Using Modified Formulation for Hydraulic Fractures Driven by Thinning Fluids

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Abstract

The modified formulation of the hydraulic fracture problem is employed to obtain analytical solutions, to compare thinning fluids and to develop efficient numerical schemes for modeling hydraulic fractures.

1 Introduction

Hydraulic fracturing is a technique widely used for increasing production of oil, gas and thermal reservoirs. Since the pioneering works [1]-[4], it has been a subject of numerous investigations (see, e. g. reviews in the papers [5]-[11]). Theoretical investigations concerned mostly with asymptotics near the fluid front and regimes of the fracture propagation (e.g., [12], [13], [10]). Benchmark solutions have been given in [4], [14], [15] for the PKN model; and in [16] for the KGD model when the fracturing fluid is Newtonian. Solution for non-Newtonian fluids was given in [8] for the KGD model when there is no lag and the fracture strength is zero ($K_{IC} = 0$). In [17], a similar problem was studied for non-zero strength ($K_{IC} \neq 0$). The solutions were obtained by involved numerical calculations; the authors used the conventional formulation of the problem which employs the opening and the net-pressure as unknowns.

Recently [18], [19] it has been shown that the conventional formulation is ill-posed in the Hadamard sense when neglecting the lag and fixing the position of the fracture front at a time step. The disclosure of this feature, which had not been reported for more than three decades, resulted in the modified formulation of the problem [19]-[21]. This formulation employs the particle velocity and modified opening as unknowns and it includes the speed equation, prescribed at each point of the front, instead of the global mass balance. Note that earlier the speed equation was clearly distinguished in 1990 by Kemp [14],

The modified formulation opens new analytical and computational options for solving problems of hydraulic fracturing. In this paper, we use them. By employing the analytical options, we consider non-Newtonian fluids, suggest criterion for their comparison as concerns with hydraulic fracturing, and discuss general features and differences caused by using various thinning fluids. By employing the computational options, we suggest a new efficient numerical approach for pseudo three-dimensional (P3D) models.

2 Modified formulation of Nordgren problem for non-Newtonian fluid

Consider a viscous fluid with the power-type viscosity law

$$\tau_{xy} = M\left(2\dot{\varepsilon}_{xy}\right)^n, \tag{1}$$

where τ_{xy} is the shear stress, $\dot{\varepsilon}_{xy}$ is the shear strain rate, M is the consistency index, n is the behavior index. For a flow in a narrow channel in the x-direction, common derivations with using the dependence (1) yield the Poiseuille type equation between the particle velocity v, averaged over the fracture opening w, and the net-pressure p:

$$v = \left(-k_f w^{n+1}\frac{\partial p}{\partial x}\right)^{1/n}, \tag{2}$$

where $k_f = 1/\{2[2(n+1)]^n M\}$.

The PKN model refers to the plane-strain conditions in cross-sections parallel to the fracture front. For it, the elasticity equation connecting the opening with the net pressure is [4]:

$$p = k_r w, \tag{3}$$

where $k_r = [2/(\pi h)]E/(1-\nu^2)$, E is the elasticity modulus, ν is the Poisson's ratio, h is the fracture height. By using (3) in (2), we have:

$$v = k\left(-\frac{\partial w^{1/\alpha}}{\partial x}\right)^{1/n}, \tag{4}$$

where $k = (k_f k_r \alpha)^{1/n}$, $\alpha = 1/(n+2)$. The speed equation [18] implies that the particle velocity at the fluid front x_* equals to the front propagation speed v_*. As the latter is neither zero, nor infinite in physically significant cases, the equation (4) yields that the function

$$y = w^{1/\alpha} \tag{5}$$

has to be *linear* in x near the front. This favorable property suggests using the modified opening y rather than the opening w itself, which has singular derivative at the front. In terms of the particle velocity v and the modified opening y, the continuity equation is written in the modified form [19], [11]:

$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x} - \frac{y}{\alpha}\frac{\partial v}{\partial x} - \frac{y^{1-\alpha}}{\alpha}q_l, \tag{6}$$

where q_l is the term accounting for the leak-off into formation. In view of (4) and (5), the dependence between v and y is:

$$v = k\left(-\frac{\partial y}{\partial x}\right)^{1/n}. \tag{7}$$

The initial condition (IC) for solving the PDE (6) is the condition of prescribed opening along the perspective propagation path. In terms of y it is:

$$y(x, t_0) = y_0(x). \tag{8}$$

There are two boundary conditions (BC). One of them is the condition of prescribed influx $q_0(t)$ at the inlet $x = 0$:

$$(y^{\alpha} v)_{x=0} = q_0(t). \tag{9}$$

The other BC expresses zero flux, and consequently zero opening, at the front $x = x_*$:

$$y(x_*) = 0. \tag{10}$$

Besides, at the front we have the speed equation (SE) [18], which in the considered problem reads:

$$v_* = \frac{dx_*}{dt} = k \left(-\frac{\partial y}{\partial x} \right)^{1/n}_{x=x_*}. \tag{11}$$

