,x_1 + 15\,(x_1-1)\,x_1 + 3\,(x_2-x_1)}{16x_1\,(x_1-1)\,(2x_1-1)\,(2x_1-3)\,(x_1-x_2)},$$

$$\beta_5 = \frac{\left(8x_2^3 - 36x_2^2 + 51x_2 - 25\right)(4x_1-3)\,x_2 + 15\,(x_2-1)\,x_2 + 3\,(x_1-x_2)}{16x_2\,(x_2-1)\,(2x_2-1)\,(2x_2-3)\,(x_1-x_2)}.$$

The Laurent series expansion of function $S(x)$ at $x = \infty$ has the form

$$S(x)|_{x=\infty} \approx -\frac{3}{16x^2} + O\left(\frac{1}{x^3}\right).$$

The direct application of the Kovacic algorithm to the equation (7.4) leads to the following result:

Theorem 6.1. The equation (6.4) has a solution expressed by liouvillian functions in unique case – when the parameters of the equation satisfy the Mushtari condition (6.1). For other values of parameters the equation (6.4) does not admit liouvillian solutions.□

Remark 6.1. By direct application of the Kovacic algorithm to the equation (6.4) it can be easily proved that this equation does not admit liouvillian solutions every time when the partial fraction expansion of function $S(x)$ has a first (odd) order pole in point $x = 0$. Under the Mushtari condition (6.1) we have $\beta_0 = 0$ and therefore the function $S(x)$ does not possess singularity at $x = 0$.

The Kovacic algorithm affords to conclude of existence of liouvillian solutions in the problem of rolling without slipping of dynamically symmetric body, bounded by a surface of revolution, on a fixed horizontal plane. In this paper we have proved the existence or absence of liouvillian solutions in the problems of motion on a perfectly rough horizontal plane of infinitely thin disk and disk of finite thickness, torus, paraboloid as well as spindle-shaped body for the first time regarded in paper [3]. The propositions proved essentially add to the earlier stated results.

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