The problem consists in solving the PDE (6), where y and v are connected by equation (7), under the IC (8) and the BC (9), (10). The SE (11) serves to trace the fracture propagation.

3 Analytical solution. Simple general solution for thinning fluids

Following the line of the paper [11], we find the solution of (6)-(11) under the initial condition of zero-opening ($y_0(x) = 0$) [22]:

$$x_* = x_{d*} x_N, \quad v_* = v_{d*} v_N, \quad v = v_d v_N, \quad t = t_d t_N, \quad w = w_d w_N, \quad y = y_d y_N, \tag{12}$$

$$p = p_d p_N, \quad q = q_d q_N, \quad q_0 = q_{0d} q_N, \quad q_l = q_{ld} q_{lN}.$$

Herein, the normalizing values, marked with the subscript N, are $x_N = (k_f k_r q_N^{n+2} t_N^{2n+2})^{1/(2n+3)}$, $v_N = x_N / t_N$, $w_N = q_N t_N / x_N$, $y_N = {w_N}^{1/\alpha}$, $p_N = k_r w_N$, $q_{lN} = q_N / x_N$; t_N and q_N are arbitrary typical values of the time and the flux, respectively. The dimensionless values, marked with the subscript d, are defined by equations:

$$x_{d*} = \xi_* t_d^{\beta_*}, \quad v_{d*} = V_* t_d^{\beta_* - 1}, \quad v_d = V(\xi) t_d^{\beta_* - 1}, \quad y_d = Y(\xi) t_d^{\beta_w / \alpha}, \quad w_d = y_d^{\alpha}, \tag{13}$$

$$q_d = Y(\xi)^{\alpha} V(\xi) t_d^{\beta_q}, \quad p_d = w_d$$

where $\xi = x_d t_d^{-\beta_*}$ is the self-similar coordinate. ξ_* is the self-similar fracture length, which is uniquely defined by the prescribed influx q_0 at the inlet. We assume that the dimensionless influx $q_{0d} = q_0 / q_N$ changes in time as $q_{0d} = t_d^{\beta_q}$. $V_* = \xi_* \beta_*$ is the self-similar fracture speed. The exponents in the time depending factors are: $\beta_* = \frac{2(n+1)+(n+2)\beta_q}{2n+3}$, $\beta_w = \frac{1+(n+1)\beta_q}{2n+3}$.

The self-similar particle velocity $V(\xi)$ and the self-similar modified opening $Y(\xi)$ are defined by the series:

$$V(\xi) = V_* \sum_{j=0}^{\infty} b_j \tau^j, \quad Y(\xi) = \frac{\xi_*^{n+1} \beta_*^n}{\alpha} \sum_{j=1}^{\infty} a_j \tau^j, \tag{14}$$

where $\tau = 1 - \xi/\xi_*$. For $j = 2, 3, ...$, the coefficients of the series are found recurrently from equations:

$$b_j = -\frac{1}{j+\alpha}\left\{\sum_{k=2}^{j}[(j-k+1+\alpha k)a_k b_{j-k+1} + (\alpha j - \frac{\beta_w}{\beta_*})a_j] - C_l \sum_{k=1}^{j} c_k q_{l(j-k)}\right\}, \quad (15)$$

$$\textstyle\sum_{k=0}^{\infty}(k+1)a_{k+1}\tau^k = \left(\sum_{j=0}^{\infty} b_j \tau^j\right)^n, \quad \sum_{k=1}^{\infty} c_k \tau^k = \tau\left(\sum_{j=0}^{\infty} a_{j+1}\tau^j\right)^\alpha$$

with $C_l = \left(\frac{\alpha}{\xi_*^{n+1}\beta_*^{n+1/\alpha}}\right)^\alpha$ and the starting values $a_1 = 1$, $b_0 = 1$, $b_1 = \frac{1}{\alpha+1}(-\alpha + \frac{\beta_w}{\beta_*} + C_l q_{l0})$, $c_1 = 1$. The solution (14), (15) accounts for leak-off prescribed by the dependence $q_{ld} = Q_l(\xi)t_d^{\beta_l}$, where $\beta_l = \beta_w - 1$; the self-similar leak-off $Q_l(\xi)$ is given by the series in $\tau = 1 - \xi/\xi_*$ as $Q_l(\xi) = \tau^\alpha \sum_{j=0}^{\infty} q_{lj}\tau^j$ with known coefficients q_{lj} (for zero leak-off, all the coefficients are zero). In the particular case of Newtonian fluid and zero leak-off, the solution (14), (15) is reduced to that obtained in [11].

Shear thinning fluids have the behavior index intermediate between those for the limiting cases of perfectly plastic ($n = 0$) and Newtonian ($n = 1$) fluids. Therefore, by continuity, we may infer conclusions for thinning fluids from the results for the limiting cases. It appears that the solutions in self-similar variables for $n = 0$ and $n = 1$ are quite close. In particular, for a constant influx ($\beta_q = 0$), the calculations give $\xi_* = \xi_{*New} = 1.00101$ for a Newtonian fluid ($n = 1$) and $\xi_* = \xi_{*P} = 1.04004$ for a perfectly plastic fluid ($n = 0$).This allows us to describe all the thinning fluids by general simple equations not depending on the behavior index. In the case of constant influx ($\beta_q = 0$) and zero leak-off, the approximate self-similar solution is:

$$\xi_* = 1.02, \quad V(\xi) = V_* = 0.74, \quad Y(\xi) = 2.20(\xi_* - \xi). \quad (16)$$

The relative error of (16) does not exceed 2% in ξ_*, 7.6% in $V(\xi)$ and 6% in $Y(\xi)$. The equations (16) show that the particle velocity is almost constant while the modified opening is almost linear along the fracture. Using (16) in (13), we have for the normalized values:

$$x_{d*} = 1.02t_d^{\beta_*}, \quad v_d = v_{d*} = 1.02\beta_* t_d^{\beta_*-1}, \quad y_d = 2.24\left(1 - \frac{x}{x_*}\right)t_d^{\beta_w/\alpha}, \quad (17)$$

$$w_d = p_d = \left[2.24\left(1 - \tfrac{x}{x_*}\right)\right]^\alpha t_d^{\beta_w}, \quad q_d = 1.02\beta_*\left[2.24\left(1 - \tfrac{x}{x_*}\right)\right]^\alpha,$$

where in the considered case of constant influx, $\beta_* = \frac{2(n+1)}{2n+3}$, $\beta_w = \frac{1}{2n+3}$; as above, $\alpha = \frac{1}{n+2}$. We see that the normalized fracture length, particle velocity, speed of propagation, opening, pressure and flux behave similarly for any behavior index. The difference is actually only in the exponents in time depending factors. The time exponents for a perfectly plastic fluids are: $\beta_* = 2/3$, $\beta_w = 1/3$, $\alpha = 1/2$; for a Newtonian fluid, they are: $\beta_* = 4/5$, $\beta_w = 1/5$, $\alpha = 1/3$. Therefore, for thinning fluids, the difference in exponents does not exceed 2/15 for both β_* and β_w.

4 Criterion of equivalence of thinning fluids. Its implications

The similarity of the solutions in the normalized variables, evident from (17), does not mean that non-normalized physical quantities also behave similarly. It is even impossible to compare non-Newtonian fluids if not making additional assumptions. Comparison becomes possible only when fixing a reference value $\dot{\varepsilon_r}$ of the strain rate $\dot{\varepsilon}_{xy}$ in the viscosity law (1). When having $\dot{\varepsilon_r}$ fixed, we may compare a fluid with given behavior n and consistency M indices with a standard fluid, having a reference behavior index n_r and a reference consistency index M_r. For convenience, the standard reference fluid may be taken as a Newtonian fluid ($n = n_r = 1$) with the reference consistency index $M_r = \mu_r$. Then, as clear from (1), to have the same shear stress for the fixed $\dot{\varepsilon}_{xy} = \dot{\varepsilon_r}$, the consistency index of the considered fluid should be [8]:

$$M = \mu_r \left(2\dot{\varepsilon}_r\right)^{1-n} . \tag{18}$$

The question is: how to properly choose the reference value $\dot{\varepsilon}_r$ for hydraulic fracture problems?

The answer is not obvious. Indeed, from the second of (17), it follows that the particle velocity is very large for small time, tending to infinity when $t \to 0$, and it is very small for large time, tending to zero when $t \to \infty$. It is easy to show that the shear strain rate is proportional to the particle velocity. Consequently, the shear strain rate changes in the entire interval $[0, \infty)$, and it is unclear which reference value to use in (18)?

In the paper [8], the authors used "an arbitrary reference value of shear strain rate $\dot{\varepsilon}_r = 25\ s^{-1}$". Following [8], the same value was adopted in the paper [17].

The solution given above provides rationale for a choice. From the definitions (12) and (13) it follows:

$$x_*(t) = \xi_*(k_f k_r q_0^{n+2})^{\beta_w} t^{\beta_*}, \tag{19}$$

where we have taken q_0 as the normalizing flux. Equation (19) suggests the needed criterion of fluid equivalence as concerns with hydraulic fracturing. *We assume fluids equivalent in their action in hydraulic fracturing when at a prescribed reference time (say, treatment time) they produce fractures of the same length.*

As mentioned, for thinning fluids, it is sufficient to consider the limiting cases of perfectly plastic and Newtonian fluids. By using (19) for these cases and equating the results we obtain the reference shear strength $\tau_r = \tau_P$:

$$\tau_r = 54^{1/5} \left(\frac{\xi_{*P}}{\xi_{*New}}\right)^3 \left(\frac{k_r\sqrt{q_0}}{t_r}\right)^{2/5} \mu_r^{3/5}. \tag{20}$$

Under prescribed k_r, q_0 and μ_r, equation (20) establishes the correspondence between the reference treatment time t_r and the reference shear strength τ_r of a thinning fluid. Since $\tau_r = \mu_r \left(2\dot{\varepsilon}_r\right)$, it can be also written as:

$$2\dot{\varepsilon}_r = 2.49 \left(\frac{k_r\sqrt{q_0}}{\mu_r t_r}\right)^{2/5}, \tag{21}$$

where we have used the evaluated values $\xi_{*New} = 1.00101$ and $\xi_{*P} = 1.04004$. Equation (21) translates the equivalence of thinning fluids in terms of their action in hydraulic fracturing into the equivalence in terms of the fluid consistency index, defined by equation (18).

With the fixed reference time t_r, we may compare differences in evolution of hydrofracture quantities caused by the difference in the consistency and behavior indices. To this end, it is sufficient to compare perfectly plastic and Newtonian fluids because for a thinning fluid, all quantities are intermediate between those for these limiting cases. The solution obtained yields the following dependencies for the ratios of major quantities:

$$\frac{x_{*New}}{x_{*P}} = \left(\frac{t}{t_r}\right)^{\beta_d}, \ \frac{v_{*New}}{v_{*P}} = \frac{\beta_{*New}}{\beta_{*P}}\left(\frac{t}{t_r}\right)^{\beta_d}, \ \frac{w_{New}(0,t)}{w_P(0,t)} = \frac{W_{0New}}{W_{0P}}\left(\frac{t}{t_r}\right)^{-\beta_d}, \quad (22)$$

where $\beta_d = \beta_{*New} - \beta_{*P} = 2/15$, $\beta_{*New}/\beta_{*P} = 5/6$, $W_{0New}/W_{0P} = 0.91959$.

From (22), it is clear that under a fixed reference time t_r and consequently under a fixed reference strain rate $\dot{\varepsilon}_r$, the curves $x_*(t)$, $v_*(t)$ and $w_*(0,t)$ for any thinning fluid intersect at the same instances equal, respectively, to $t_{x*} = t_r$ for the fracture length $x_*(t)$, $t_{v*} = (\beta_{*P}/\beta_{*New})^{1/\beta_d} t_r = 0.25476 t_r$ for the propagation speed $v_*(t)$, and $t_w = (W_{0New}/W_{0P})^{1/\beta_d} t_r = 0.53328 t_r$ for the opening at the inlet $w(0,t)$. Before these instances, the fracture length and the propagation speed is greater, while the opening is less for a thinning fluid than for the equivalent Newtonian fluid. After these instances, the relations are opposite.

Still, as the exponent β_d is quite small, the differences are not really great for the time within the range of practical significance ($10s < t < 10^5 s$) given the reference time is of order of the treatment time ($t_r \simeq 10^4 s$) This shows that there is no decisive differences to choose between fluids with various behavior indices. At most, the differences may serve to have some quantity greater (less) at time notably less or greater than the reference time. Therefore, the choice between fluids, which have various behavior indices, while providing the same fracture length at the same reference time t_r, is to be made primarily from technological and/or economic considerations. Meanwhile, when using such considerations, one needs to know the consistency indices of the compared fluids, for which the fluids are equivalent in providing the same mechanical effect. The equation (21) (or, equivalently, (20)) offers an answer. It gives the reference shear rate, which via equation (18) defines the consistency index of a fluid.

Note that these conclusions are actually obtained due to the possibility to use the self-similar variables for the PKN model. The similar option being available for the KGD model, the conclusions stay true (with obvious changes in time exponents) for the latter model. Numerical results and graphs presented in the papers [8] and [17] for the KGD model evidently confirm them. In particular, for any thinning fluid, calculations performed under a fixed reference strain rate, result in graphs $x_*(t)$, which intersect at the same point (Figure 7 of the paper [8], Figure 7a of the paper [17]). The same refers to the propagation speed $v_*(t)$ (Figure 8 of the paper [8]). The behavior of quantities before and after the intersection points is in complete agreement with the analysis above, as well.

5 Efficient numerical solution of problems for P3D models

The modified formulation also serves us to revisit pseudo three-dimensional (P3D) models. These models, discussed in detail in [6], extend the PKN model to the case when the fracture propagates into the layers embedding the pay-layer. Then the fracture height is not constant, while in-situ stresses are various in various layers. Consequently, the net-pressure, defined as the difference between the fluid pressure and the normal in-situ traction, changes along the height. Meanwhile, in P3D models, the fluid pressure is assumed constant in a vertical cross section Therefore, to keep track with the PKN-model, we may employ a fixed reference in-situ stress, say that in the pay-layer, to define the net-pressure. Below we use this agreement and conditionally call the difference the net-pressure.

For a vertical cross-section, it is assumed that we have a crack of the length h_f in plane-strain conditions in an elastic plane. In contrast with the PKN model, the normal traction, which opens the crack, is now not constant on the crack surfaces. Still the dependence between the opening and the traction is quite simple (it is defined by the classical Muskhelishvili's solution). The positions of the lower z_l and upper z_u crack tips are found from the conditions of linear fracture mechanics: $K_{Il} = K_{IC}$, $K_{Iu} = K_{IC}$, where K_{Il} and K_{Iu} are stress intensity factors (SIF) at the lower and upper tip, respectively; K_{IC} is the critical value of the SIF, defining the strength of a layer where a tip is presently located. For prescribed elastic properties, critical SIFs and in-situ normal tractions in each of the layers, the opening $w(z)$, locations z_l and z_u of the tips and consequently the fracture height $h_f = z_l - z_u$ are functions of the fluid pressure. These functions may be evaluated in advance.

With known distribution $w(z)$ of the opening along the height, we obtain the opening w_{av}, averaged over a cross section: $w_{av} = \frac{1}{h_f}\int_{z_l}^{z_u} w(z)dz$. It is a known function of the fluid pressure for a prescribed system of layers. Therefore, to keep connection with the PKN model, we may use w_{av}, rather than the net-pressure p, as the argument in the mentioned functions: $w(z, w_{av})$, $z_l(w_{av})$, $z_u(w_{av})$, $h_f(w_{av})$. From now on, we use this option. Then for the net-pressure, defined as agreed, we can write equation (3) extended to the P3D model as

$$p = k_r w_{av} F_p(w_{av}), \tag{23}$$

where $F_p(w_{av})$ is a function evaluated in advance. It equals to the unit for sufficiently small opening, in particular, near the fluid front.

Obviously, the cross-sectional area $A = w_{av}h_f$ is also a known function of w_{av}. The flux Q through the cross section is:

$$Q = \int_{z_l}^{z_u} v(z)w(z)dz, \tag{24}$$

where the distribution $v(z)$ of the particle velocity along the height is defined by the Poiseuille type equation (2). Its substitution into (24) allows us to evaluate the particle velocity averaged over a cross section:

$$v_{av} = \frac{Q}{A} = \left(-k_f w_{av}^{n+1}\frac{\partial p}{dx}\right)^{1/n} F_v(w_{av}), \tag{25}$$

where $F_v(w_{av}) = \int_{z_l}^{z_u}[w(z)/w_{av}]^{1+1/n}dz$ is a function, which may be evaluated in advance; it equals to the unit near the fluid front.

Turning to the continuity equation, we write it in terms of the cross-sectional area A and the average velocity as:

$$\frac{\partial A}{\partial t} + \frac{\partial(v_{av}A)}{\partial x} + Q_l = 0, \tag{26}$$

where Q_l is leak-off through the entire cross-section. To keep track with the PKN model, we refer A and Q_l to a reference height h_r, say the height of the pay-layer. Denote $\widetilde{w} = \frac{A}{h_r} = w_{av}\frac{h_f}{h_r}$, $q = \frac{Q}{h_r}$, $q_l = \frac{Q_l}{h_r}$. In terms of these quantities, the continuity equation (26) becomes:

$$\frac{\partial \widetilde{w}}{\partial t} + \frac{\partial(v_{av}\widetilde{w})}{\partial x} + q_l = 0. \tag{27}$$

Since h_f is a known function of w_{av}, we have $w_{av} = \widetilde{w}\widetilde{F}_w(\widetilde{w})$ with $\widetilde{F}_w(\widetilde{w})$ being a fuction to evaluate in advance. By using $\widetilde{w}$ as the argument instead of w_{av}, we may write (23) as

$$p = k_r \widetilde{w}\widetilde{F}_p(\widetilde{w}), \tag{28}$$

where $\widetilde{F}_p(\widetilde{w})$ is a known function evaluated in advance. Substitution (28) into (25) yields the equation for the averaged velocity in the form:

$$v_{av} = k\left(-\frac{\partial \widetilde{w}^{n+2}}{dx}\right)^{1/n} \widetilde{H}(\widetilde{w}), \tag{29}$$

where $\widetilde{H}(\widetilde{w}), = \widetilde{F}_w \left[\frac{d(\widetilde{w}\widetilde{F}_p)}{\partial \widetilde{w}}\right]^{1/n} \widetilde{F}_v(\widetilde{w})$, $\widetilde{F}_v(\widetilde{w}) = F_v(w_{av}(\widetilde{w}))$. Note that $\widetilde{H}(\widetilde{w}) = 1$ for sufficiently small $\widetilde{w}$. Again, since the propagation speed is neither zero, nor infinite, equation (29) implies that the function $y = \widetilde{w}^{n+2}$, which presents the modified opening, should be linear in the distance from the fracture front.

In terms of the modified opening and the averaged particle velocity, the continuity equation (27) obtains the same form as (6):

$$\frac{\partial y}{\partial t} = -v_{av}\frac{\partial y}{\partial x} - \frac{y}{\alpha}\frac{\partial v_{av}}{\partial x} - \frac{y^{1-\alpha}}{\alpha}q_l \tag{30}$$

with the dependence between y and v_{av} similar to (7):

$$v_{av} = k\left(-\frac{\partial y}{dx}\right)^{1/n} H(y), \tag{31}$$

where $H(y) = \widetilde{H}(y^\alpha)$ and, as above, $\alpha = 1/(n+2)$. The PDE (30) is to be solved under the IC:

$$y(x, t_0) = y_0(x). \tag{32}$$

where $y_0(x)$ is prescribed initial modified opening.

Since $q = Q/h_r$, for the flux $Q_0(t)$, prescribed at the inlet $x = 0$, we have the BC $q(0,t) = q_0 = Q_0/h_r$. In terms of y and v_{av}, the BC at the inlet obtains the form similar to (9):

$$(y^\alpha v_{av})_{x=0} = q_0(t). \tag{33}$$

The other BC expresses zero flux, and consequently zero opening, at the fracture front $x = x_*$. In terms of the modified opening y it is similar to (10):

$$y(x_*) = 0. \tag{34}$$

Besides, at the front we have the speed equation $v_*(t) = v_{av}(x_*, t)$, which in terms of y and v_{av} reads similar to (11):

$$v_* = \frac{dx_*}{dt} = k\left(-\frac{\partial y}{\partial x}\right)^{1/n}_{x=x_*}. \tag{35}$$

The problem consists in solving the PDE (30), where y and v_{av} are connected by equation (31), under the IC (32) and BC (33), (34). The SE (35) serves to trace the fracture propagation.

Obviously, the problem (30)-(35) for the P3D models is similar to that (6)-(11) for the PKN-model. The only difference is: the expression (31) for the averaged particle velocity contains the multiplier $H(y)$. Since the function $H(y)$ is a smooth function equal to the unity near the fracture front, this difference is not significant as concerns with methods used for solving the problem.

We come to the major conclusion: the highly efficient numerical methods, developed in the papers [19], [21] for the PKN model, may serve for stable, accurate and robust solving problems for the P3D models, as well. This opens new options for modeling hydraulic fractures in real time.

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